THE NUMBER OF CONNECTED COMPONENTS OF CERTAIN REAL ALGEBRAIC CURVES

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(Received 7 June 2000 and in revised form 7 August 2000)

ABSTRACT. For an integer $n \ge 2$, let $p(z) = \prod_{k=1}^{n} (z - \alpha_k)$ and $q(z) = \prod_{k=1}^{n} (z - \beta_k)$, where α_k, β_k are real. We find the number of connected components of the real algebraic curve $\{(x, y) \in \mathbb{R}^2 : |p(x + iy)| - |q(x + iy)| = 0\}$ for some α_k and β_k . Moreover, in these cases, we show that each connected component contains zeros of p(z) + q(z), and we investigate the locus of zeros of p(z) + q(z).

2000 Mathematics Subject Classification. Primary 26C10; Secondary 30C15.

1. Introduction. Throughout the paper, *n* is an integer ≥ 2 . Let f(x, y) be an integral polynomial of degree *n*. Let *A* be the real algebraic curve defined by $A = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$. It is known that *A* consists of at most finitely many connected components. More precisely, when the curve is real nonsingular, each unbounded component is homeomorphic to a line and each bounded component is homeomorphic to a circle. We will call a bounded component an oval, and an unbounded component an ∞ -component. Also, we will write "component" instead of "connected component" for convenience. Let $p(z) = \prod_{k=1}^{n} (z - \alpha_k)$ and $q(z) = \prod_{k=1}^{n} (z - \beta_k)$, where α_k, β_k are real. The zeros of g(z) := p(z) + q(z) are clearly contained in the locus of the real algebraic curve

$$C := \{ (x, y) \in \mathbb{R}^2 : | p(x + iy) | - | q(x + iy) | = 0 \}.$$
(1.1)

In fact, in their study of "cylindrical algebraic decomposition," Arnon, Collins, and McCallum [1, 2] provide an algorithm for calculating the number of components given a specific example. However, we do not know the answer in the general case. We provide a different idea in this paper from that in [1, 2]. With the above terminology, here are some general questions.

(a) Given P(x, y) = 0 for real variables x and y, how many components are there? It is still unclear how to describe all possibilities for the topological nature of all components of an arbitrary P(x, y) = 0; this is the essence of the Hilbert's 16th problem. On the other hand, one of the most significant theorems of real algebraic geometry (Harnack (see [3, pages 257–258]), 1876) tells us that the number of components is at most one more than the genus.

(b) The curve *C* has finitely many components. Must each component have zeros of g(z) = 0?

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We answer the questions (a) and (b) for some real algebraic curves of the form (1.1). Define, for real variables x and y,

$$P(x,y) := |p(x+iy)|^{2} - |q(x+iy)|^{2}, \qquad (1.2)$$

where $\{\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n\} \subseteq \{1, 2, ..., 2n\}$. The simplest case for the questions (a) and (b) is $\{\alpha_k\} = \{1, 2, 3, ..., n\}$ and $\{\beta_k\} = \{n + 1, n + 2, ..., 2n\}$. Then all zeros of P(x, y) obviously lie on the vertical line x = n + 1/2, so P(x, y) has only one component. We will study the case $\{\alpha_k\} = \{2, 2, ..., 2\}$ and $\{\beta_k\} = \{1, n + 1, n + 1, ..., n + 1\}$ in Section 3. Moreover, in Section 2, we will investigate the locus of zeros of the more general polynomial equation

$$g(x,t) := (x-2)^n + (x-1)(x-t)^{n-1} = 0, \quad t \ge 3.$$
(1.3)

2. The zeros of g(x,t) = 0. We need the following two lemmas. First, Lemma 2.1 easily follows from the theorems of Hurwitz (see [4, page 4]) and Rouché (see [4, page 2]).

LEMMA 2.1. Let n > m > 0 be integers. Let A, B, and C be real numbers with $C \neq 0$. If a trinomial equation

$$Az^{n} + Bz^{m} + C = 0 \quad with |B| \ge |A| + |C|$$
 (2.1)

has no zeros on |z| = 1, then it has exactly *m* zeros strictly inside |z| = 1.

LEMMA 2.2. The zeros of g(x,t) are $(2+a_{n,t})/(1+a_{n,t})$, where each $a_{n,t}^{-1/(n-1)}$ is a zero of the trinomial equation $(2-t)z^n + (1-t)z + 1 = 0$.

PROOF. From g(x,t) = 0, we obtain $-(x-2)/(x-1) = ((x-t)/(x-2))^{n-1}$. Let

$$-\frac{x-2}{x-1} = \left(\frac{x-t}{x-2}\right)^{n-1} = a,$$
(2.2)

where $a := a_{n,t}$ is a complex number. From -(x-2)/(x-1) = a, we find that x = (2+a)/(1+a), and it easily follows from $((x-t)/(x-2))^{n-1} = a$ that $x = (2a^{1/(n-1)}-t)/(a^{1/(n-1)}-1)$. Equating these two formulae for x leads to $a^{n/n-1} + (1-t)a + 2 - t = 0$. The result follows by multiplying each side by $a^{-n/(n-1)}$.

Now we find a relation between *x* (a zero of g(x,t) = 0) and *z* (a zero of $(2-t)z^n + (1-t)z + 1 = 0$) as follows:

$$x = \frac{2z^{n-1} + 1}{z^{n-1} + 1} = 1 + \frac{1}{1 + 1/z^{n-1}}.$$
(2.3)

So

$$z^{n-1} = \frac{x-1}{2-x}$$
, that is, $z = \left(\frac{x-1}{2-x}\right)^{1/(n-1)}$. (2.4)

Using Lemmas 2.1 and 2.2, we have the following proposition.

PROPOSITION 2.3. The function g(x,t) has only one zero x_0 in $\Re x < 3/2$, and has no zeros in $3/2 \le \Re x \le (t+2)/2$.

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PROOF. Observe that the strip $3/2 \le \Re x \le (t+2)/2$ is zero-free, since, for such x, $|x-2| \le |x-t|$ and |x-2| < |x-1|. Now we consider the trinomial equation $(2-t)z^n + (1-t)z + 1 = 0$. It has no zero on |z| = 1, since, if there were such a zero z, then by (2.4), $1 = |z^{n-1}| = |(x-1)/(2-x)|$, that is, $x = 3/2 + i\beta$ for some real number β . This is a contradiction. Hence, by Lemma 2.1, the trinomial equation $(2-t)z^n + (1-t)z + 1 = 0$ has exactly one zero z_0 interior to |z| = 1. Then $|z_0| = |((x_0 - 1)/(2-x_0))^{1/(n-1)}| < 1$, that is, $|x_0 - 1| < |2 - x_0|$ for some real number x_0 . Hence $\Re x_0 < 3/2$ which proves the proposition.

Next, we study further the unique zero x_0 given by Proposition 2.3.

PROPOSITION 2.4. Let *n* be an integer ≥ 3 and $t \ge 3$. Then the only zero x_0 of g(x,t) in $\Re x \le (t+2)/2$ is real and

$$\frac{1+2(-\epsilon+1/n)^{n-1}}{1+(-\epsilon+1/n)^{n-1}} < x_0 < \frac{1+2(\epsilon+1/n)^{n-1}}{1+(\epsilon+1/n)^{n-1}},$$
(2.5)

where $\epsilon = \epsilon(n, t) = 2^n (t-2) / (t-1)^{n+1}$.

PROOF. For *n* an integer ≥ 3 , let $\epsilon = \epsilon(n,t) = 2^n(t-2)/(t-1)^{n+1}$. Then $0 < \epsilon \leq 1/(t-1)$, since $(2/(t-1))^n < 1/(t-2)$ and $n \geq 3$. Then the trinomial equation $(2-t)z^n + (1-t)z + 1 = 0$ has at least one real zero z_0 in $(1/(t-1)-\epsilon, 1/(t-1)+\epsilon)$. In fact, by algebra, we can see that the left side of the trinomial equation is

$$-(2^{n}+(1+2^{n}(-2+t)(-1+t)^{-n})^{n})(-2+t)(-1+t)^{-n}<0$$
(2.6)

at $z = 1/(t-1) + \epsilon$, and

$$-\left(-2^{n}+\left(1-2^{n}(-2+t)(-1+t)^{-n}\right)^{n}\right)(-2+t)(-1+t)^{-n}>0$$
(2.7)

at $z = 1/(t-1) - \epsilon$. Set $z_0 = ((x_0-1)/(2-x_0))^{1/(n-1)}$. Since z_0 is real, so is x_0 . Now we obtain the inequality $|((x_0-1)/(2-x_0))^{1/(n-1)} - 1/(t-1)| < \epsilon$, and from this we have the inequality (2.5). A simple calculation yields that (1+2A)/(1+A) < (t+2)/2 for A > 0. This proves the result.

REMARK 2.5. (a) For n = 2 and $t \ge 3$, we can easily check that g(x,t) has two real zeros. Here the smaller zero is $\le (t+2)/2$, but it does not satisfy (2.5).

(b) In Lemma 2.2, we encountered a trinomial equation $(t-2)z^n + (t-1)z - 1 = 0$ $(t \ge 3)$. Here we define a more general polynomial

$$h(z) = (t-2)z^{n} + (t-1)z - s \quad (s \ge 0).$$
(2.8)

Then we have the following zero distributions. The function h(z) has

all its zeros with modulus > 1if s > 2t - 3,one (real) zero with modulus = 1 and all others > 1if s = 2t - 3,one (real) zero with modulus < 1 and all others > 1if $0 \le s \le 1$.

This can be proved by elementary calculation, Lemma 2.1, and Eneström-Kakeya theorem (see [4, page 136]). However, we did not consider the case 1 < s < 2t - 3. We conjecture that, for 1 < s < 2t - 3, h(z) has one (real) zero with modulus < 1 and all others > 1, as the case $0 \le s \le 1$, but it remains an open problem.

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3. The number of components of $|(z-2)^n| = |(z-1)(z-(n+1))^{n-1}|$. Let

$$g(z) := (z-2)^{n} + (z-1)(z-(n+1))^{n-1}.$$
(3.1)

If g(z) = 0, then $|(z-1)(z-(n+1))^{n-1}/(z-2)^n|^2 = 1$. This motivates, for real variables *x* and *y*, the introduction of

$$G(x,y) := \frac{((x-1)^2 + y^2)((x-(n+1))^2 + y^2)^{n-1}}{((x-2)^2 + y^2)^n} - 1.$$
 (3.2)

Here G(x, y) is obviously symmetric about the *x*-axis. In this section, we find the number of components of G(x, y) = 0 and show that each component has zeros of g(z) = 0. First, using Proposition 2.3, we find that the number of components of G(x, y) = 0 is at least two.

PROPOSITION 3.1. The locus of

$$|(z-2)^{n}| = |(z-1)(z-t)^{n-1}|, \quad t \ge 3$$
(3.3)

has at least two components.

PROOF. We showed in Proposition 2.3 that g(x,t) has one real zero < 2 and n-1 zeros with real part > (t+2)/2 > 2. So it suffices to show that, on z = 2 + is (*s* real), the two absolute values are never equal. On z = 2 + is (*s* real),

$$\left| (z-1)(z-t)^{n-1} \right|^{2} - \left| (z-2)^{n} \right|^{2} = (1+s^{2})((t-2)^{2}+s^{2})^{n-1} - s^{2n} \ge (t-2)^{2} > 0.$$
(3.4)

Next, we show that the points where the locus of G(x, y) = 0 has vertical tangents lie on the real axis. We use this later to show that the locus consists of either one oval, one ∞ -component or three ∞ -components. In order to prove this, we need the following lemma.

LEMMA 3.2. Let *n* be an integer \geq 3. Define

$$f(x) := \left(\frac{-2x+3}{(n-1)(-2x+n+2)}\right)^{n-1} - \frac{-2x+n+2}{(n-1)(-2x+n+3)}.$$
(3.5)

Then all real zeros of f(x) are

$$\begin{cases} \frac{n^2 + n - 5}{2n - 4}, & n \text{ even,} \\ \frac{n^2 + n - 5}{2n - 4}, r(n), & n \text{ odd,} \end{cases}$$
(3.6)

where $(n^2 + n - 5)/(2n - 4)$ is a double zero in each case and $3/2 < r = r(n) < (n^2 + n + 1)/2n$.

PROOF. From f(x) = 0, we find that

$$\left(\frac{-2x+3}{(n-1)(-2x+n+2)}\right)^{n-1} = \frac{-2x+n+2}{(n-1)(-2x+n+3)} = a,$$
(3.7)

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where $a := a_n$ is a complex number. From $(-2x+3)/(n-1)(-2x+n+2) = a^{1/(n-1)}$, we get

$$x = -\frac{3 - a^{1/(n-1)}(n-1)(n+2)}{-2 + 2a^{1/(n-1)}(n-1)},$$
(3.8)

and also

$$x = -\frac{n+2-a(n-1)(n+3)}{-2+2a(n-1)}$$
(3.9)

from (-2x+n+2)/(n-1)(-2x+n+3) = a. Equating these two formulae for x leads to $(n-1)a^{n/(n-1)} - na + 1 = 0$, and so $a^{1/(n-1)}$ is a zero of the trinomial equation $w(y) := (n-1)y^n - ny^{n-1} + 1 = 0$. Now, we have

$$\frac{w(y)}{(y-1)^2} = (n-1)y^{n-2} + (n-2)y^{n-3} + (n-3)y^{n-4} + \dots + 2y + 1.$$
(3.10)

Since $a^{1/(n-1)}$ is real if and only if the corresponding x in (3.7) is real, the number of real zeros of f(x) is equal to that of w(y). By (3.10), w(y) has a real double zero at 1, and its corresponding x is $(n^2+n-5)/(2n-4)$, since (-2x+3)/(n-1)(-2x+n+2) = 1. On the other hand, it follows from Eneström-Kakeya theorem that $w(y)/(y-1)^2$ has no zero for |y| > 1. Also it is obvious that $w(y)/(y-1)^2$ has no real zero ≥ 0 . In order to find the real zeros of f(x), we first need to determine whether w(y) has a real zero on (-1,0) or not. We see that $w'(y) = n(n-1)y^{n-2}(y-1)$. So if n is even, then w'(y) < 0 for -1 < y < 0. Moreover, w(0) = 1 > 0, which implies there are no real zeros of w(y) other than 1. Hence f(x) has only one (double) real zero $(n^2+n-5)/(2n-4)$. Suppose that n is odd. Then w'(y) > 0 on -1 < y < 0, w(-1) = 2(1-n) < 0, and w(0) > 0. This implies that there must be exactly one real zero on (-1,0). Say x_0 is its corresponding real number. Then by (3.7)

$$-1 < \frac{-2x_0 + 3}{(n-1)(-2x_0 + n + 2)} < 0.$$
(3.11)

Simple calculations yield that $3/2 < x_0 < (n^2 + n + 1)/2n$. This completes the proof.

Now we have the following Proposition.

PROPOSITION 3.3. The points where the locus of G(x, y) = 0 has vertical tangents lie on the real axis.

PROOF. It suffices to show that $(0,1) \cdot \nabla G(x,y) = 0$ and G(x,y) = 0 implies y = 0. A calculation shows that $(0,1) \cdot \nabla G(x,y) = \partial G/\partial y = 0$ if and only if y = 0 or $y^2 = A(x)$, where

$$A(x) = \frac{2(n-2)x^3 - (n^2 + 5n - 17)x^2 + 2(n^2 + n - 12)x - (n^2 - 2n - 11)}{-2(n-2)x + n^2 + n - 5}.$$
 (3.12)

Suppose that $y^2 = A(x)$. Then

$$f(x) := G(x, y) = \begin{cases} \frac{1}{4x^2 - 16x + 15}, & n = 2, \\ \left(\frac{-2x + 3}{(n-1)(-2x + n + 2)}\right)^{n-1} - \frac{-2x + n + 2}{(n-1)(-2x + n + 3)}, & n \ge 3, \end{cases}$$
(3.13)

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by simplifying the equations. So it is clear that there are no zeros of f(x) in the case of n = 2. Suppose that $n \ge 3$. By Lemma 3.2, $(n^2 + n - 5)/(2n - 4)$ is a (double) real zero of f(x) and, in particular, if n is even, such a real zero is unique. But $A((n^2 + n - 5)/(2n - 4))$ is not defined. So this is a contradiction. Suppose that n is odd. Then by Lemma 3.2, all zeros of f(x) are $(n^2 + n - 5)/(2n - 4)$ and r(n), where $3/2 < r(n) < (n^2 + n + 1)/2n$. As above, $A((n^2 + n - 5)/(2n - 4))$ is not defined. So it is enough to consider r(n). Now, we have that A(3/2) = -1/4 < 0 and $A((n^2 + n + 1)/2n) = -(n^4 - 2n^3 + 5n^2 - 4n + 1)/4n^2 < 0$. So if we show that A'(x) < 0 on $3/2 < x < (n^2 + n + 1)/2n$, then $y^2 = A(x) < 0$, which is a contradiction. We see that

$$A'(x) = -\frac{2s(x)}{\left(-2(n-2)x + n^2 + n - 5\right)^2},$$
(3.14)

where $s(x) = 4(n-2)^2 x^3 - 4(n-2)^2 (n+4) x^2 + (n^2+5n-17)(n^2+n-5)x - n^4 - n^3 + 12n^2 + 10n - 38$. So it is enough to show that s(x) > 0 on $3/2 < x < (n^2+n+1)/2n$. Now

$$s\left(\frac{3}{2}\right) = \frac{1}{2}(n-1)^{3}(n+1) > 0,$$

$$s\left(\frac{n^{2}+n+1}{2n}\right) = \frac{(n-1)^{3}(2n-1)(n^{2}-2n+2)}{n^{3}} > 0,$$

$$s'(x) = (6(2-n)x + n^{2} + 5n - 17)(2(2-n)x + n^{2} + n - 5).$$

(3.15)

Hence, $(n^2 + 5n - 17)/6(n-2)$ and $(n^2 + n - 5)/2(n-2)$ are the zeros of s'(x), and we can check that

$$\begin{cases} \frac{n^2 + 5n - 17}{6(n-2)} < \frac{3}{2} < \frac{n^2 + n + 1}{2n} < \frac{n^2 + n - 5}{2(n-2)}, & n = 3, \\ \frac{3}{2} < \frac{n^2 + 5n - 17}{6(n-2)} < \frac{n^2 + n + 1}{2n} < \frac{n^2 + n - 5}{2(n-2)}, & n \ge 4. \end{cases}$$
(3.16)

This proves the result, since s(3/2) > 0 and $s((n^2 + n + 1)/2n) > 0$.

Next we establish the following Proposition.

PROPOSITION 3.4. For fixed $y_0 \neq 0$,

- (a) $\lim_{x \to \pm \infty} G(x, y_0) = 0,$
- (b) for |x| large, the limit is approached from above for $x \to -\infty$ and the limit is approached from below for $x \to +\infty$,
- (c) G(x,0) has exactly three real zeros. Moreover, $(\partial G/\partial x)(x, y_0)$ has at most four real zeros,

$$\frac{\partial^2 G}{\partial x^2}(x, y_0) \begin{cases} \geq 0 & \text{as } x \to -\infty, \\ \leq 0 & \text{as } x \to \infty. \end{cases}$$
(3.17)

PROOF. Let y_0 be nonzero and fixed. It is obvious that $\lim_{x \to \pm \infty} G(x, y_0) = 0$. By a calculation, we have

$$\frac{\partial G}{\partial x}(x, y_0) = \frac{-2n((x-n-1)^2 + y_0^2)^n B(x, y_0)}{((x-2)^2 + y_0^2)^{n+1}((x-n-1)^2 + y_0^2)^2},$$
(3.18)

⁽d)

where

$$B(x) = B(x, y_0)$$

= $(n-2)y_0^4 + (n^2 - n + 1)(x-2)y_0^2 - (x-1)(x-2)(x - (n+1))((n-2)x - n + 3)$
(3.19)

is a polynomial in *x* of degree 4 whose leading coefficient is 2 - n. So it follows from the positivity of the leading coefficient of the numerator of the right side of (3.18) that, for |x| large, $(\partial G/\partial x)(x, y_0) > 0$, that is, $G(x, y_0)$ is increasing on (x_1, ∞) and $(-\infty, -x_1)$ for x_1 is sufficiently large. On the other hand, by (a), $\lim_{x\to\pm\infty} G(x, y_0) = 0$. Hence (b) holds. For (c), we observe that $(\partial G/\partial x)(x, 0)$ has the three real zeros 1, n + 1, (n - 3)/(n - 2), and we can check that G(1, 0) = G(n + 1, 0) = -1 < 0 and G((n - 3)/(n - 2), 0) > 0. So G(x, 0) has exactly three real zeros. The second assertion of (c) is easily seen from deg B(x) = 4, since $(x, y) \neq (n + 1, 0)$. Finally, we see that

$$\frac{\partial^2 G}{\partial x^2}(x, y_0) = \frac{2n((x-n-1)^2 + y_0^2)^n C(x)}{((x-2)^2 + y_0^2)^{n+2}((x-n-1)^2 + y_0^2)^3},$$
(3.20)

where C(x) is a polynomial in x of degree 7 whose leading coefficient is 2(2-n). So it follows from the negativity of the leading coefficient of the numerator of the right side of (3.20) that (d) holds.

By Proposition 3.4(c), G(x,0) has exactly three real zeros, and for fixed $y \neq 0$ the graph of G(x,y) indicates that the value 0 can be taken on at most three times. Thus, by Propositions 3.1 and 3.3, the locus consists of

{one oval, one
$$\infty$$
-component} or {three ∞ -components}. (3.21)

Next we examine the number of real zeros of $(\partial G/\partial x)(x, y)$ for |y| sufficiently large.

LEMMA 3.5. For $|y_0|$ sufficiently large, $(\partial G/\partial x)(x, y_0)$ has exactly two real zeros.

PROOF. Let y_0 be sufficiently large and fixed. From (3.18),

$$\frac{\partial G}{\partial x}(x, y_0) = \frac{-2n((x-n-1)^2 + y_0^2)^n B(x, y_0)}{((x-2)^2 + y_0^2)^{n+1}((x-n-1)^2 + y_0^2)^2}.$$
(3.22)

Since $(x - n - 1)^2 + y_0^2 \neq 0$, $(\partial G / \partial x)(x, y_0) = 0$ is equivalent to $B(x, y_0) = 0$. Then

$$B(x) = B(x, y_0) = (ux + v) - (x - 1)(x - 2)(x - (n + 1))((n - 2)x - n + 3), \quad (3.23)$$

where *u* and *v* are positive numbers with v/u large. Observe that the zeros of ux + vand -(x-1)(x-2)(x-(n+1))((n-2)x-n+3) are -v/u, (n-3)/(n-2), 1, 2, n+1. By sign changes, we observe that there are no real zeros of B(x) on $(-\infty, -v/u) \cup$ $((n-3)/(n-2), 1) \cup (2, n+1)$, and there is at least one real zero of B(x) on (-v/u, (n-3)/(n-2)). Also there are no real zeros of B(x) on $[0, (n-3)/(n-2)] \cup (1, 2)$, since v/u is large. On the other hand, we can check that B(-x) has only one sign change in its coefficients. Hence, by Descartes' rule of signs and the above, there is only one real zero of B(x) on (-v/u, 0). But the degree of B(x) is four, so the number of real zeros on $(n+1,\infty)$ is either one or three. It is obvious that more than two real zeros are not on $(n+1,\infty)$. Hence $(\partial G/\partial x)(x, y_0)$ has exactly two real zeros. By Proposition 3.4(a), (b) and Lemma 3.5, there is only one real x with G(x, y) = 0 for |y| sufficiently large. This shows that originally there could have been at most one ∞ -component. Hence, by the above, equation (3.21), Proposition 2.3, and the proof of Proposition 3.1, we have the following theorem.

THEOREM 3.6. The locus of

$$|(z-2)^{n}| = |(z-1)(z-(n+1))^{n-1}|$$
 (3.24)

has exactly two components; one oval and one ∞ -component. Each component has zeros of $(z-2)^n + (z-1)(z-(n+1))^{n-1} = 0$.

Here Figure 3.1 (n = 3) is enlightening.



FIGURE 3.1. $|(z-2)^3| = |(z-1)(z-4)^2|$.

REMARK 3.7. Let *n* and *m* be positive integers with $1 \le k < n$. If we choose $\{\alpha_k\} = \{1, 2, ..., m, n + m + 1, n + m + 2, ..., 2n\}$ and $\{\beta_k\} = \{m + 1, m + 2, ..., m + n\}$ in (1.2), we can show that the locus of P(x, y) = 0 has at least two components.

ACKNOWLEDGEMENT. This work was supported by the Brain Korea 21 Project. The author wishes to thank Professor Kenneth B. Stolarsky for his help and encouragement.

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