IJMMS 26:11 (2001) 707–712
PII. S0161171201006299
http://ijmms.hindawi.com

ON FUZZY POINTS IN SEMIGROUPS

KYUNG HO KIM

(Received 4 December 2000)

ABSTRACT. We consider the semigroup $\underline{S}$ of the fuzzy points of a semigroup S, and discuss the relation between the fuzzy interior ideals and the subsets of $\underline{S}$ in an (intra-regular) semigroup S.

2000 Mathematics Subject Classification. 03E72, 20M12.

1. Introduction. After the introduction of the concept of fuzzy sets by Zadeh [8], several researches were conducted on the generalizations of the notion of fuzzy sets. Pu and Liu [5] introduced the notion of fuzzy points. In [6, 7, 8], authors characterized fuzzy ideals as fuzzy points of semigroups. In [1, 2, 3], Kuroki discussed the properties of fuzzy ideals and fuzzy bi-ideals in a semigroup and a regular semigroup. In this paper, we consider the semigroup $\underline{S}$ of the fuzzy points of a semigroup S, and discuss the relation between the fuzzy interior ideals and the subsets of $\underline{S}$ in an (intra-regular) semigroup S.

2. Preliminaries. Let S be a semigroup with a binary operation "$\cdot$". A nonempty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$, a *left* (resp., *right*) *ideal* of S if $SA \subseteq A$ (resp., $AS \subseteq A$), and a *two-sided ideal* (or simply *ideal*) of S if A is both a left and a right ideal of S. A subsemigroup A of S is called a *bi-ideal* of S if $ASA \subseteq A$. Let S be a semigroup. A nonempty subset A of S is called an interior ideal of S if $SAS \subseteq A$. A function f from a set X to $[0,1]$ is called a *fuzzy subset* of X. The set $\{x \in X \mid f(x) > 0\}$ is called the *support*, denoted by $\operatorname{supp} f$, of f. The closed interval $[0,1]$ is a complete lattice with two binary operations "$\vee$" and "$\wedge$", where $\alpha \vee \beta = \sup\{\alpha,\beta\}$ and $\alpha \wedge \beta = \inf\{\alpha,\beta\}$ for each $\alpha,\beta \in [0,1]$. For any $\alpha \in (0,1]$ and $x \in X$, a fuzzy subset x_α of X is called a *fuzzy point* in X if

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

for each $y \in X$. If f is a fuzzy subset of X, then a fuzzy point x_α is said to be *contained in* f, denoted by $x_\alpha \in f$, if $\alpha \le f(x)$. It is clear that $x_\alpha \in f$ for some $\alpha \in (0,1]$ if and only if $x \in \operatorname{supp} f$.

A fuzzy subset f of a semigroup S is called a *fuzzy subsemigroup* of S if

$$f(xy) \ge f(x) \wedge f(y), \tag{2.2}$$

for all $x,y \in S$, a *fuzzy left* (resp., *right*) *ideal* of S if

$$f(xy) \geq f(y) \text{ (resp., } f(xy) \geq f(x)), \tag{2.3}$$

for all $x, y \in S$, and a *fuzzy ideal* of S if f is both a fuzzy left and a fuzzy right ideal of S. It is clear that f is a fuzzy ideal of a semigroup S if and only if $f(xy) \geq f(x) \vee f(y)$ for all $x, y \in S$, and that every fuzzy left (right, two-sided) ideal of S is a fuzzy subsemigroup of S.

3. Interior ideals of fuzzy points. Let $\mathcal{F}(S)$ be the set of all fuzzy subsets of a semigroup S. For each $f, g \in \mathcal{F}(S)$, the product of f and g is a fuzzy subset $f \circ g$ defined as follows:

$$(f \circ g)(x) = \begin{cases} \bigvee (f(y) \wedge g(z)) & \text{if } x = yz \ (y, z \in S), \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

for each $x \in S$. It is clear that $(f \circ g) \circ h = f \circ (g \circ h)$, and that if $f \subseteq g$, then $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$ for any f, g, and $h \in \mathcal{F}(S)$. Thus $\mathcal{F}(S)$ is a semigroup with the product "$\circ$".

Let $\underline{S}$ be the set of all fuzzy points in a semigroup S. Then $x_\alpha \circ y_\beta = (xy)_{\alpha \wedge \beta} \in \underline{S}$ and $x_\alpha \circ (y_\beta \circ z_\gamma) = (xyz)_{\alpha \wedge \beta \wedge \gamma} = (x_\alpha \circ y_\beta) \circ z_\gamma$ for any x_α, y_β, and $z_\gamma \in \underline{S}$. Thus $\underline{S}$ is a subsemigroup of $\mathcal{F}(S)$.

For any $f \in \mathcal{F}(S)$, $\underline{f}$ denotes the set of all fuzzy points contained in f, that is, $\underline{f} = \{x_\alpha \in \underline{S} \mid f(x) \geq \alpha\}$. If $x_\alpha \in \underline{S}$, then $\alpha > 0$.

For any $A, B \subseteq \underline{S}$, we define the product of two sets A and B as $A \circ B = \{x_\alpha \circ y_\beta \mid x_\alpha \in A, y_\beta \in B\}$.

Lemma 3.1 (see [7, Lemma 4.1]). *Let f be a nonzero fuzzy subset of a semigroup S. Then the following conditions are equivalent:*

(1) *f is a fuzzy left (right, two-sided) ideal of S.*
(2) *$\underline{f}$ is a left (right, two-sided) ideal of $\underline{S}$.*

Lemma 3.2 (see [7, Lemma 4.2]). *Let f and g be two fuzzy subsets of a semigroup S. Then*

(1) $\underline{f \cup g} = \underline{f} \cup \underline{g}$.
(2) $\underline{f \cap g} = \underline{f} \cap \underline{g}$.
(3) $\underline{f \circ g} \supseteq \underline{f} \circ \underline{g}$.

A fuzzy subsemigroup f of a semigroup S is called a fuzzy interior ideal of S if $f(xay) \geq f(a)$ for all $x, a, y \in S$.

Lemma 3.3. *Let f be a nonzero fuzzy subset of a semigroup S. Then the following conditions are equivalent:*

(1) *f is a fuzzy interior ideal of S.*
(2) *$\underline{f}$ is an interior ideal of $\underline{S}$.*

Proof. Let f be a fuzzy interior ideal of S, and let $x_\alpha, z_\gamma \in \underline{S}$ and $y_\beta \in \underline{f}$. Then since $\alpha > 0$, $\gamma > 0$, and $0 < \beta \leq f(y)$, we have

$$0 < \alpha \wedge \beta \wedge \gamma \leq \alpha \wedge f(y) \wedge \gamma \leq f(y) \leq f(xyz). \tag{3.2}$$

Hence $x_\alpha \circ y_\beta \circ z_\gamma = (xyz)_{\alpha\wedge\beta\wedge\gamma} \in \underline{f}$. This implies that $\underline{S} \circ \underline{f} \circ \underline{S} \subseteq \underline{f}$, thus $\underline{f}$ is an interior ideal of $\underline{S}$. Conversely, suppose that $\underline{f}$ is an interior ideal of $\underline{S}$. Let $x, y, z \in S$. If $f(y) = 0$, then $f(y) = 0 \leq f(xyz)$. If $f(y) \neq 0$, then $y_{f(y)} \in \underline{f}$ and $x_{f(y)}, z_{f(y)} \in \underline{S}$. Since $\underline{f}$ is an interior ideal of $\underline{S}$, we have

$$(xyz)_{f(y)} = (xyz)_{f(y)\wedge f(y)\wedge f(y)} = x_{f(y)} \circ y_{f(y)} \circ z_{f(y)} \in \underline{f}. \tag{3.3}$$

This implies that $f(xyz) \geq f(y)$, and hence f is a fuzzy interior ideal of S. □

It is clear that any ideal of a semigroup S is an interior ideal of S. It is also clear that any fuzzy ideal of S is a fuzzy interior ideal of S. A semigroup S is called regular if, for each element a of S, there exists an element x in S such that $a = axa$.

Theorem 3.4. *Let f be any fuzzy set in a regular semigroup S. Then the following conditions are equivalent:*

(1) *f is a fuzzy right (resp., left) ideal of S.*

(2) *$\underline{f}$ is an interior ideal of $\underline{S}$.*

Proof. It suffices to show that (2) implies (1). Assume that (2) holds. Let x be any element in S. Then since S is regular, there exists element a in S such that $x = xax$. If $f(x) = 0$, $f(x) = 0 \leq f(xy)$. If $f(x) \neq 0$, then $x_{f(x)} \in \underline{f}$ and $y_{f(x)} \in \underline{S}$. Since $\underline{f}$ is an interior ideal of $\underline{S}$, we have

$$\begin{aligned} (xy)_{f(x)} &= (xaxy)_{f(x)} \\ &= \big((xa)xy\big)_{f(x)\wedge f(x)\wedge f(x)} \\ &= (xa)_{f(x)} \circ x_{f(x)} \circ y_f(x) \in \underline{f}. \end{aligned} \tag{3.4}$$

This implies that $f(xy) \geq f(x)$, and hence f is a fuzzy right ideal of S. □

Theorem 3.5 (see [7, Theorem 3.3])**.** *Let S be a semigroup. If for a fixed $\alpha \in (0,1]$, $f_\alpha : S \to \underline{S}$ is a function defined by $f_\alpha(x) = x_\alpha$, then f_α is a one-to-one homomorphism of semigroups.*

From Theorem 3.5, we can consider $\underline{S}$ as an extension of a semigroup S.

Let f be a fuzzy subset of a semigroup S. If $\mathcal{R}_f$ is the subset of $\underline{S} \times \underline{S}$ given as following:

$$\mathcal{R}_f = \{(x_\alpha, x_\alpha) \mid x_\alpha \notin \underline{f}\} \cup \{(x_\alpha, x_\beta) \mid x_\alpha,\ x_\beta \in \underline{f}\}, \tag{3.5}$$

then the set $\mathcal{R}_f$ is an equivalence relation on $\underline{S}$. We can consider the quotient set $\underline{S}/\mathcal{R}_f$, with the equivalence classes $\overline{x}_\alpha$ for each $x \in S$. We will denote the subset $\{\overline{x}_\alpha \mid x_\alpha \in \underline{f}\}$ of $\underline{S}/\mathcal{R}_f$ by $E(\underline{f})$. If $\overline{x}_\alpha \in E(\underline{f})$, then $\overline{x}_\alpha = \overline{x}_{f(x)} = \{x_\lambda \mid 0 < \lambda \leq f(x)\}$. If $\overline{x}_\alpha \notin E(\underline{f})$, then $\overline{x}_\alpha = \{x_\alpha\}$ (singleton set).

Let f be a fuzzy subsemigroup of S. If the product "$*$" on $E(\underline{f})$ is defined by $\overline{x}_\alpha * \overline{y}_\beta = \overline{(xy)}_{\alpha\wedge\beta}$ for each $\overline{x}_\alpha, \overline{y}_\beta \in E(\underline{f})$, then $E(\underline{f})$ is a semigroup under the operation "$*$".

Theorem 3.6. *Let f be a fuzzy interior ideal of S. Then $E(\underline{f})$ is an interior ideal of $(\underline{S}/\mathcal{R}_f, *)$.*

Proof. Let $\overline{x}_\alpha, \overline{y}_\beta \in \underline{S}/\mathcal{R}_f$ and $\overline{a}_\gamma \in E(\underline{f})$. Then since $x_\alpha, y_\beta \in \underline{S}$, $a_\gamma \in \underline{f}$ and $\underline{f}$ is an interior ideal of $\underline{S}$, $(xay)_{\alpha\wedge\gamma\wedge\beta} = x_\alpha \circ a_\gamma \circ y_\beta \in \underline{f}$. Hence $\overline{x}_\alpha * \overline{a}_\gamma * \overline{y}_\beta = \overline{(xay)}_{\alpha\wedge\gamma\wedge\beta} \in E(\underline{f})$. It follows that $E(\underline{f})$ is an interior ideal of $\underline{S}/\mathcal{R}_f$. □

A semigroup S is called intra-regular if, for each element a of S, there exists elements x and y in S such that $a = xa^2y$.

Theorem 3.7. *A semigroup S is intra-regular if and only if the semigroup $\underline{S}$ is intra-regular.*

Proof. Let $a_\alpha \in \underline{S}$. Then since S is intra-regular and $a \in S$, there exist x, y in S such that $a = xa^2y$. Thus $x_\alpha \in \underline{S}$ and $y_\alpha \in \underline{S}$ and

$$x_\alpha \circ a_\alpha \circ a_\alpha \circ y_\alpha = x_\alpha \circ (a^2)_\alpha \circ y_\alpha = (xa^2y)_\alpha = a_\alpha. \tag{3.6}$$

Hence $\underline{S}$ is intra-regular. Conversely, let $\underline{S}$ be intra-regular and $a \in S$. Then for any $\alpha \in (0,1]$, there exist elements $x_\beta, y_\delta \in \underline{S}$ such that

$$a_\alpha = x_\beta \circ a_\alpha \circ a_\alpha \circ y_\delta = (xa^2y)_{\beta\wedge\alpha\wedge\delta}. \tag{3.7}$$

This implies that $a = xa^2y$ and $x, y \in S$. □

Theorem 3.8. *For a fuzzy set f of an intra-regular semigroup S the following conditions are equivalent:*

(1) *f is a right (resp., left) ideal of S.*

(2) *$\underline{f}$ is an interior ideal of $\underline{S}$.*

Proof. It is clear that (1) implies (2). Assume that (2) holds. Let x, y be any elements in S. Then since S is intra-regular, there exist elements a, b in S such that $x = ax^2b$. If $f(x) = 0$, $f(x) = 0 \le f(xy)$. If $f(x) \ne 0$, then $x_{f(x)} \in \underline{f}$ and $y_{f(x)} \in \underline{S}$. Since $\underline{f}$ is an interior ideal of $\underline{S}$, we have

$$\begin{aligned}(xy)_{f(x)} &= (ax^2by)_{f(x)} \\ &= ((ax)x(by))_{f(x)\wedge f(x)\wedge f(x)} \\ &= (ax)_{f(x)} \circ x_{f(x)} \circ (by)_f(x) \in \underline{f}.\end{aligned} \tag{3.8}$$

This implies that $f(xy) \ge f(x)$, and hence f is a fuzzy right ideal of S. □

Lemma 3.9 (see [3, Lemma 4.1]). *For a semigroup S, the following conditions are equivalent:*

(1) *S is intra-regular.*

(2) *$L \cap R \subset LR$ holds for every left ideal L and right ideal R of S.*

Lemma 3.10 (see [3, Lemma 4.2]). *For a semigroup S, the following conditions are equivalent:*

(1) *S is intra-regular.*

(2) *$f \cap g \subset g \circ f$ holds for every fuzzy right ideal f and fuzzy left ideal g of S.*

Theorem 3.11. *For a semigroup S, the following conditions are equivalent:*

(1) *S is intra-regular.*

(2) *$\underline{f} \cap \underline{g} \subset \underline{g} \circ \underline{f}$ for every fuzzy right ideal f and every fuzzy left ideal g of S.*

Proof. Let f be a fuzzy right ideal and g a left ideal of S. Since $\underline{S}$ is intra-regular, $\underline{f}$ is a right ideal, and $\underline{g}$ is a left ideal of $\underline{S}$, $\underline{f} \cap \underline{g} \subset \underline{g} \circ \underline{f}$ by Lemma 3.9.

Conversely, let f be a fuzzy right ideal and g a fuzzy left ideal of S. Let $x \in S$. If $f(x) = 0$ or $g(x) = 0$, then

$$0 = f(x) \wedge g(x) \subseteq (g \circ f)(x). \tag{3.9}$$

If $f(x) \neq 0$ and $g(x) \neq 0$, then $x_{f(x)\wedge g(x)} \in \underline{f}$ and $x_{f(x)\wedge g(x)} \in \underline{g}$. Hence

$$x_{f(x)\wedge g(x)} \in \underline{f} \cap \underline{g} \subset \underline{g} \circ \underline{f} \subseteq \underline{g \circ f}. \tag{3.10}$$

It follows that $f(x) \wedge g(x) \subseteq (g \circ f)(x)$. Hence $(f \cap g)(x) = f(x) \wedge g(x) \subseteq (g \circ f)(x)$ for all $x \in S$ and $f \cap g \subset g \circ f$. By Lemma 3.10, S is intra-regular. □

Lemma 3.12 (see [4, Lemma 4.3])**.** *For a semigroup S the following conditions are equivalent:*

(1) *S is both regular and intra-regular.*
(2) *$B^2 = B$ for every bi-ideal B of S.*
(3) *$A \cap B \subset AB \cap BA$ for all bi-ideals A and B of S.*
(4) *$B \cap L \subset BL \cap LB$ for every bi-ideal B and every left ideal L of S.*
(5) *$B \cap R \subset BR \cap RB$ for every bi-ideal B and every right ideal R of S.*
(6) *$L \cap R \subset LR \cap RL$ for every right ideal R and every left ideal L of S.*

A fuzzy subsemigroup f of S is called a *fuzzy bi-ideal* of S if $f(xyz) \geq f(x) \wedge f(z)$ for all x, y and $z \in S$.

Corollary 3.13. *For a semigroup S the following conditions are equivalent:*

(1) *S is both regular and intra-regular.*
(2) *$\underline{f} \circ \underline{f} = \underline{f}$ for every fuzzy bi-ideal f of S.*
(3) *$\underline{f} \cap \underline{g} \subset \underline{f} \circ \underline{g} \cap \underline{g} \circ \underline{f}$ for all fuzzy bi-ideals f and g of S.*
(4) *$\underline{f} \cap \underline{g} \subset \underline{f} \circ \underline{g} \cap \underline{g} \circ \underline{f}$ for every fuzzy bi-ideal f and every fuzzy left ideal g of S.*
(5) *$\underline{f} \cap \underline{g} \subset \underline{f} \circ \underline{g} \cap \underline{g} \circ \underline{f}$ for every fuzzy bi-ideal f and every fuzzy right ideal g of S.*
(6) *$\underline{f} \cap \underline{g} \subset \underline{f} \circ \underline{g} \cap \underline{g} \circ \underline{f}$ for every fuzzy right ideal f and every fuzzy left ideal g of S.*

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Kyung Ho Kim: Department of Mathematics, Chungju National University, Chungju 380-702, Korea

E-mail address: ghkim@gukwon.chungju.ac.kr