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ON MATRIX TRANSFORMATIONS CONCERNING THE NAKANO VECTOR-VALUED SEQUENCE SPACE

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ABSTRACT. We give the matrix characterizations from Nakano vector-valued sequence space $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces E_r, ℓ_∞, $\underline{\ell}_\infty(q)$, bs, and cs, where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers such that $p_k>1$ for all $k\in\mathbb{N}$ and $r\geq 0$.

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1. Introduction. Let $(X,\|\cdot\|)$ be a Banach space, $r\geq 0$ and $p=(p_k)$ a bounded sequence of positive real numbers. We write $x=(x_k)$ with x_k in X for all $k\in\mathbb{N}$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $E_r(X,p)$, $F_r(X,p)$, and $\underline{\ell}_\infty(X,p)$ are defined as

$$\begin{aligned}
c_0(X,p)&=\Big\{x=(x_k):\lim_{k\to\infty}\|x_k\|^{p_k}=0\Big\},\\
c(X,p)&=\Big\{x=(x_k):\lim_{k\to\infty}\|x_k-a\|^{p_k}=0,\ \text{for some } a\in X\Big\},\\
\ell_\infty(X,p)&=\Big\{x=(x_k):\sup_k\|x_k\|^{p_k}<\infty\Big\},\\
\ell(X,p)&=\Big\{x=(x_k):\sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty\Big\},\\
E_r(X,p)&=\Big\{x=(x_k):\sup_k\frac{\|x_k\|^{p_k}}{k^r}<\infty\Big\},\\
F_r(X,p)&=\Big\{x=(x_k):\sum_{k=1}^{\infty}k^r\|x_k\|^{p_k}<\infty\Big\},\\
\underline{\ell}_\infty(X,p)&=\bigcap_{n=1}^{\infty}\Big\{x=(x_k):\sup_k\|x_k\|n^{1/p_k}\Big\}.
\end{aligned}\tag{1.1}$$

When $X=K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, $\ell(p)$, $E_r(p)$, $F_r(p)$, and $\underline{\ell}_\infty(p)$, respectively. The spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ are known as the sequence spaces of Maddox. These spaces were first introduced and studied by Simons [7] and Maddox [4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and it is known as the Nakano sequence space and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. When $p_k=1$ for all $k\in\mathbb{N}$, the spaces $E_r(p)$ and $F_r(p)$ are written as E_r and F_r, respectively. These two sequence spaces were first introduced by Cooke [1]. The space $\underline{\ell}_\infty(p)$ was first defined by Grosse-Erdmann [2] and he has given in [3] characterizations of infinite matrices mapping between scalar-valued sequence spaces of Maddox. Wu and Liu [10] gave necessary and sufficient conditions for infinite matrices mapping from $c_0(X,p)$ and $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$. Suantai [8] has given characterizations of infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\underline{\ell}_\infty(q)$ when $p_k \le 1$ for all $k \in \mathbb{N}$ and he has also given in [9] characterizations of those infinite matrices mapping from $\ell(X,p)$ into the sequence space E_r when $p_k \le 1$ for all $k \in \mathbb{N}$.

In this paper, we extend the results of [8, 9] in case $p_k > 1$ for all $k \in \mathbb{N}$. Moreover, we also give the matrix characterizations from $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces bs and cs.

2. Notations and definitions. Let $(X,\|\cdot\|)$ be a Banach space, the space of all sequences in X is denoted by $W(X)$, and $\Phi(X)$ denotes the space of all finite sequences in X. When $X = K$, the scalar field of X, the corresponding spaces are written as w and Φ.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in \mathbb{N}$, x_k stands for the kth term of x. For $k \in \mathbb{N}$, we denote by e_k the sequence $(0,0,\dots,0,1,0,\dots)$ with 1 in the kth position and by e the sequence $(1,1,1,\dots)$. For $x \in X$ and $k \in \mathbb{N}$, let $e^k(x)$ be the sequence $(0,0,\dots,0,x,0,\dots)$ with x in the kth position and let $e(x)$ be the sequence $(x,x,x,\dots)$. We call a sequence space E normal if $(t_kx_k) \in E$ for all $x = (x_k) \in E$ and $t_k \in K$ with $|t_k| = 1$ for all $t_k \in \mathbb{N}$. A normed sequence space $(E,\|\cdot\|)$ is said to be *norm monotone* if $x = (x_k)$, $y = (y_k) \in E$ with $\|x_k\| \le \|y_k\|$ for all $k \in \mathbb{N}$ we have $\|x\| \le \|y\|$. For a fixed scalar sequence $\mu = (\mu_k)$, the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\}. \tag{2.1}$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$, if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^\infty f_k^n(x_k)$ converges for each $n \in \mathbb{N}$, and the sequence $Ax = (A_n(x)) \in F$. Let (E,F) denote the set of all infinite matrices mapping from E into F.

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a K-space if for each $k \in \mathbb{N}$, the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E,τ) is a Fréchet (Banach) space, then E is called an FK- (BK-) space. Now, suppose that E contains $\Phi(X)$. Then E is said to have *property* AB if the set $\{\sum_{k=1}^n e^k(x_k) : n \in \mathbb{N}\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property* AK if $\sum_{k=1}^n e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property* AD if $\Phi(X)$ is dense in E.

It is known that the Nakano sequence space $\ell(X,p)$ is an FK-space with property AK under the paranorm $g(x) = (\sum_{k=1}^\infty \|x_k\|^{p_k})^{1/M}$, where $M = \max\{1, \sup_k p_k\}$. If $p_k > 1$ for all $k \in \mathbb{N}$, then $\ell(X,p)$ is a BK-space with the Luxemburg norm defined by

$$\|(x_k)\| = \inf\left\{\varepsilon > 0 : \sum_{k=1}^\infty \left\|\frac{x_k}{\varepsilon}\right\|^{p_k} \le 1\right\}. \tag{2.2}$$

3. Main results. We first give a characterization of an infinite matrix mapping from $\ell(X,p)$ into E_r when $p_k > 1$ for all $k \in \mathbb{N}$. To do this, we need the following lemma.

Lemma 3.1. *Let E be an X-valued BK-space which is normal and norm monotone and let $A = (f_k^n)$ be an infinite matrix. Then $A : E \to E_r$ if and only if $\sup_n \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r < \infty$ for every $x = (x_k) \in E$.*

Proof. If the condition holds true, it follows that

$$\sup_n \frac{\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|}{n^r} \le \sup_n \sum_{k=1}^{\infty} \frac{\left|f_k^n(x_k)\right|}{n^r} < \infty \tag{3.1}$$

for every $x = (x_k) \in E$, hence $A : E \to E_r$.

Conversely, assume that $A : E \to E_r$. Since E and E_r are BK-spaces, by Zeller's theorem, $A : E \to E_r$ is bounded, so there exists $M > 0$ such that

$$\sup_{\substack{n\in\mathbb{N} \\ \|(x_k)\|\le 1}} \frac{\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|}{n^r} \le M. \tag{3.2}$$

Let $x = (x_k) \in E$ be such that $\|x\| = 1$. For each $n \in \mathbb{N}$, we can choose a scalar sequence (t_k) with $|t_k| = 1$ and $f_k^n(t_k x_k) = |f_k^n(x_k)|$ for all $k \in \mathbb{N}$. Since E is normal and norm monotone, we have $(t_k x_k) \in E$ and $\|(t_k x_k)\| \le 1$. It follows from (3.2) that

$$\sum_{k=1}^{\infty} \frac{\left|f_k^n(x_k)\right|}{n^r} = \frac{\left|\sum_{k=1}^{\infty} f_k^n(t_k x_k)\right|}{n^r} \le M, \tag{3.3}$$

which implies

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} \frac{\left|f_k^n(x_k)\right|}{n^r} \le M. \tag{3.4}$$

It follows from (3.4) that for every $x = (x_k) \in E$,

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} \frac{\left|f_k^n(x_k)\right|}{n^r} \le M\|x\|. \tag{3.5}$$

This completes the proof. □

Theorem 3.2. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$, and let $r \ge 0$. For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), E_r)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < \infty. \tag{3.6}$$

Proof. Let $x = (x_k) \in \ell(X,p)$. By (3.6), there are $m_0 \in \mathbb{N}$ and $K > 1$ such that

$$\sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < K, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

Note that for $a, b \ge 0$, we have

$$ab \le a^{p_k} + b^{q_k}. \tag{3.8}$$

It follows by (3.7) and (3.8) that for $n \in \mathbb{N}$,

$$\begin{aligned} n^{-r}\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right| &= n^{-r}\left|\sum_{k=1}^{\infty} f_k^n(m_0^{-1}\cdot m_0 x_k)\right| \\ &\le \sum_{k=1}^{\infty}\left(n^{-r} m_0^{-1}\|f_k^n\|\right)\left(\|m_0 x_k\|\right) \\ &\le \sum_{k=1}^{\infty} n^{-rq_k} m_0^{-q_k}\|f_k^n\|^{q_k} + m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k} \\ &\le K + m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k}, \quad \text{where } \alpha = \sup_k p_k. \end{aligned} \tag{3.9}$$

Hence $\sup n^{-r}|\sum_{k=1}^{\infty} f_k^n(x_k)| < \infty$, so that $Ax \in E_r$.

For necessity, assume that $A \in (\ell(X,p), E_r)$. For each $k \in \mathbb{N}$, we have $\sup_n n^{-r}|f_k^n(x)| < \infty$ for all $x \in X$ since $e^{(k)}(x) \in \ell(X,p)$. It follows by the uniform bounded principle that for each $k \in \mathbb{N}$ there is $C_k > 1$ such that

$$\sup_n n^{-r}\|f_k^n\| \le C_k. \tag{3.10}$$

Suppose that (3.6) is not true. Then

$$\sup_n \sum_{k=1}^{\infty}\|f_k^n\|^{q_k} n^{-rq_k} m^{-q_k} = \infty, \quad \forall m \in \mathbb{N}. \tag{3.11}$$

For $n \in \mathbb{N}$, we have by (3.10) that for $k, m \in \mathbb{N}$,

$$\begin{aligned} \sum_{j=1}^{\infty}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j} &= \sum_{j=1}^{k}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j} + \sum_{j>k}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j} \\ &\le \sum_{j=1}^{k} C_j^{q_j} m^{-q_j} + \sum_{j>k}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j}. \end{aligned} \tag{3.12}$$

This together with (3.11) give

$$\sup_n \sum_{j>k}\left\|f_j^n\right\|^{q_j} n^{-rq_j} m^{-q_j} = \infty, \quad \forall k, m \in \mathbb{N}. \tag{3.13}$$

By (3.13) we can choose $0 = k_0 < k_1 < k_2 < \cdots$, $m_1 < m_2 < \cdots$, $m_i > 4^i$ and a subsequence (n_i) of positive integers such that for all $i \ge 1$,

$$\sum_{k_{i-1} < j \le k_i}\left\|f_j^{n_i}\right\|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i. \tag{3.14}$$

For each $i \in \mathbb{N}$, we can choose $x_j \in X$ with $\|x_j\| = 1$, for $k_{i-1} < j \le k_i$ such that

$$\sum_{k_{i-1} < j \le k_i}\left|f_j^{n_i}(x_j)\right|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i. \tag{3.15}$$

For each $i \in \mathbb{N}$, let $F_i : (0,\infty) \to (0,\infty)$ be defined by

$$F_i(M) = \sum_{k_{i-1}<j\le k_i} \left| f_j^{n_i}(x_j) \right|^{q_j} n_i^{-rq_j} M^{-q_j}. \tag{3.16}$$

Then F_i is continuous and non-increasing such that $F(M) \to 0$ as $M \to \infty$. Thus there exists $M_i > 0$ such that $M_i > m_i$ and

$$F(M_i) = \sum_{k_{i-1}<j\le k_i} \left| f_j^{n_i}(x_j) \right|^{q_j} n_i^{-rq_j} M_i^{-q_j} = 2^i. \tag{3.17}$$

Put

$$y = (y_j), \quad y_j = 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j/p_j} \left| f_j^{n_i}(x_j) \right|^{q_j-1} x_j \text{ for } k_{i-1} < j \le k_i. \tag{3.18}$$

Thus

$$\begin{aligned}
\sum_{j=1}^{\infty} \|y_j\|^{p_j} &= \sum_{i=1}^{\infty} \sum_{k_{i-1}<j\le k_i} 4^{-ip_j} M_i^{-p_j(q_j-1)} n_i^{-rq_j} \left| f_j^{n_i}(x_j) \right|^{p_j(q_j-1)} \\
&\le \sum_{i=1}^{\infty} 4^{-i} \sum_{k_{i-1}<j\le k_i} M_i^{-q_j} n_i^{-rq_j} \left| f_j^{n_i}(x_j) \right|^{q_j} \\
&= \sum_{i=1}^{\infty} 4^{-i} \cdot 2^i \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.
\end{aligned} \tag{3.19}$$

Thus $y = (y_j) \in \ell(X,p)$. Since $\ell(X,p)$ is a BK-space which is normal and norm monotone under the Luxemburg norm, by Lemma 3.1, we obtain that

$$\sup_n \sum_{k=1}^{\infty} \frac{|f_k^n(y_k)|}{n^r} < \infty. \tag{3.20}$$

But we have

$$\begin{aligned}
\sup_n \sum_{j=1}^{\infty} \frac{\left| f_j^n(y_j) \right|}{n^r} &\ge \sup_i \sum_{j=1}^{\infty} \frac{\left| f_j^{n_i}(y_j) \right|}{n_i^r} \ge \sup_i \sum_{k_{i-1}<j\le k_i} \frac{\left| f_j^{n_i}(y_j) \right|}{n_i^r} \\
&= \sup_i \sum_{k_{i-1}<j\le k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-r(q_j/p_j+1)} \left| f_j^{n_i}(x_j) \right|^{q_j} \\
&= \sup_i \sum_{k_{i-1}<j\le k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j} \left| f_j^{n_i}(x_j) \right|^{q_j} \\
&= \sup_i \sum_{k_{i-1}<j\le k_i} \left(\left| f_j^{n_i}(x_j) \right|^{q_j} n_i^{-rq_j} M_i^{-q_j} \right) 4^{-i} M_i \\
&\ge \sup_i 2^i = \infty, \quad \text{because } M_i > 4^i.
\end{aligned} \tag{3.21}$$

This is contradictory with (3.20). Therefore (3.6) is satisfied. □

Theorem 3.3. *Let $p=(p_k)$ be a bounded sequence of positive real numbers such that $p_k>1$ for all $k\in\mathbb{N}$, $1/p_k+1/q_k=1$ for all $k\in\mathbb{N}$, $r\geq 0$ and $s\geq 0$. Then for an infinite matrix $A=(f_k^n)$, $A\in(F_r(X,p),E_s)$ if and only if there is $m_0\in\mathbb{N}$ such that*

$$\sup_n\sum_{k=1}^{\infty}\left(k^{-rq_k/p_k}\|f_k^n\|^{q_k}n^{-sq_k}m_0^{-q_k}\right)<\infty. \tag{3.22}$$

Proof. Since $F_r(X,p)=\ell(X,p)_{(k^{r/p_k})}$, it is easy to see that

$$A\in(F_r(X,p),E_s)\Longleftrightarrow(k^{-r/p_k}f_k^n)_{n,k}\in(\ell(X,p)E_s). \tag{3.23}$$

By Theorem 3.2, we have $(k^{-r/p_k}f_k^n)_{n,k}\in(\ell(X,p)E_s)$ if and only if there is $m_0\in\mathbb{N}$ such that

$$\sup_n\sum_{k=1}^{\infty}\left(k^{-rq_k/p_k}\|f_k^n\|^{q_k}n^{-sq_k}m_0^{-q_k}\right)<\infty. \tag{3.24}$$

Thus the theorem is proved. □

Since $E_0=\ell_\infty$, the following two results are obtained directly from Theorems 3.2 and 3.3, respectively.

Corollary 3.4. *Let $p=(p_k)$ be a bounded sequence of positive real numbers with $p_k>1$ for all $k\in\mathbb{N}$ and let $1/p_k+1/q_k=1$ for all $k\in\mathbb{N}$. Then for an infinite matrix $A=(f_k^n)$, $A\in(\ell(X,p),\ell_\infty)$ if and only if there is $m_0\in\mathbb{N}$ such that*

$$\sup_n\sum_{k=1}^{\infty}\|f_k^n\|^{q_k}m_0^{-q_k}<\infty. \tag{3.25}$$

Corollary 3.5. *Let $p=(p_k)$ be a bounded sequence of positive real numbers with $p_k>1$ for all $k\in\mathbb{N}$ and let $1/p_k+1/q_k=1$ for all $k\in\mathbb{N}$. Then for an infinite matrix $A=(f_k^n)$, $A\in(F_r(X,p),\ell_\infty)$ if and only if there is $m_0\in\mathbb{N}$ such that*

$$\sup_n\sum_{k=1}^{\infty}\left(k^{-rq_k/p_k}\|f_k^n\|^{q_k}m_0^{-q_k}\right)<\infty. \tag{3.26}$$

Theorem 3.6. *Let $p=(p_k)$ and $q=(q_k)$ be bounded sequences of positive real numbers with $p_k>1$ for all $k\in\mathbb{N}$ and let $1/p_k+1/t_k=1$ for all $k\in\mathbb{N}$. Then for an infinite matrix $A=(f_k^n)$, $A\in(\ell(X,p),\underline{\ell}_\infty(q))$ if and only if for each $r\in\mathbb{N}$, there is $m_r\in\mathbb{N}$ such that*

$$\sup_{n,k}\sum_{k=1}^{\infty}r^{t_k/q_n}\|f_k^n\|^{t_k}m_r^{-t_k}<\infty. \tag{3.27}$$

Proof. Since $\underline{\ell}_\infty(q)=\cap_{r=1}^{\infty}\ell_{\infty(r^{1/q_k})}$, it follows that

$$A\in(\ell(X,p),\underline{\ell}_\infty(q))\Longleftrightarrow A\in\left(\ell(X,p),\ell_{\infty(r^{1/q_k})}\right),\quad\forall r\in\mathbb{N}. \tag{3.28}$$

It is easy to show that for $r\in\mathbb{N}$,

$$A\in\left(\ell(X,p),\ell_{\infty(r^{1/q_k})}\right)\Longleftrightarrow(r^{1/q_n}f_k^n)_{n,k}\in(\ell(X,p),\ell_\infty). \tag{3.29}$$

We obtain by Corollary 3.4 that for $r \in \mathbb{N}$, $(r^{1/q_n} f_k^n)_{n,k} \in (\ell(X,p), \ell_\infty)$ if and only if there is $m_r \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} r^{t_k/q_n} \|f_k^n\|^{t_k} m_r^{-t_k} < \infty. \tag{3.30}$$

Thus the theorem is proved. □

Theorem 3.7. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and let $1/p_k + 1/t_k = 1$ for all $k \in \mathbb{N}$. For an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p), \underline{\ell}_\infty(q))$ if and only if for each $i \in \mathbb{N}$, there is $m_i \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty. \tag{3.31}$$

Proof. Since $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$, it implies that

$$A \in (F_r(X,p), \underline{\ell}_\infty(q)) \Longleftrightarrow (k^{-r/p_k} f_k^n)_{n,k} \in (\ell(X,p), \underline{\ell}_\infty(q)). \tag{3.32}$$

It follows from Theorem 3.6 that $A \in (F_r(X,p), \underline{\ell}_\infty(q))$ if and only if for each $i \in \mathbb{N}$, there is $m_i \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty. \tag{3.33}$$

□

Theorem 3.8. *Let $p = (p_k)$ be bounded sequence of positive real numbers with $p_k > 1$ for all $n \in \mathbb{N}$ and let $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), bs)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} f_k^i \right\|^{q_k} m_0^{-q_k} < \infty. \tag{3.34}$$

Proof. For an infinite matrix $A = (f_k^n)$, we can easily show that

$$A \in (\ell(X,p), bs) \Longleftrightarrow \left(\sum_{i=1}^{n} f_k^i \right)_{n,k} \in (\ell(X,p), \ell_\infty). \tag{3.35}$$

This implies by Corollary 3.4 that $A \in (\ell(X,p), bs)$ if and only if there is $m_0 \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} f_k^i \right\|^{q_k} m_0^{-q_k} < \infty. \tag{3.36}$$

□

Theorem 3.9. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and let $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), cs)$ if and only if*

(1) *there is $m_0 \in \mathbb{N}$ such that $\sup_n \sum_{k=1}^{\infty} \| \sum_{i=1}^{n} f_k^i \|^{q_k} m_0^{-q_k} < \infty$ and*
(2) *for each $k \in \mathbb{N}$ and $x \in X$, $\sum_{n=1}^{\infty} f_k^n(x)$ converges.*

Proof. The necessity is obtained by Theorem 3.8 and by the fact that $e^{(k)}(x) \in \ell(X,p)$ for every $k \in \mathbb{N}$ and $x \in X$.

Now, suppose that (1) and (2) hold. By Theorem 3.8, we have $A : \ell(X,p) \to bs$. Let $x = (x_k) \in \ell(X,p)$. Since $\ell(X,p)$ has the AK property, we have $x = \lim_{n\to\infty} \sum_{k=1}^{n} e^{(k)}(x_k)$. By Zeller's theorem, $A : \ell(X,p) \to bs$ is continuous. It implies that

$$Ax = \lim_{n\to\infty} \sum_{k=1}^{n} Ae^{(k)}(x_k). \tag{3.37}$$

By (2), $Ae^{(k)}(x_k) \in cs$ for all $k \in \mathbb{N}$. Since cs is a closed subspace of bs, it implies that $Ax \in cs$, that is, $A : \ell(X,p) \to cs$. □

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