# **COUNTABLY** *I*-COMPACT SPACES

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ABSTRACT. We introduce the class of countably *I*-compact spaces as a proper subclass of countably *S*-closed spaces. A topological space (X, T) is called countably *I*-compact if every countable cover of *X* by regular closed subsets contains a finite subfamily whose interiors cover *X*. It is shown that a space is countably *I*-compact if and only if it is extremally disconnected and countably *S*-closed. Other characterizations are given in terms of covers by semiopen subsets and other types of subsets. We also show that countable *I*-compactness is invariant under almost open semi-continuous surjections.

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**1. Introduction.** A topological space (X,T) is called *S*-closed by Thompson [8] if every cover of *X* by semiopen subsets contains a finite subfamily whose union is dense in (X,T). Cameron [1] showed that (X,T) is *S*-closed if and only if every cover of *X* by regular closed sets has a finite subcover (accordingly, *S*-closed spaces are called rc-compact). A topological space (X,T) is called *I*-compact by Cameron [2] if every cover of *X* by regular closed sets contains a finite subfamily whose interiors cover *X*. *I*-compact spaces were further studied by Sivaraj in [7].

In [3] the class of countably *S*-closed spaces was introduced and studied. A space (X,T) is called countably *S*-closed if every countable cover of *X* by regular closed subsets has a finite subcover for *X*. It was studied further in [5], under the name countably rc-compact.

In the present paper, after the preliminaries in Section 2, we define in Section 3 the class of countably *I*-compact spaces, where a space (X, T) is called countably *I*-compact if every countable cover of *X* by regular closed subsets contains a countable subfamily whose interiors cover *X*. Then we provide characterizations of countably *I*-compact spaces in terms of semiopen covers or semipreopen covers. Also it is shown that (X, T) is countably *I*-compact if and only if it is countably *S*-compact and extremally disconnected.

In Section 4, we include main properties of countably *I*-compact spaces, while we deal in Section 5 with mappings of countably *I*-spaces.

Throughout this paper, a space will mean a topological space with no separation axiom assumed. We always use (X, T) and (Y, M) to denote topological spaces, and  $\mathbb{N}$  denotes the set of natural numbers.

**2. Preliminaries.** Let (X, T) be a space and let  $A \subseteq X$ . Then  $\operatorname{int}_T(A)$  and  $\operatorname{cl}_T(A)$  (or simply  $\operatorname{int}(A)$  and  $\operatorname{cl}(A)$ ) denote the interior of A and the closure of A in (X, T), respectively. The subset  $A \subseteq X$  is *regular open* (*regular closed*) if  $A = \operatorname{int} \operatorname{cl}(A)$  ( $A = \operatorname{clint}(A)$ ).

It is clear that *A* is regular open if and only if its complement is regular closed. Also we include the following easy facts.

**REMARK 2.1.** Let A be a subset of a space (X, T). Then

(a) *A* is regular open if and only if A = int(F) for some closed subset *F* of (X, T).

(b) *A* is regular closed if and only if A = cl(U) for some  $U \subseteq T$ .

We use RO(X,T) and RC(X,T) to denote the family of all regular open subsets and the family of all regular closed subsets of (X,T), respectively.

A subset *A* of a space (X,T) is called *semiopen* (resp., *preopen*,  $\alpha$ -*open*) if  $A \subseteq$  clint(*A*) (resp.,  $A \subseteq$  int cl(*A*),  $A \subseteq$  int clint(*A*)). We let SO(*X*,*T*) (resp., PO(*X*,*T*),  $T^{\alpha}$ ) denote the family of semiopen (resp., preopen,  $\alpha$ -open) subsets of a space (*X*,*T*). We point here to the fact that  $T^{\alpha}$  is a topology on *X* with  $T \subseteq T^{\alpha}$ .

**REMARK 2.2.** For a space (X, T) it is well known that

(a)  $\operatorname{RO}(X,T) \bigcup \operatorname{RC}(X,T) \subseteq \operatorname{SO}(X,T)$ ,

(b)  $T^{\alpha} = SO(X,T) \cap PO(X,T)$ .

A space (X, T) is called *extremally disconnected* (abbreviated e.d.) if cl(U) is open for any open subset U of X. We include for later use the following well-known facts.

**LEMMA 2.3.** A space (X,T) is e.d. if and only if whenever  $U, V \in T$  and  $U \cap V = \emptyset$  then  $cl(U) \cap cl(V) = \emptyset$ .

**LEMMA 2.4** (see [3]). If P is a preopen ( $\equiv$  locally dense) subset of a space (X, T), then

$$\operatorname{RC}(P,T|_P) = \{F \cap P : F \in \operatorname{RC}(X,T)\}.$$
(2.1)

**3.** Countably *I*-compact spaces. In [3], a space (X, T) is called *countably S-closed* if every countable cover of *X* by regular closed subsets has a finite subcover (such a space is also studied in [5] where it was called *countably rc-compact*). A space (X, T) is called *feebly compact* if every countable open cover of *X* contains a finite subfamily whose union is dense in (X, T). It is known that every countably *S*-closed space is feebly compact (see [3, Proposition 2.1]) while [3, Example 4.3] provides several feebly compact spaces which are not countably *S*-closed.

We define now the class of countably *I*-compact spaces.

**DEFINITION 3.1.** A space (X, T) is called *countably I-compact* if every countable cover  $\{F_n : n \in \mathbb{N}\}$  of *X* by regular closed subsets contains a finite subfamily

$$\{F_k : k = 1, \dots, m\}$$
(3.1)

such that

$$X = \bigcup_{k=1}^{m} \operatorname{int}(F_k).$$
(3.2)

We start with the following characterization of countably I-compact spaces.

**THEOREM 3.2.** A space (X,T) is countably *I*-compact if and only if it is countably *S*-compact and *e.d.* 

## Proof

**NECESSITY.** Let (X, T) be countably *I*-compact. It is immediate from the definition that (X, T) is countably *S*-closed. Now, suppose that (X, T) is not e.d. We find an open set *U* such that cl(U) is not open and therefore  $cl(U) - intcl(U) \neq \emptyset$ . We put V = X - cl(U). Then  $\{cl(U), cl(V)\}$  is a countable cover of *X* by regular closed subsets but  $X \neq intcl(U) \bigcup intcl(V)$ , a contradiction.

**SUFFICIENCY.** Let  $\{F_n : n \in \mathbb{N}\}$  be a countable cover of X by regular closed subsets. Since (X,T) is countably S-closed then there exists  $m \in \mathbb{N}$  such that  $X = \bigcup_{k=1}^{m} F_k$ . For each k = 1, ..., m, we pick  $U_k \in T$  such that  $F_k = \operatorname{cl}(U_k)$ . Since (X,T) is e.d., then  $\operatorname{cl}(U_k)$  is open for each k = 1, ..., m. Thus  $X = \bigcup_{k=1}^{m} F_k = \bigcup_{k=1}^{m} \operatorname{int}(F_k)$  and (X,T) is countably I-compact.

We point here that [3, Example 4.2] provides a countably *S*-closed space which is not e.d. Thus the class of countably *I*-compact spaces is a proper subclass of the countably *S*-closed spaces.  $\Box$ 

**DEFINITION 3.3** (see [3]). A space (X,T) is called *km-perfect* if for every  $G \in$ RO(X,T) and for every point  $x \in X - G$  there exists a sequence  $\{U_n : n \in \mathbb{N}\}$  of open subsets of X such that  $\bigcup_{n \in \mathbb{N}} U_n \subseteq G \subseteq \bigcup_{n \in \mathbb{N}} cl(U_n)$  and  $x \notin \bigcup_{n \in \mathbb{N}} cl(U_n)$ .

We include the following fact from [3].

**PROPOSITION 3.4** (see [3, Theorem 3.2]). Let (X,T) be countably S-closed and kmperfect. Then (X,T) is e.d.

We now have the following result.

**THEOREM 3.5.** A space (X, T) is countably *I*-compact if and only if (X, T) is countably *S*-closed and km-perfect.

#### Proof

**NECESSITY.** Follows easily from the fact that every e.d. space is km-perfect (see [3, Theorem 3.1(i)]).

**SUFFICIENCY.** Follows from Theorem 3.2 and Proposition 3.4.

**THEOREM 3.6.** The following conditions are equivalent for an e.d. space (X,T):

- (a) (X,T) is countably *I*-compact,
- (b) (X,T) is countably S-closed,
- (c) (X,T) is feebly compact.

**PROOF.** (a) $\Rightarrow$ (b). Is clear.

(b) $\Rightarrow$ (c). Is clear.

(c)⇒(a). Since (*X*, *T*) is feebly compact and e.d. then it is countably *S*-closed (see [3, Corollary 3.3(i)]). Thus (*X*, *T*) is countably *I*-compact by Theorem 3.2.  $\Box$ 

To state our final characterization of countably *I*-compact spaces we recall that a subset *A* of a space (X, T) is called *regular semiopen* if there exists  $G \in \text{RO}(X, T)$  such that  $G \subseteq A \subseteq \text{cl}(G)$ . The subset *A* is called *semipreopen* (see [4]) if  $A \subseteq \text{clintcl}(A)$ .

Let RSO(X,T) and SPO(X,T) denote, respectively, the family of all regular semiopen subsets of (X,T) and the family of all semipreopen subsets of (X,T). It is

easy to check the following inclusions for a space (X, T):

$$\operatorname{RC}(X,T) \subseteq \operatorname{RSO}(X,T) \subseteq \operatorname{SO}(X,T) \subseteq \operatorname{SPO}(X,T).$$
 (3.3)

Also, the following is an easy fact whose proof is omitted.

**PROPOSITION 3.7.** A subset A of a space (X,T) is semipreopen if and only if cl(A) is regular closed.

We have now the following characterization.

**THEOREM 3.8.** The following conditions are equivalent for a space (X, T):

(a) (X,T) is countably *I*-compact.

(b) For every countable cover  $\{A_n : n \in \mathbb{N}\}$  of X by semipreopen subsets there exists  $m \in n$  such that  $X = \bigcup_{k=1}^{m} \operatorname{int} \operatorname{cl}(A_k)$ .

(c) For every countable cover  $\{S_n : n \in \mathbb{N}\}$  of X by semiopen subsets there exists  $m \in n$  such that  $X = \bigcup_{k=1}^{m} \operatorname{int} \operatorname{cl}(S_k)$ .

(d) For every countable cover  $\{R_n : n \in \mathbb{N}\}$  of X by regular semiopen subsets there exists  $m \in n$  such that  $X = \bigcup_{k=1}^{m} \operatorname{int} \operatorname{cl}(R_k)$ .

**PROOF.** Follows easily from Proposition 3.7 and the remark preceding it involving the stated inclusions.

**4. Properties of countably** *I***-compact spaces.** To begin with, we point to the fact that, given a space (X, T), the family RO(X, T) is a base for a topology  $T_s \subseteq T$  on X called the *semiregularization* of (X, T). A property P of topological spaces is called a *semiregular property* if a space (X, T) has property P if and only if  $(X, T_s)$  has property P. Countable *S*-closedness is a semiregular property (see [3, Proposition 2.6]). Also, it is a well-known fact that extremal disconnectedness is a semiregular property. Now, these remarks, together with Theorem 3.2, form the proof of the following result.

**THEOREM 4.1.** The property of being a countably *I*-compact space is a semiregular property.

The remaining results of this section deal with subsets of countably *I*-compact spaces, or with those subsets which are countably *I*-compact.

**PROPOSITION 4.2.** Let (X,T) be a countably *I*-compact space and let *S* be a regular semiopen subset of (X,T). Then (S,T | S) is countably *I*-compact.

**PROOF.** Since extremal disconnectedness is semiopen hereditary (see [6, Corollary 4.3]), it follows that (S, T | S) is e.d.

We show that (S, T | S) is countably *S*-closed. Choose  $U \in \text{RO}(X, T)$  such that  $U \subseteq S \subseteq \text{cl}(U)$ . We have that *U* is, by [3, Proposition 2.9(i)], countably *S*-closed. It follows that (S, T | S) is, by [3, Proposition 2.9(ii)], countably *S*-closed. We conclude that (S, T | S) is, by Theorem 3.2, countably *I*-compact.

**COROLLARY 4.3.** Let (X,T) be a countably *I*-compact space. (a) If  $G \in RO(X,T)$  then (G,T | G) is countably *I*-compact. (b) If  $F \in RC(X,T)$  then (F,T | F) is countably *I*-compact. **PROOF.** Follows easily from the obvious facts that  $RO(X,T) \subseteq RSO(X,T)$  and  $RC(X,T) \subseteq RSO(X,T)$ .

**PROPOSITION 4.4.** If a space (X,T) is a finite union of regular open countably *I*-compact subspaces  $G_k$ , k = 1,...,n, then (X,T) is countably *I*-compact.

**PROOF.** Let  $\{F_n : n \in \mathbb{N}\}$  be a countable cover of *X* by regular closed subsets of the space (X, T). For  $1 \le k \le m$ , we let  $\mathcal{F}_k = \{G_k \cap F_n : n \in \mathbb{N}\}$ . Again, by Lemma 2.4,  $\mathcal{F}_k$  is a cover of  $G_k$  by regular closed subsets of the countably *I*-compact subspace  $(G_k, T|_{G_k})$ . Thus there exists  $\ell_k \in \mathbb{N}$  such that

$$G_k = \bigcup_{j=1}^{\ell_k} \operatorname{int}_{G_k} (G_k I F_j) \subseteq \bigcup_{j=1}^{\ell_k} \operatorname{int}_T (F_j).$$

$$(4.1)$$

We let  $\ell = \max\{\ell_k : k = 1, ..., n\}$ . Then  $X = \bigcup_{k=1}^m G_k = \bigcup_{j=1}^\ell \operatorname{int} F_j$ , and the proof is complete.

**5. Mappings of countably** *I***-compact spaces.** A function  $f : (X,T) \rightarrow (Y,M)$  is called *irresolute* (resp., *semi-continuous*) if  $f^{-1}(V)$  is a semiopen subset of (X,T) for each semiopen (resp., open) subset *V* of (Y,M). The function *f* is called *almost open* if  $f^{-1}(cl(B)) \subseteq cl(f^{-1}(B))$  for every  $B \in M$ .

It is well known (see [3, Proposition 2.7(i)]) that the irresolute image of a countably *S*-closed space is countably *S*-closed. So the next result follows easily from Theorem 3.2.

**THEOREM 5.1.** Let f be an irresolute function from a countably I-compact space (X,T) onto an e.d. space (Y,M). Then (Y,M) is countably I-compact.

Recall that a subset *A* of (X, T) is called *semiclosed* if X - A is semiopen. The *semiclosure* of a subset *A* of a space (X, T), written scl(A), is the intersection of all semiclosed subsets of (X, T) that contain *A*.

**PROPOSITION 5.2** (see [9, Theorem 3.1]). A function  $f : (X,T) \rightarrow (Y,M)$  is semicontinuous if and only if  $f(\operatorname{scl}(A)) \subseteq \operatorname{cl}(f(A))$  for every  $A \subseteq X$ .

**PROPOSITION 5.3** (see [7, Corollary 2.3]). Let (X,T) be e.d. If  $A \in SO(X,T)$  then scl(A) = cl(A)

Now, we state our main result of this section.

**THEOREM 5.4.** Let  $f : (X,T) \rightarrow (Y,M)$  be a semi-continuous almost open surjection. If (X,T) is countably *I*-compact then so is (Y,M).

**PROOF.** First, we show that (Y,M) is countably *S*-closed. Let  $\{S_n : n \in \mathbb{N}\}$  be a countable semiopen cover of the space (Y,M). For each  $n \in \mathbb{N}$  we choose  $V_n \in M$  such that  $V_n \subseteq S_n \subseteq cl(V_n)$ . Note that  $cl(S_n) = cl(V_n)$  for each  $n \in \mathbb{N}$ . Since *f* is semicontinuous,  $f^{-1}(V_n) \in SO(X,T)$  and hence  $cl(f^{-1}(V_n)) \in RC(X,T)$  for each  $n \in \mathbb{N}$ .

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Note that

$$X = f^{-1}\left(\bigcup_{n \in \mathbb{N}} S_n\right) \subseteq f^{-1}\left(\bigcup_{n \in \mathbb{N}} \operatorname{cl}\left(V_n\right)\right) = \bigcup_{n \in \mathbb{N}} f^{-1}(\operatorname{cl}\left(V_n\right)) \subseteq \bigcup_{n \in \mathbb{N}} \operatorname{cl}\left(f^{-1}(V_n)\right) \quad (5.1)$$

(since f is almost open).

So the family  $\{cl(f^{-1}(V_n)) : n \in \mathbb{N}\}$  is a countable cover of X by regular closed subsets. So there exists  $m \in \mathbb{N}$  such that  $X = \bigcup_{k=1}^{m} cl(f^{-1}(V_k))$ . It follows that

$$Y = f\left(\bigcup_{k=1}^{m} \operatorname{cl}\left(f^{-1}(V_{k})\right)\right) = (\text{by Proposition 5.3}) f\left(\bigcup_{k=1}^{m} \operatorname{scl}\left(f^{-1}(V_{k})\right)\right)$$
$$= \bigcup_{k=1}^{m} f\left(\operatorname{scl}\left(f^{-1}(V_{k})\right)\right) \subseteq (\text{by Proposition 5.2}) \bigcup_{k=1}^{m} \operatorname{cl}\left(f\left(f^{-1}(V_{k})\right)\right)$$
$$= \bigcup_{k=1}^{m} \operatorname{cl}\left(V_{k}\right) = \bigcup_{k=1}^{m} \operatorname{cl}\left(S_{k}\right).$$
(5.2)

This proves that (Y, M) is countably *S*-closed.

Next, we prove that (Y, M) is e.d. Let  $G, H \in M$  with  $G \cap H = \emptyset$ . It is enough, by Lemma 2.3, to show that  $cl(G) \cap cl(H) = \emptyset$ . Now, we have

$$f^{-1}(\operatorname{cl}(G) \cap \operatorname{cl}(H)) = f^{-1}(\operatorname{cl}(G)) \cap f^{-1}(\operatorname{cl}(H))$$
  

$$\subseteq (\text{as } f \text{ is almost open)} \operatorname{cl}(f^{-1}(G)) \cap \operatorname{cl}(f^{-1}(H)).$$
(5.3)

But *f* is semi-continuous, so  $f^{-1}(G)$ ,  $f^{-1}(H) \in SO(X,T)$ . We choose  $U, V \in T$  such that  $U \subseteq f^{-1}(G) \subseteq cl(U)$  while  $V \subseteq f^{-1}(H) \subseteq cl(V)$ . We note that  $U \cap V = \emptyset$  (as  $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ ) and, since (X,T) is e.d., then (by Lemma 2.3) we have  $cl(U) \cap cl(V) = \emptyset$ . It is clear that  $cl(U) = cl(f^{-1}(G))$  and  $cl(V) = cl(f^{-1}(H))$ . We conclude that  $f^{-1}(cl(G) \cap cl(H)) \subseteq cl(f^{-1}(G)) \cap cl(f^{-1}(H)) = \emptyset$  and therefore  $cl(G) \cap cl(H) = \emptyset$ , as required. The proof of the theorem is now complete.

**COROLLARY 5.5.** Let  $f : (X,T) \rightarrow (Y,M)$  be an open continuous surjection. If (X,T) is countably *I*-compact then so is the space (Y,M).

**COROLLARY 5.6.** If a product space  $\prod_{\alpha \in \Delta} X_{\alpha}$  is countably *I*-compact then  $(X_{\alpha}, T_{\alpha})$  is countably *I*-compact, for each  $\alpha \in \Delta$ .

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