# THE NUMBER OF CONNECTED COMPONENTS OF CERTAIN REAL ALGEBRAIC CURVES 

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(Received 7 June 2000 and in revised form 7 August 2000)


#### Abstract

For an integer $n \geq 2$, let $p(z)=\prod_{k=1}^{n}\left(z-\alpha_{k}\right)$ and $q(z)=\prod_{k=1}^{n}\left(z-\beta_{k}\right)$, where $\alpha_{k}, \beta_{k}$ are real. We find the number of connected components of the real algebraic curve $\left\{(x, y) \in \mathbb{R}^{2}:|p(x+i y)|-|q(x+i y)|=0\right\}$ for some $\alpha_{k}$ and $\beta_{k}$. Moreover, in these cases, we show that each connected component contains zeros of $p(z)+q(z)$, and we investigate the locus of zeros of $p(z)+q(z)$.


2000 Mathematics Subject Classification. Primary 26C10; Secondary 30C15.

1. Introduction. Throughout the paper, $n$ is an integer $\geq 2$. Let $f(x, y)$ be an integral polynomial of degree $n$. Let $A$ be the real algebraic curve defined by $A=\{(x, y) \in$ $\left.\mathbb{R}^{2}: f(x, y)=0\right\}$. It is known that $A$ consists of at most finitely many connected components. More precisely, when the curve is real nonsingular, each unbounded component is homeomorphic to a line and each bounded component is homeomorphic to a circle. We will call a bounded component an oval, and an unbounded component an $\infty$-component. Also, we will write "component" instead of "connected component" for convenience. Let $p(z)=\prod_{k=1}^{n}\left(z-\alpha_{k}\right)$ and $q(z)=\prod_{k=1}^{n}\left(z-\beta_{k}\right)$, where $\alpha_{k}, \beta_{k}$ are real. The zeros of $g(z):=p(z)+q(z)$ are clearly contained in the locus of the real algebraic curve

$$
\begin{equation*}
C:=\left\{(x, y) \in \mathbb{R}^{2}:|p(x+i y)|-|q(x+i y)|=0\right\} . \tag{1.1}
\end{equation*}
$$

In fact, in their study of "cylindrical algebraic decomposition," Arnon, Collins, and McCallum [1, 2] provide an algorithm for calculating the number of components given a specific example. However, we do not know the answer in the general case. We provide a different idea in this paper from that in [1, 2]. With the above terminology, here are some general questions.
(a) Given $P(x, y)=0$ for real variables $x$ and $y$, how many components are there? It is still unclear how to describe all possibilities for the topological nature of all components of an arbitrary $P(x, y)=0$; this is the essence of the Hilbert's 16 th problem. On the other hand, one of the most significant theorems of real algebraic geometry (Harnack (see [3, pages 257-258]), 1876) tells us that the number of components is at most one more than the genus.
(b) The curve $C$ has finitely many components. Must each component have zeros of $g(z)=0$ ?

We answer the questions (a) and (b) for some real algebraic curves of the form (1.1). Define, for real variables $x$ and $y$,

$$
\begin{equation*}
P(x, y):=|p(x+i y)|^{2}-|q(x+i y)|^{2}, \tag{1.2}
\end{equation*}
$$

where $\left\{\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n}\right\} \subseteq\{1,2, \ldots, 2 n\}$. The simplest case for the questions (a) and (b) is $\left\{\alpha_{k}\right\}=\{1,2,3, \ldots, n\}$ and $\left\{\beta_{k}\right\}=\{n+1, n+2, \ldots, 2 n\}$. Then all zeros of $P(x, y)$ obviously lie on the vertical line $x=n+1 / 2$, so $P(x, y)$ has only one component. We will study the case $\left\{\alpha_{k}\right\}=\{2,2, \ldots, 2\}$ and $\left\{\beta_{k}\right\}=\{1, n+1, n+1, \ldots, n+1\}$ in Section 3. Moreover, in Section 2, we will investigate the locus of zeros of the more general polynomial equation

$$
\begin{equation*}
g(x, t):=(x-2)^{n}+(x-1)(x-t)^{n-1}=0, \quad t \geq 3 . \tag{1.3}
\end{equation*}
$$

2. The zeros of $g(x, t)=0$. We need the following two lemmas. First, Lemma 2.1 easily follows from the theorems of Hurwitz (see [4, page 4]) and Rouché (see [4, page 2]).

Lemma 2.1. Let $n>m>0$ be integers. Let $A, B$, and $C$ be real numbers with $C \neq 0$. If a trinomial equation

$$
\begin{equation*}
A z^{n}+B z^{m}+C=0 \quad \text { with }|B| \geq|A|+|C| \tag{2.1}
\end{equation*}
$$

has no zeros on $|z|=1$, then it has exactly $m$ zeros strictly inside $|z|=1$.
Lemma 2.2. The zeros of $g(x, t)$ are $\left(2+a_{n, t}\right) /\left(1+a_{n, t}\right)$, where each $a_{n, t}^{-1 /(n-1)}$ is a zero of the trinomial equation $(2-t) z^{n}+(1-t) z+1=0$.

Proof. From $g(x, t)=0$, we obtain $-(x-2) /(x-1)=((x-t) /(x-2))^{n-1}$. Let

$$
\begin{equation*}
-\frac{x-2}{x-1}=\left(\frac{x-t}{x-2}\right)^{n-1}=a \tag{2.2}
\end{equation*}
$$

where $a:=a_{n, t}$ is a complex number. From $-(x-2) /(x-1)=a$, we find that $x=(2+a) /(1+a)$, and it easily follows from $((x-t) /(x-2))^{n-1}=a$ that $x=$ $\left(2 a^{1 /(n-1)}-t\right) /\left(a^{1 /(n-1)}-1\right)$. Equating these two formulae for $x$ leads to $a^{n / n-1}+$ $(1-t) a+2-t=0$. The result follows by multiplying each side by $a^{-n /(n-1)}$.

Now we find a relation between $x$ (a zero of $g(x, t)=0$ ) and $z$ (a zero of $(2-t) z^{n}+$ $(1-t) z+1=0)$ as follows:

$$
\begin{equation*}
x=\frac{2 z^{n-1}+1}{z^{n-1}+1}=1+\frac{1}{1+1 / z^{n-1}} . \tag{2.3}
\end{equation*}
$$

So

$$
\begin{equation*}
z^{n-1}=\frac{x-1}{2-x}, \quad \text { that is, } z=\left(\frac{x-1}{2-x}\right)^{1 /(n-1)} . \tag{2.4}
\end{equation*}
$$

Using Lemmas 2.1 and 2.2, we have the following proposition.
Proposition 2.3. The function $g(x, t)$ has only one zero $x_{0}$ in $\Re x<3 / 2$, and has no zeros in $3 / 2 \leq \Re x \leq(t+2) / 2$.

Proof. Observe that the strip $3 / 2 \leq \mathfrak{R} x \leq(t+2) / 2$ is zero-free, since, for such $x,|x-2| \leq|x-t|$ and $|x-2|<|x-1|$. Now we consider the trinomial equation $(2-t) z^{n}+(1-t) z+1=0$. It has no zero on $|z|=1$, since, if there were such a zero $z$, then by (2.4), $1=\left|z^{n-1}\right|=|(x-1) /(2-x)|$, that is, $x=3 / 2+i \beta$ for some real number $\beta$. This is a contradiction. Hence, by Lemma 2.1, the trinomial equation (2$t) z^{n}+(1-t) z+1=0$ has exactly one zero $z_{0}$ interior to $|z|=1$. Then $\left|z_{0}\right|=\mid\left(\left(x_{0}-\right.\right.$ 1) $\left./\left(2-x_{0}\right)\right)^{1 /(n-1)} \mid<1$, that is, $\left|x_{0}-1\right|<\left|2-x_{0}\right|$ for some real number $x_{0}$. Hence $\Re x_{0}<3 / 2$ which proves the proposition.

Next, we study further the unique zero $x_{0}$ given by Proposition 2.3.
Proposition 2.4. Let $n$ be an integer $\geq 3$ and $t \geq 3$. Then the only zero $x_{0}$ of $g(x, t)$ in $\Re x \leq(t+2) / 2$ is real and

$$
\begin{equation*}
\frac{1+2(-\epsilon+1 / n)^{n-1}}{1+(-\epsilon+1 / n)^{n-1}}<x_{0}<\frac{1+2(\epsilon+1 / n)^{n-1}}{1+(\epsilon+1 / n)^{n-1}} \tag{2.5}
\end{equation*}
$$

where $\epsilon=\epsilon(n, t)=2^{n}(t-2) /(t-1)^{n+1}$.
Proof. For $n$ an integer $\geq 3$, let $\epsilon=\epsilon(n, t)=2^{n}(t-2) /(t-1)^{n+1}$. Then $0<\epsilon \leq$ $1 /(t-1)$, since $(2 /(t-1))^{n}<1 /(t-2)$ and $n \geq 3$. Then the trinomial equation $(2-t) z^{n}+(1-t) z+1=0$ has at least one real zero $z_{0}$ in $(1 /(t-1)-\epsilon, 1 /(t-1)+\epsilon)$. In fact, by algebra, we can see that the left side of the trinomial equation is

$$
\begin{equation*}
-\left(2^{n}+\left(1+2^{n}(-2+t)(-1+t)^{-n}\right)^{n}\right)(-2+t)(-1+t)^{-n}<0 \tag{2.6}
\end{equation*}
$$

at $z=1 /(t-1)+\epsilon$, and

$$
\begin{equation*}
-\left(-2^{n}+\left(1-2^{n}(-2+t)(-1+t)^{-n}\right)^{n}\right)(-2+t)(-1+t)^{-n}>0 \tag{2.7}
\end{equation*}
$$

at $z=1 /(t-1)-\epsilon$. Set $z_{0}=\left(\left(x_{0}-1\right) /\left(2-x_{0}\right)\right)^{1 /(n-1)}$. Since $z_{0}$ is real, so is $x_{0}$. Now we obtain the inequality $\left|\left(\left(x_{0}-1\right) /\left(2-x_{0}\right)\right)^{1 /(n-1)}-1 /(t-1)\right|<\epsilon$, and from this we have the inequality (2.5). A simple calculation yields that $(1+2 A) /(1+A)<(t+2) / 2$ for $A>0$. This proves the result.

REMARK 2.5. (a) For $n=2$ and $t \geq 3$, we can easily check that $g(x, t)$ has two real zeros. Here the smaller zero is $\leq(t+2) / 2$, but it does not satisfy (2.5).
(b) In Lemma 2.2, we encountered a trinomial equation $(t-2) z^{n}+(t-1) z-1=0$ $(t \geq 3)$. Here we define a more general polynomial

$$
\begin{equation*}
h(z)=(t-2) z^{n}+(t-1) z-s \quad(s \geq 0) . \tag{2.8}
\end{equation*}
$$

Then we have the following zero distributions. The function $h(z)$ has

$$
\begin{cases}\text { all its zeros with modulus }>1 & \text { if } s>2 t-3,  \tag{2.9}\\ \text { one (real) zero with modulus }=1 \text { and all others }>1 & \text { if } s=2 t-3, \\ \text { one (real) zero with modulus }<1 \text { and all others }>1 & \text { if } 0 \leq s \leq 1\end{cases}
$$

This can be proved by elementary calculation, Lemma 2.1, and Eneström-Kakeya theorem (see [4, page 136]). However, we did not consider the case $1<s<2 t-3$. We conjecture that, for $1<s<2 t-3, h(z)$ has one (real) zero with modulus $<1$ and all others $>1$, as the case $0 \leq s \leq 1$, but it remains an open problem.
3. The number of components of $\left|(z-2)^{n}\right|=\left|(z-1)(z-(n+1))^{n-1}\right|$. Let

$$
\begin{equation*}
g(z):=(z-2)^{n}+(z-1)(z-(n+1))^{n-1} . \tag{3.1}
\end{equation*}
$$

If $g(z)=0$, then $\left|(z-1)(z-(n+1))^{n-1} /(z-2)^{n}\right|^{2}=1$. This motivates, for real variables $x$ and $y$, the introduction of

$$
\begin{equation*}
G(x, y):=\frac{\left((x-1)^{2}+y^{2}\right)\left((x-(n+1))^{2}+y^{2}\right)^{n-1}}{\left((x-2)^{2}+y^{2}\right)^{n}}-1 . \tag{3.2}
\end{equation*}
$$

Here $G(x, y)$ is obviously symmetric about the $x$-axis. In this section, we find the number of components of $G(x, y)=0$ and show that each component has zeros of $g(z)=0$. First, using Proposition 2.3, we find that the number of components of $G(x, y)=0$ is at least two.

Proposition 3.1. The locus of

$$
\begin{equation*}
\left|(z-2)^{n}\right|=\left|(z-1)(z-t)^{n-1}\right|, \quad t \geq 3 \tag{3.3}
\end{equation*}
$$

has at least two components.
Proof. We showed in Proposition 2.3 that $g(x, t)$ has one real zero $<2$ and $n-1$ zeros with real part $>(t+2) / 2>2$. So it suffices to show that, on $z=2+i s(s$ real $)$, the two absolute values are never equal. On $z=2+i s$ ( $s$ real),

$$
\begin{equation*}
\left|(z-1)(z-t)^{n-1}\right|^{2}-\left|(z-2)^{n}\right|^{2}=\left(1+s^{2}\right)\left((t-2)^{2}+s^{2}\right)^{n-1}-s^{2 n} \geq(t-2)^{2}>0 . \tag{3.4}
\end{equation*}
$$

Next, we show that the points where the locus of $G(x, y)=0$ has vertical tangents lie on the real axis. We use this later to show that the locus consists of either one oval, one $\infty$-component or three $\infty$-components. In order to prove this, we need the following lemma.

Lemma 3.2. Let $n$ be an integer $\geq 3$. Define

$$
\begin{equation*}
f(x):=\left(\frac{-2 x+3}{(n-1)(-2 x+n+2)}\right)^{n-1}-\frac{-2 x+n+2}{(n-1)(-2 x+n+3)} . \tag{3.5}
\end{equation*}
$$

Then all real zeros of $f(x)$ are

$$
\begin{cases}\frac{n^{2}+n-5}{2 n-4}, & n \text { even },  \tag{3.6}\\ \frac{n^{2}+n-5}{2 n-4}, r(n), & n \text { odd }\end{cases}
$$

where $\left(n^{2}+n-5\right) /(2 n-4)$ is a double zero in each case and $3 / 2<r=r(n)<$ $\left(n^{2}+n+1\right) / 2 n$.

Proof. From $f(x)=0$, we find that

$$
\begin{equation*}
\left(\frac{-2 x+3}{(n-1)(-2 x+n+2)}\right)^{n-1}=\frac{-2 x+n+2}{(n-1)(-2 x+n+3)}=a, \tag{3.7}
\end{equation*}
$$

where $a:=a_{n}$ is a complex number. From $(-2 x+3) /(n-1)(-2 x+n+2)=a^{1 /(n-1)}$, we get

$$
\begin{equation*}
x=-\frac{3-a^{1 /(n-1)}(n-1)(n+2)}{-2+2 a^{1 /(n-1)}(n-1)} \tag{3.8}
\end{equation*}
$$

and also

$$
\begin{equation*}
x=-\frac{n+2-a(n-1)(n+3)}{-2+2 a(n-1)} \tag{3.9}
\end{equation*}
$$

from $(-2 x+n+2) /(n-1)(-2 x+n+3)=a$. Equating these two formulae for $x$ leads to $(n-1) a^{n /(n-1)}-n a+1=0$, and so $a^{1 /(n-1)}$ is a zero of the trinomial equation $w(y):=(n-1) y^{n}-n y^{n-1}+1=0$. Now, we have

$$
\begin{equation*}
\frac{w(y)}{(y-1)^{2}}=(n-1) y^{n-2}+(n-2) y^{n-3}+(n-3) y^{n-4}+\cdots+2 y+1 . \tag{3.10}
\end{equation*}
$$

Since $a^{1 /(n-1)}$ is real if and only if the corresponding $x$ in (3.7) is real, the number of real zeros of $f(x)$ is equal to that of $w(y)$. By (3.10), $w(y)$ has a real double zero at 1 , and its corresponding $x$ is $\left(n^{2}+n-5\right) /(2 n-4)$, since $(-2 x+3) /(n-1)(-2 x+n+2)=1$. On the other hand, it follows from Eneström-Kakeya theorem that $w(y) /(y-1)^{2}$ has no zero for $|y|>1$. Also it is obvious that $w(y) /(y-1)^{2}$ has no real zero $\geq 0$. In order to find the real zeros of $f(x)$, we first need to determine whether $w(y)$ has a real zero on $(-1,0)$ or not. We see that $w^{\prime}(y)=n(n-1) y^{n-2}(y-1)$. So if $n$ is even, then $w^{\prime}(y)<0$ for $-1<y<0$. Moreover, $w(0)=1>0$, which implies there are no real zeros of $w(y)$ other than 1 . Hence $f(x)$ has only one (double) real zero $\left(n^{2}+n-5\right) /(2 n-4)$. Suppose that $n$ is odd. Then $w^{\prime}(y)>0$ on $-1<y<0, w(-1)=$ $2(1-n)<0$, and $w(0)>0$. This implies that there must be exactly one real zero on $(-1,0)$. Say $x_{0}$ is its corresponding real number. Then by (3.7)

$$
\begin{equation*}
-1<\frac{-2 x_{0}+3}{(n-1)\left(-2 x_{0}+n+2\right)}<0 . \tag{3.11}
\end{equation*}
$$

Simple calculations yield that $3 / 2<x_{0}<\left(n^{2}+n+1\right) / 2 n$. This completes the proof.

Now we have the following Proposition.
Proposition 3.3. The points where the locus of $G(x, y)=0$ has vertical tangents lie on the real axis.

Proof. It suffices to show that $\langle 0,1\rangle \cdot \nabla G(x, y)=0$ and $G(x, y)=0$ implies $y=0$. A calculation shows that $\langle 0,1\rangle \cdot \nabla G(x, y)=\partial G / \partial y=0$ if and only if $y=0$ or $y^{2}=$ $A(x)$, where

$$
\begin{equation*}
A(x)=\frac{2(n-2) x^{3}-\left(n^{2}+5 n-17\right) x^{2}+2\left(n^{2}+n-12\right) x-\left(n^{2}-2 n-11\right)}{-2(n-2) x+n^{2}+n-5} . \tag{3.12}
\end{equation*}
$$

Suppose that $y^{2}=A(x)$. Then

$$
f(x):=G(x, y)= \begin{cases}\frac{1}{4 x^{2}-16 x+15}, & n=2  \tag{3.13}\\ \left(\frac{-2 x+3}{(n-1)(-2 x+n+2)}\right)^{n-1}-\frac{-2 x+n+2}{(n-1)(-2 x+n+3)}, & n \geq 3\end{cases}
$$

by simplifying the equations. So it is clear that there are no zeros of $f(x)$ in the case of $n=2$. Suppose that $n \geq 3$. By Lemma 3.2, $\left(n^{2}+n-5\right) /(2 n-4)$ is a (double) real zero of $f(x)$ and, in particular, if $n$ is even, such a real zero is unique. But $A\left(\left(n^{2}+n-5\right) /(2 n-4)\right)$ is not defined. So this is a contradiction. Suppose that $n$ is odd. Then by Lemma 3.2, all zeros of $f(x)$ are $\left(n^{2}+n-5\right) /(2 n-4)$ and $r(n)$, where $3 / 2<r(n)<\left(n^{2}+n+1\right) / 2 n$. As above, $A\left(\left(n^{2}+n-5\right) /(2 n-4)\right)$ is not defined. So it is enough to consider $r(n)$. Now, we have that $A(3 / 2)=-1 / 4<0$ and $A\left(\left(n^{2}+n+1\right) / 2 n\right)=-\left(n^{4}-2 n^{3}+5 n^{2}-4 n+1\right) / 4 n^{2}<0$. So if we show that $A^{\prime}(x)<0$ on $3 / 2<x<\left(n^{2}+n+1\right) / 2 n$, then $y^{2}=A(x)<0$, which is a contradiction. We see that

$$
\begin{equation*}
A^{\prime}(x)=-\frac{2 s(x)}{\left(-2(n-2) x+n^{2}+n-5\right)^{2}} \tag{3.14}
\end{equation*}
$$

where $s(x)=4(n-2)^{2} x^{3}-4(n-2)^{2}(n+4) x^{2}+\left(n^{2}+5 n-17\right)\left(n^{2}+n-5\right) x-n^{4}-n^{3}+$ $12 n^{2}+10 n-38$. So it is enough to show that $s(x)>0$ on $3 / 2<x<\left(n^{2}+n+1\right) / 2 n$. Now

$$
\begin{gather*}
s\left(\frac{3}{2}\right)=\frac{1}{2}(n-1)^{3}(n+1)>0, \\
s\left(\frac{n^{2}+n+1}{2 n}\right)=\frac{(n-1)^{3}(2 n-1)\left(n^{2}-2 n+2\right)}{n^{3}}>0,  \tag{3.15}\\
s^{\prime}(x)=\left(6(2-n) x+n^{2}+5 n-17\right)\left(2(2-n) x+n^{2}+n-5\right) .
\end{gather*}
$$

Hence, $\left(n^{2}+5 n-17\right) / 6(n-2)$ and $\left(n^{2}+n-5\right) / 2(n-2)$ are the zeros of $s^{\prime}(x)$, and we can check that

$$
\begin{cases}\frac{n^{2}+5 n-17}{6(n-2)}<\frac{3}{2}<\frac{n^{2}+n+1}{2 n}<\frac{n^{2}+n-5}{2(n-2)}, & n=3  \tag{3.16}\\ \frac{3}{2}<\frac{n^{2}+5 n-17}{6(n-2)}<\frac{n^{2}+n+1}{2 n}<\frac{n^{2}+n-5}{2(n-2)}, & n \geq 4\end{cases}
$$

This proves the result, since $s(3 / 2)>0$ and $s\left(\left(n^{2}+n+1\right) / 2 n\right)>0$.
Next we establish the following Proposition.
Proposition 3.4. For fixed $y_{0} \neq 0$,
(a) $\lim _{x \rightarrow \pm \infty} G\left(x, y_{0}\right)=0$,
(b) for $|x|$ large, the limit is approached from above for $x \rightarrow-\infty$ and the limit is approached from below for $x \rightarrow+\infty$,
(c) $G(x, 0)$ has exactly three real zeros. Moreover, $(\partial G / \partial x)\left(x, y_{0}\right)$ has at most four real zeros,
(d)

$$
\frac{\partial^{2} G}{\partial x^{2}}\left(x, y_{0}\right) \begin{cases}\geq 0 & \text { as } x \rightarrow-\infty  \tag{3.17}\\ \leq 0 & \text { as } x \rightarrow \infty\end{cases}
$$

Proof. Let $y_{0}$ be nonzero and fixed. It is obvious that $\lim _{x \rightarrow \pm \infty} G\left(x, y_{0}\right)=0$. By a calculation, we have

$$
\begin{equation*}
\frac{\partial G}{\partial x}\left(x, y_{0}\right)=\frac{-2 n\left((x-n-1)^{2}+y_{0}^{2}\right)^{n} B\left(x, y_{0}\right)}{\left((x-2)^{2}+y_{0}^{2}\right)^{n+1}\left((x-n-1)^{2}+y_{0}^{2}\right)^{2}} \tag{3.18}
\end{equation*}
$$

where

$$
\begin{align*}
B(x) & =B\left(x, y_{0}\right) \\
& =(n-2) y_{0}^{4}+\left(n^{2}-n+1\right)(x-2) y_{0}^{2}-(x-1)(x-2)(x-(n+1))((n-2) x-n+3) \tag{3.19}
\end{align*}
$$

is a polynomial in $x$ of degree 4 whose leading coefficient is $2-n$. So it follows from the positivity of the leading coefficient of the numerator of the right side of (3.18) that, for $|x|$ large, $(\partial G / \partial x)\left(x, y_{0}\right)>0$, that is, $G\left(x, y_{0}\right)$ is increasing on $\left(x_{1}, \infty\right)$ and $\left(-\infty,-x_{1}\right)$ for $x_{1}$ is sufficiently large. On the other hand, by (a), $\lim _{x \rightarrow \pm \infty} G\left(x, y_{0}\right)=$ 0 . Hence (b) holds. For (c), we observe that $(\partial G / \partial x)(x, 0)$ has the three real zeros $1, n+1,(n-3) /(n-2)$, and we can check that $G(1,0)=G(n+1,0)=-1<0$ and $G((n-3) /(n-2), 0)>0$. So $G(x, 0)$ has exactly three real zeros. The second assertion of (c) is easily seen from $\operatorname{deg} B(x)=4$, since $(x, y) \neq(n+1,0)$. Finally, we see that

$$
\begin{equation*}
\frac{\partial^{2} G}{\partial x^{2}}\left(x, y_{0}\right)=\frac{2 n\left((x-n-1)^{2}+y_{0}^{2}\right)^{n} C(x)}{\left((x-2)^{2}+y_{0}^{2}\right)^{n+2}\left((x-n-1)^{2}+y_{0}^{2}\right)^{3}}, \tag{3.20}
\end{equation*}
$$

where $C(x)$ is a polynomial in $x$ of degree 7 whose leading coefficient is $2(2-n)$. So it follows from the negativity of the leading coefficient of the numerator of the right side of (3.20) that (d) holds.

By Proposition 3.4(c), $G(x, 0)$ has exactly three real zeros, and for fixed $y \neq 0$ the graph of $G(x, y)$ indicates that the value 0 can be taken on at most three times. Thus, by Propositions 3.1 and 3.3, the locus consists of
\{one oval, one $\infty$-component\} or \{three $\infty$-components\}.
Next we examine the number of real zeros of $(\partial G / \partial x)(x, y)$ for $|y|$ sufficiently large.
Lemma 3.5. For $\left|y_{0}\right|$ sufficiently large, $(\partial G / \partial x)\left(x, y_{0}\right)$ has exactly two real zeros.
Proof. Let $y_{0}$ be sufficiently large and fixed. From (3.18),

$$
\begin{equation*}
\frac{\partial G}{\partial x}\left(x, y_{0}\right)=\frac{-2 n\left((x-n-1)^{2}+y_{0}^{2}\right)^{n} B\left(x, y_{0}\right)}{\left((x-2)^{2}+y_{0}^{2}\right)^{n+1}\left((x-n-1)^{2}+y_{0}^{2}\right)^{2}} . \tag{3.22}
\end{equation*}
$$

Since $(x-n-1)^{2}+y_{0}^{2} \neq 0,(\partial G / \partial x)\left(x, y_{0}\right)=0$ is equivalent to $B\left(x, y_{0}\right)=0$. Then

$$
\begin{equation*}
B(x)=B\left(x, y_{0}\right)=(u x+v)-(x-1)(x-2)(x-(n+1))((n-2) x-n+3), \tag{3.23}
\end{equation*}
$$

where $u$ and $v$ are positive numbers with $v / u$ large. Observe that the zeros of $u x+v$ and $-(x-1)(x-2)(x-(n+1))((n-2) x-n+3)$ are $-v / u,(n-3) /(n-2), 1,2$, $n+1$. By sign changes, we observe that there are no real zeros of $B(x)$ on $(-\infty,-v / u) \cup$ $((n-3) /(n-2), 1) \cup(2, n+1)$, and there is at least one real zero of $B(x)$ on $(-v / u,(n-$ $3) /(n-2))$. Also there are no real zeros of $B(x)$ on $[0,(n-3) /(n-2)] \cup(1,2)$, since $v / u$ is large. On the other hand, we can check that $B(-x)$ has only one sign change in its coefficients. Hence, by Descartes' rule of signs and the above, there is only one real zero of $B(x)$ on $(-v / u, 0)$. But the degree of $B(x)$ is four, so the number of real zeros on $(n+1, \infty)$ is either one or three. It is obvious that more than two real zeros are not on $(n+1, \infty)$. Hence $(\partial G / \partial x)\left(x, y_{0}\right)$ has exactly two real zeros.

By Proposition 3.4(a), (b) and Lemma 3.5, there is only one real $x$ with $G(x, y)=0$ for $|y|$ sufficiently large. This shows that originally there could have been at most one $\infty$-component. Hence, by the above, equation (3.21), Proposition 2.3, and the proof of Proposition 3.1, we have the following theorem.

Theorem 3.6. The locus of

$$
\begin{equation*}
\left|(z-2)^{n}\right|=\left|(z-1)(z-(n+1))^{n-1}\right| \tag{3.24}
\end{equation*}
$$

has exactly two components; one oval and one $\infty$-component. Each component has zeros of $(z-2)^{n}+(z-1)(z-(n+1))^{n-1}=0$.

Here Figure $3.1(n=3)$ is enlightening.


Figure 3.1. $\left|(z-2)^{3}\right|=\left|(z-1)(z-4)^{2}\right|$.
Remark 3.7. Let $n$ and $m$ be positive integers with $1 \leq k<n$. If we choose $\left\{\alpha_{k}\right\}=$ $\{1,2, \ldots, m, n+m+1, n+m+2, \ldots, 2 n\}$ and $\left\{\beta_{k}\right\}=\{m+1, m+2, \ldots, m+n\}$ in (1.2), we can show that the locus of $P(x, y)=0$ has at least two components.

Acknowledgement. This work was supported by the Brain Korea 21 Project. The author wishes to thank Professor Kenneth B. Stolarsky for his help and encouragement.

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