COMMON COINCIDENCE POINTS OF R -WEAKLY COMMUTING MAPS

TAYYAB KAMRAN

(Received 22 May 2000)

ABSTRACT. A common coincidence point theorem for R -weakly commuting mappings is obtained. Our result extend several ones existing in literature.

2000 Mathematics Subject Classification. 54H25.

1. Introduction. Throughout this paper, X denotes a metric space with metric d . For $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $CB(X)$ the class of all nonempty bounded closed subsets of X . Let H be the Hausdorff metric with respect to d , that is,

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} \quad (1.1)$$

for every $A, B \in CB(X)$. The mappings $T : X \rightarrow CB(X)$, $f : X \rightarrow X$ are said to be commuting if, $fTx \subseteq TfX$. A point $p \in X$ is said to be a fixed point of $T : X \rightarrow CB(X)$ if $p \in Tp$. The point p is called a coincidence point of f and T if $fp \in Tp$. The mappings $f : X \rightarrow X$ and $T : X \rightarrow CB(X)$ are called weakly commuting if, for all $x \in X$, $fTx \in CB(X)$ and $H(fTx, Tfx) \leq d(fx, Tx)$.

Recently Daffer and Kaneko [2] reaffirmed the positive answer [5] to the conjecture of Reich [8] by giving an alternative proof to Theorem 5 of Mizoguchi and Takahashi [5]. We state Theorem 2.1 of Daffer and Kaneko [2] for convenience.

THEOREM 1.1. *Let X be a complete metric space and $T : X \rightarrow CB(X)$. If α is a function of $(0, \infty)$ to $(0, 1]$ such that $\limsup_{r \rightarrow t^+} \alpha(r) < 1$ for each $t \in [0, \infty)$ and if*

$$H(Tx, Ty) \leq \alpha(d(x, y)) d(x, y) \quad (1.2)$$

for each $x, y \in X$, then T has a fixed point in X .

The purpose of this paper is to obtain a coincidence point theorem for R -weakly commuting multivalued mappings analogous to Theorem 1.1. We follow the same technique used in [2]. The notion of R -weak commutativity for single-valued mappings was defined by Pant [7] to generalize the concept of commuting and weakly commuting mappings [9]. Recently, Shahzad and Kamran [10] extended this concept to the setting of single and multivalued mappings, and studied the structure of common fixed points.

DEFINITION 1.2 (see [10]). The mappings $f : X \rightarrow X$ and $T : X \rightarrow CB(X)$ are called R -weakly commuting if for all $x \in X$, $fTx \in CB(X)$ and there exists a positive real number R such that

$$H(Tfx, Tfx) \leq R d(fx, Tx). \quad (1.3)$$

2. Main result. Before giving our main result, we state the following lemmas which are noted in Nadler [6], and Assad and Kirk [1].

LEMMA 2.1. If $A, B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \varepsilon. \quad (2.1)$$

LEMMA 2.2. If $\{A_n\}$ is a sequence in $CB(X)$ and $\lim_{n \rightarrow \infty} H(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$, then $x \in A$.

Now, we prove our main result.

THEOREM 2.3. Let X be a complete metric space, $f, g : X \rightarrow X$ and $S, T : X \rightarrow CB(X)$ are continuous mappings such that $SX \subseteq gX$ and $TX \subseteq fX$. Let $\alpha : (0, \infty) \rightarrow (0, 1]$ be such that $\limsup_{r \rightarrow t^+} \alpha(r) < 1$ for each $t \in [0, \infty)$ and

$$H(Sx, Ty) \leq \alpha(d(gx, fy)) d(gx, fy) \quad (2.2)$$

for each $x, y \in X$. If the pairs (g, T) and (f, S) are R -weakly commuting, then g, S and f, T have a common coincidence point.

PROOF. Our method is constructive. We construct sequences $\{x_n\}$, $\{y_n\}$, and $\{A_n\}$ in X and $CB(X)$, respectively as follows. Let x_0 be an arbitrary point of X and $y_0 = fx_0$. Since $Sx_0 \subseteq gX$, there exists a point $x_1 \in X$ such that $y_1 = gx_1 \in Sx_0 = A_0$. Choose a positive integer n_1 such that

$$\alpha^{n_1}(d(y_0, y_1)) < \{1 - \alpha(d(y_0, y_1))\} d(y_0, y_1). \quad (2.3)$$

Now Lemma 2.1 and the fact $TX \subseteq fX$ guarantee that there is a point $y_2 = fy_1 \in Tx_1 = A_1$ such that

$$d(y_2, y_1) \leq H(A_1, A_0) + \alpha^{n_1}(d(y_0, y_1)). \quad (2.4)$$

The above inequality in view of (2.2) and (2.3) implies that $d(y_2, y_1) < d(y_0, y_1)$. Now choose a positive integer $n_2 > n_1$ such that

$$\alpha^{n_2}(d(y_2, y_1)) < \{1 - \alpha(d(y_2, y_1))\} d(y_2, y_1). \quad (2.5)$$

Again using Lemma 2.1 and the fact $SX \subseteq gX$, we get a point $y_3 = gx_3 \in Sx_2 = A_2$ such that

$$d(y_3, y_2) \leq H(A_2, A_1) + \alpha^{n_2}(d(y_2, y_1)). \quad (2.6)$$

Now (2.2) and (2.5) further imply that $d(y_3, y_2) < d(y_2, y_1)$.

By induction we obtain sequences $\{x_n\}$, $\{y_n\}$, and $\{A_n\}$ in X and $CB(X)$, respectively, such that

$$y_{2k+1} = gx_{2k+1} \in Sx_{2k} = A_{2k}, \quad y_{2k} = fx_{2k} \in Tx_{2k-1} = A_{2k-1}, \quad (2.7)$$

$$d(y_{2k+1}, y_{2k}) \leq H(A_{2k}, A_{2k-1}) + \alpha^{n_k}(d(y_{2k}, y_{2k-1})), \quad (2.8)$$

where

$$\alpha^{n_{2k}}(d(y_{2k}, y_{2k-1})) < \{1 - \alpha(d(y_{2k}, y_{2k-1}))\} d(y_{2k}, y_{2k-1}) \quad (2.9)$$

for each k . So we have $d(y_{2k+1}, y_{2k}) < d(y_{2k}, y_{2k-1})$. Therefore, the sequence $\{d(y_{2k+1}, y_{2k})\}$ is monotone nonincreasing. Then, as in the proof of Theorem 2.1 in [2], $\{y_n\}$ is a Cauchy sequence in X . Further, equation (2.2) ensures that $\{A_n\}$ is a Cauchy sequence in $CB(X)$. It is well known that if X is complete, then so is $CB(X)$. Therefore, there exist $z \in X$ and $A \in CB(X)$ such that $y_n \rightarrow z$ and $A_n \rightarrow A$. Moreover, $gx_{2k+1} \rightarrow z$ and $fx_{2k} \rightarrow z$. Since

$$d(z, A) = \lim_{n \rightarrow \infty} d(y_n, A_n) \leq \lim_{n \rightarrow \infty} H(A_{n-1}, A_n) = 0, \quad (2.10)$$

it follows from Lemma 2.2 that $z \in A$. Also

$$\lim_{k \rightarrow \infty} fx_{2k} = z \in A = \lim_{k \rightarrow \infty} Sx_{2k}, \quad \lim_{k \rightarrow \infty} gx_{2k+1} = z \in A = \lim_{k \rightarrow \infty} Tx_{2k-1}. \quad (2.11)$$

Using (2.7) and R -weak commutativity of the pairs (g, T) and (f, S) , we have

$$\begin{aligned} d(gfx_{2k+2}, Tgx_{2k+1}) &\leq H(gTx_{2k+1}, Tgx_{2k+1}) \leq Rd(gx_{2k+1}, Tx_{2k+1}), \\ d(fgx_{2k+1}, Sfx_{2k}) &\leq H(fSx_{2k}, Sfx_{2k}) \leq Rd(fx_{2k}, Sx_{2k}). \end{aligned} \quad (2.12)$$

Now it follows from the continuity of f , g , T , and S that $gz \in Tz$ and $fz \in Sz$. $\square$

If we put $T = S$ and $f = g$ in Theorem 2.3, we get the following corollary.

COROLLARY 2.4. Let X be a complete metric space, and let $f : X \rightarrow X$ be a continuous mapping and $T : X \rightarrow CB(X)$ be a mapping such that $TX \subseteq fX$. Let $\alpha : (0, \infty) \rightarrow (0, 1]$ be such that $\limsup_{r \rightarrow t^+} \alpha(r) < 1$ for each $t \in [0, \infty)$ and

$$H(Tx, Ty) \leq \alpha(d(fx, fy)) d(fx, fy) \quad (2.13)$$

for each $x, y \in X$. If the mappings f and T are R -weakly commuting, then f and T have coincidence point.

REMARK 2.5. (1) Theorem 2.3 improves and extends some known results of Hu [3], Kaneko [4], Mizoguchi and Takahashi [5], and Nadler [6].

(2) In Corollary 2.4, T is not assumed to be continuous. In fact the continuity of T follows from the continuity of f .

(3) If we put $f = I$, the identity map, in Corollary 2.4, we obtain Theorem 1.1.

ACKNOWLEDGEMENT. The author is grateful to Dr. Naseer Shahzad for his useful suggestions.

REFERENCES

- [1] N. A. Assad and W. A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math. **43** (1972), 553–562. [MR 49#6210](#). [Zbl 239.54032](#).
- [2] P. Z. Daffer and H. Kaneko, *Fixed points of generalized contractive multi-valued mappings*, J. Math. Anal. Appl. **192** (1995), no. 2, 655–666. [MR 96d:54040](#). [Zbl 835.54028](#).
- [3] T. Hu, *Fixed point theorems for multivalued mappings*, Canad. Math. Bull. **23** (1980), no. 2, 193–197. [MR 82d:54052](#). [Zbl 436.54037](#).
- [4] H. Kaneko, *Single-valued and multi-valued f -contractions*, Boll. Un. Mat. Ital. A (6) **4** (1985), no. 1, 29–33. [MR 86e:54054](#). [Zbl 568.54031](#).
- [5] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl. **141** (1989), no. 1, 177–188. [MR 90f:47086](#). [Zbl 688.54028](#).
- [6] S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–488. [MR 40#8035](#). [Zbl 187.45002](#).
- [7] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188** (1994), no. 2, 436–440. [MR 95j:54037](#). [Zbl 830.54031](#).
- [8] S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital. (4) **5** (1972), 26–42. [MR 46#8206](#). [Zbl 249.54026](#).
- [9] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. (Beograd) (N.S.) **32(46)** (1982), 149–153. [MR 85f:54107](#). [Zbl 523.54030](#).
- [10] N. Shahzad and T. Kamran, *Coincidence points and R -weakly commuting maps*, to appear.

TAYYAB KAMRAN: DEPARTMENT OF MATHEMATICS, QUAID-I-AZAM UNIVERSITY, ISLAMABAD, PAKISTAN

E-mail address: tayyab_k@hotmail.com