IJMMS 28:6 (2001) 359–365
PII. S0161171201011966
http://ijmms.hindawi.com

SOME REMARKS ON THE INVARIANT SUBSPACE PROBLEM FOR HYPONORMAL OPERATORS

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(Received 11 February 2001)

ABSTRACT. We make some remarks concerning the invariant subspace problem for hyponormal operators. In particular, we bring together various hypotheses that must hold for a hyponormal operator without nontrivial invariant subspaces, and we discuss the existence of such operators.

2000 Mathematics Subject Classification. 47B20, 47A15.

Let $\mathcal{H}$ be a separable, infinite-dimensional, complex Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all linear and bounded operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *hyponormal* (notation: $T \in H(\mathcal{H})$) if $[T^*,T] := T^*T - TT^* \geq 0$, or equivalently, if $\|T^*x\| \leq \|Tx\|$ for every $x \in \mathcal{H}$.

The purpose of this paper is to use several results that may be applied to the invariant subspace problem (ISP) for hyponormal operators and thus to bring into focus what remains to be done to solve the problem completely. We begin by recalling some standard notation and terminology to be used. For a (nonempty) compact subset $K \subset \mathbb{C}$, we denote by $C(K)$ the Banach algebra of all continuous complex-valued functions on K with the supremum norm, by $\operatorname{Rat}(K)$ the subalgebra of $C(K)$ consisting of all rational functions with poles off the set K, and by $R(K)$ the closure in $C(K)$ of $\operatorname{Rat}(K)$. For $T \in \mathcal{L}(\mathcal{H})$, the spectrum of T is denoted by $\sigma(T)$ and the algebra $\{r(T) : r \in \operatorname{Rat}(\sigma(T))\}$ by $\operatorname{Rat}(T)$. The *rational cyclic multiplicity* of T (notation: $m(T)$) is the smallest cardinal number m with the property that there are m vectors $\{x_i\}_{0\leq i<m}$ in $\mathcal{H}$ such that $\vee\{Ax_i \mid 0 \leq i < m,\ A \in \operatorname{Rat}(T)\} = \mathcal{H}$. For a bounded (nonempty) open subset $U \subset \mathbb{C}$, one denotes by $\mathbf{H}^\infty(U)$ the Banach algebra of those analytic complex-valued functions on U with the property that $\|f\|_{\infty,U} := \sup_{z\in U}|f(z)| < \infty$. The ideal of all compact operators on $\mathcal{H}$ will be denoted by $\mathbb{K} = \mathbb{K}(\mathcal{H})$. Since $\mathbb{K}$ is a two-sided, norm-closed ideal in $\mathcal{L}(\mathcal{H})$, the quotient algebra $\mathcal{L}(\mathcal{H})/\mathbb{K}$ is a C^*-algebra, which is called the *Calkin* algebra, and the quotient map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})/\mathbb{K}$ will be denoted by π. For T in $\mathcal{L}(\mathcal{H})$, we write $\sigma_e(T)$ (resp., $\sigma_{\text{re}}(T), \sigma_{\text{le}}(T)$) for the *essential* (resp., *r*ight-, left-essential) spectrum of T (i.e., the spectrum (resp., right, left spectrum) of $\pi(T)$). An operator $A \in \mathbb{K}(\mathcal{H})$ is called a *trace-class* operator (notation: $A \in \mathcal{C}_1(\mathcal{H})$) if the series $\operatorname{tr}(|A|) := \sum_{i\in\mathbb{N}}\langle |A|e_i, e_i\rangle$ is convergent, where $|A| = (A^*A)^{1/2}$ and $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. An operator $A \in \mathbb{K}(\mathcal{H})$ is *Hilbert-Schmidt* operator (notation: $A \in \mathcal{C}_2(\mathcal{H})$) if $A^*A \in \mathcal{C}_1(\mathcal{H})$. For a selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$, A_- will denote its negative part $(|A| - A)/2$. Finally, μ will denote planar Lebesgue measure defined on the Borel subsets of $\mathbb{R}^2$.

A beautiful generalization of the Berger-Shaw inequality for hyponormal operators [3] was given by Voiculescu.

Theorem 1 (see [17]). *If* $[T^*,T]_- \in \mathscr{C}_1(\mathscr{H})$ *and* $X \in \mathscr{C}_2(\mathscr{H})$ *is such that* $m(T+X) < +\infty$*, then* $[T^*,T] \in \mathscr{C}_1(\mathscr{H})$ *and* $\pi\,\mathrm{tr}([T^*,T]) \le m(T+X)\mu(\sigma(T+X))$.

For purposes of finding a nontrivial invariant subspace (n.i.s.) for an arbitrary operator $T \in \mathscr{L}(\mathscr{H})$, one may assume that every nonzero vector in $\mathscr{H}$ is cyclic for T, and hence that $m(T) = 1$. Thus the following corollary (of Theorem 1 or the earlier Berger-Shaw inequality) is useful.

Corollary 2. *Every hyponormal operator* T *in* $\mathscr{L}(\mathscr{H})$ *with a rational cyclic vector* $(i.e., m(T) = 1)$ *is essentially normal (i.e.,* $\pi(T)$ *is normal in the Calkin algebra).*

When looking for a n.i.s. for an arbitrary T in $\mathscr{L}(\mathscr{H})$, one knows (cf. [10]) from a deep theorem of Apostol, Foiaş, and Voiculescu [2] that T may be assumed to belong to the class $\mathscr{BQT}(\mathscr{H})$ of *biquasitriangular operators*, (see [10] for the definition and a characterization). If we denote by $\mathscr{EN}(\mathscr{H})$ the collection of essentially normal operators on $\mathscr{H}$ and by $(N+K)(\mathscr{H})$ the collection of those operators in $\mathscr{L}(\mathscr{H})$ which can be written as a sum of a normal and a compact operator, then a deep result of Brown-Douglas-Fillmore can also be applied.

Theorem 3 (see [5]). *The equality* $\mathscr{BQT}(\mathscr{H}) \cap \mathscr{EN}(\mathscr{H}) = (N+K)(\mathscr{H})$ *holds.*

It is thus immediate from Theorems 1 and 3 and Corollary 2 that if there exists T in $H(\mathscr{H})$ without a n.i.s., then $[T^*,T] \in \mathscr{C}_1(\mathscr{H})$ and $T \in (N+K)(\mathscr{H})$.

A hyponormal operator is called *pure* if it does not have a nonzero reducing subspace to which its restriction is a normal operator. Obviously an operator in $H(\mathscr{H})$ without a n.i.s. is pure. The following result of Putnam [14] leads to another reduction of the ISP for hyponormal operators.

Theorem 4. *Let* $T \in \mathscr{L}(\mathscr{H})$ *be a pure hyponormal operator. If* $\Delta \subset \mathbb{C}$ *is an open set, then* $\mu(\Delta \cap \sigma(T)) > 0$ *whenever* $\Delta \cap \sigma(T) \neq \emptyset$.

This says that each point of the spectrum of such a T has *positive planar density*, and thus we may assume of a hyponormal operator T without a n.i.s. that $\sigma(T)$ has not only positive μ-measure but positive planar density at each point.

Let $A \in \mathscr{L}(\mathscr{H})$ be a selfadjoint operator and denote by E the spectral measure of the operator A. To every vector $x \in \mathscr{H}$ one may associate the Borel measure ν_x on $\mathbb{R}$ defined by $\nu_x(\Omega) = \langle E(\Omega)x, x\rangle$ for every Borel set $\Omega \subset \mathbb{R}$. The vector x is called *absolutely continuous* with respect to A if the measure ν_x is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. The selfadjoint operator A is called *absolutely continuous* if every vector of $\mathscr{H}$ is absolutely continuous with respect to A. The following result can be found in [13], (see also [9, page 135]).

Proposition 5. *If* $T = X + iY \in \mathscr{L}(\mathscr{H})$ *is the Cartesian decomposition of a pure hyponormal operator, then* X *and* Y *are both absolutely continuous operators.*

Next, recall that a subset Δ of a nonempty open set U in $\mathbb{C}$ is called *dominating* for U if $\|f\|_{\infty,U} = \sup_{\lambda\in\Delta}|f(\lambda)|$, $f \in \mathbf{H}^\infty(U)$.

The deep invariant subspace theorem for hyponormal operators obtained by Brown in [6] on the basis of the beautiful structure theorem for such operators by Putinar [12] is the following.

Theorem 6. *Let $T \in \mathcal{L}(\mathcal{H})$ be a hyponormal operator. If there is a nonempty open set $U \subset \mathbb{C}$ such that $\sigma(T) \cap U$ is dominating for U, then T has a n.i.s.*

Since one knows (cf. [1] or [6]) that if K is a (nonempty) compact set in $\mathbb{C}$ such that $R(K) \neq C(K)$, then K is dominating on some nonempty open set, one gets immediately the following corollary.

Corollary 7 (see [6]). *Any hyponormal operator $T \in \mathcal{L}(\mathcal{H})$ with $R(\sigma(T)) \neq C(\sigma(T))$ has a n.i.s.*

Thus if there exist hyponormal operators T without a n.i.s., then as noted above, $T \in (N+K)(\mathcal{H})$ and $\sigma(T)$ must satisfy $R(\sigma(T)) = C(\sigma(T))$. Moreover, it is a consequence of elementary Fredholm theory (cf. [10]) that if $T \in \mathcal{L}(\mathcal{H})$ and $\sigma_{\mathrm{le}}(T) \neq \sigma(T)$, then T^* has point spectrum and thus T has a n.i.s. Hence when looking for invariant subspaces for an arbitrary operator T we may always suppose that $\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T) = \sigma(T)$. This allows one to apply a result of Stampfli [16] to the problem.

Theorem 8. *Suppose $T \in \mathcal{EN}(\mathcal{H})$ is such that $\sigma(T) = \sigma_{\mathrm{le}}(T)$ and the Calkin map $\pi : \mathrm{Rat}(T) \to \mathcal{L}(\mathcal{H})/\mathbb{K}$ is bounded below. Then T has a n.i.s.*

Proof. By hypothesis, there exists a constant $M > 0$ such that $\|\pi(r(T))\|_e \geq M\|r(T)\|$ for every $r \in \mathrm{Rat}(\sigma(T))$, where $\|\cdot\|_e$ is the norm in the Calkin algebra. On the other hand,

$$\|\pi(r(T))\|_e = \|r(\pi(T))\|_e = \sup_{z\in\sigma_e(T)} |r(z)| = \sup_{z\in\sigma(T)} |r(z)|. \tag{1}$$

Thus $\|r(T)\| \leq (1/M)\|r\|_{\sigma(T)}$, $r \in \mathrm{Rat}(\sigma(T))$, so $\sigma(T)$ is a $(1/M)$-spectral set for T. The result now follows from [16]. □

Corollary 9. *If $T \in H(\mathcal{H})$ and T has no n.i.s., then there exist sequences $\{r_n(T)\}_{n\in\mathbb{N}}$ in the algebra $\mathrm{Rat}(T)$ and $\{K_n\}_{n\in\mathbb{N}}$ in $\mathbb{K}$ such that $\|r_n(T)\| = 1$, $n \in \mathbb{N}$, and $\|r_n(T) - K_n\| \to 0$. Moreover, any such sequence $\{r_n(T)\}_{n\in\mathbb{N}}$ has no subnet converging in the weak operator topology (WOT) to a nonzero operator.*

Proof. Since $T \in H(\mathcal{H})$ has no n.i.s., $m(T) = 1$, and according to Corollary 2, $T \in \mathcal{EN}(\mathcal{H})$. According to Theorem 8, $\pi : \mathrm{Rat}(T) \to \mathcal{L}(\mathcal{H})/\mathbb{K}$ must not be bounded below. Thus, there are sequences $\{r_n(T)\}_{n\in\mathbb{N}}$ in the algebra $\mathrm{Rat}(T)$ and $\{K_n\}_{n\in\mathbb{N}}$ in $\mathbb{K}$ such that $\|r_n(T)\| = 1$, $n \in \mathbb{N}$, and $\|r_n(T) - K_n\| \to 0$. Moreover, if there is a subnet $\{r_{n_k}(T)\}_{k\in\mathbb{N}}$ converging in the WOT to a nonzero operator, then T has a nontrivial hyperinvariant subspace according to [8]. □

The following proposition simply summarizes the results mentioned above.

Proposition 10. *If there exists $T \in H(\mathcal{H})$ such that T has no n.i.s., then T has the following properties:*

(a) $\sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{re}}(T) = \sigma(T)$,

(b) *$\sigma(T)$ is a connected and perfect subset of $\mathbb{C}$ such that every point of $\sigma(T)$ has positive planar density,*
(c) *$R(\sigma(T)) = C(\sigma(T))$, and, more generally, for every nonempty bounded open set $U \subset \mathbb{C}$, $U \cap \sigma(T)$ is not dominating for U,*
(d) *$T = N + K$, where N is a normal operator and K is compact,*
(e) *$T = X + iY$, where X and Y are absolutely continuous selfadjoint operators,*
(f) *$[T^*, T]$ is a positive semi-definite operator in $\mathcal{C}_1(\mathcal{H})$ with $\mathrm{tr}([T^*, T]) > 0$,*
(g) *$m(T) = 1$,*
(h) *there exist sequences $\{r_n(T)\}_{n\in\mathbb{N}} \subset \mathrm{Rat}(T)$ and $\{K_n\}_{n\in\mathbb{N}} \in \mathbb{K}$ such that $\|r_n(T)\| = 1$, $n \in \mathbb{N}$, and $\|r_n(T) - K_n\| \to 0$. Moreover, any such sequence has no subnet converging in the WOT to a nonzero operator.*

This proposition raises the interesting question:

Problem 11. Are there *any* operators in $H(\mathcal{H})$ satisfying (a)–(h)?

This is perhaps a difficult question, which we are unable to answer at present. Moreover, the list of necessary conditions for a hyponormal operator that has no n.i.s. is larger and, of course, not all results are included in this paper. The remainder of this paper is devoted to making some progress on this question. We first recall a result from [11].

Proposition 12. *Given a nonempty compact set $K \subset \mathbb{C}$ with positive density at each point, there is an irreducible, hyponormal operator with rank one self-commutator whose spectrum is K.*

This proposition shows that to make a start toward answering Problem 11 in the affirmative, we need to construct a compact set K which has properties (b) and (c) of Proposition 10 (with $K = \sigma(T)$). A collection of such sets K may be constructed by slight variation of the following.

Example 13. We first specify a Cantor set $C_{\{\theta_n\}} \subset [0,1]$ which has positive linear Lebesgue measure and, in fact, has positive linear density at each point. For this purpose we follow the notation and terminology of [4, Example 6P] and set $\theta_n = 1/3^n$ for each $n \in \mathbb{N}$. Note that for each n the closed intervals in the collection $\mathcal{F}_n$ have the same length—say l_n. Let $p \in C_{\{\theta_n\}}$ and let (a,b) be an open interval containing the point p. Since $C_{\{\theta_n\}} = \cap_n(\cup\{I : I \in \mathcal{F}_n\})$ and $l_n \to 0$, there exists n_0 sufficiently large that some interval $I_0 \in \mathcal{F}_{n_0}$ satisfies $p \in I_0 \subset (a,b)$. Since $C_{\{\theta_n\}} \cap I_0$ is another Cantor set, its measure can be easily calculated to be greater than $[l_{n_0} - l_{n_0}(\sum_{k\in\mathbb{N}} 1/3^{n_0+k})] = l_{n_0}(1 - 1/(2\cdot 3^{n_0})) > 0$, and thus $C_{\{\theta_n\}}$ has positive density at each point p. Let now z_0 be the point $(1/2, 1)$ in $\mathbb{C}$, and consider the planar set

$$K_1 := \{tp + (1-t)z_0 \mid 0 \le t \le 1,\ p \in C_{\{\theta_n\}}\}. \tag{2}$$

We will show that K_1 has the properties (b) and (c) above. By construction, the set K_1 is arcwise connected and perfect (since $C_{\{\theta_n\}}$ is perfect). Next, we show that every point q of K_1 has positive planar density. Let Δ be an open disc in $\mathbb{C}$ such that $q \in \Delta$. Since K_1 is perfect, there exists a point $t_0p_0 + (1-t_0)x_0 \in \Delta \cap K_1$ such that

$0 < t_0 < 1$. Clearly there exist positive real numbers t_1, t_2 such that $0 < t_1 < t_0 < t_2 < 1$ and some nondegenerate interval $[a,b]$ with $p_0 \in [a,b]$ such that the trapezoid-like figure $\Gamma := \{tp + (1-t)z_0 : t_1 \le t \le t_2,\ p \in C_{\{\theta_n\}} \cap [a,b]\}$ is contained in Δ. Let $\alpha > 0$ be the linear measure of the set $C_{\{\theta_n\}} \cap [a,b]$. Then the intersection of the set $K_1 \cap \Gamma$ with each horizontal line $y = t$, $t_1 \le t \le t_2$ has (linear) measure $(1-t)\alpha$. Thus, by Fubini's theorem, the μ-measure of the set $K_1 \cap \Gamma$ is $\int_{t_1}^{t_2} (1-t)\alpha\, dt = (t_2 - t_1)(1-(t_1+t_2)/2)\alpha > 0$, and hence K_1 has property (b). To check that K_1 has property (c), the following lemma is useful.

Lemma 14. *If U is any bounded open set such that $U \cap K_1 \neq \emptyset$, then there exist a point p_0 belonging to the outer boundary of U, an $\varepsilon > 0$, and a disc $D = \{z \in \mathbb{C} : |z - p_0| < \varepsilon\}$ such that $D \cap K_1 = \emptyset$.*

Proof. By construction, $[0,1] \setminus C_{\{\theta_n\}} = \cup_{n=1}^{\infty} (a_n, b_n)$ where $\{(a_n, b_n)\}_{n=1}^{\infty}$ is the disjoint sequence of "excluded" open intervals. Thus each open triangular domain $T_n = \{tz_0 + (1-t)p : -\infty < t < 1,\ p \in (a_n, b_n)\}$ in $\mathbb{C}$ is disjoint from K_1. Since $U \cap K_1 \neq \emptyset$ and every point of $C_{\{\theta_n\}}$ is a limit point of end points of arbitrarily short excluded intervals, there exists some triangular domain T_{n_0} such that $U \cap T_{n_0} \neq \emptyset$, and clearly any half-line joining a point of $U \cap T_{n_0}$ to the ideal point $|z| = +\infty$ and lying entirely in T_{n_0} must intersect ∂U in some last point (since ∂U is compact), which clearly satisfies the desired conclusions. □

We next show that K_1 satisfies (c) of Proposition 10. Let U be a bounded open set in $\mathbb{C}$ such that $U \cap K_1 \neq \emptyset$ and set $C = U^-$. Then the outer boundary of C coincides with the outer boundary of U, and applying Lemma 14 to U, we get a point p_0 of the outer boundary of U and an open disc D centered at p_0 with radius $\varepsilon > 0$ such that $D \cap K_1 = \emptyset$. By [7, Corollary 13.3], p_0 is a peak point of $R(C)$, that is, there exists an $f_0 \in R(C)$ such that $f_0(p_0) = 1$ and $|f_0(z)| < 1$ for $z \in C \setminus \{p_0\}$. Clearly $f_0 \in \mathbf{H}^{\infty}(U)$ and $\sup_{\lambda \in U} |f_0(\lambda)| = 1$ (since $p_0 \in \partial U$) while $\sup_{\lambda \in K_1 \cap U} |f_0(\lambda)| < 1$ (since $(K_1 \cap U)^-$ is at positive distance from p_0 and $|f_0| < 1$ on $(K_1 \cap U)^- \subset C \setminus D$). Thus $K_1 \cap U$ is not dominating for U, and K_1 has property (c).

Let T_1 be an irreducible hyponormal operator with rank one self-commutator whose spectrum $\sigma(T_1)$ is the compact set K_1 described in Example 13, and whose existence is guaranteed by Proposition 12. Thus property (f) is also satisfied. One observes that property (a) is satisfied too. Indeed, if one assumes that there exists $\lambda_0 \in \sigma(T_1) \setminus \sigma_{\mathrm{le}}(T_1)$, then $T_1 - \lambda_0$ is semi-Fredholm operator with nonpositive index (since $T_1 \in H(\mathcal{H})$). Since T_1 is pure, $\sigma_p(T_1) = \emptyset$, and thus the index is negative, which implies that $\sigma(T_1)$ contains a nonempty open set. Obviously this is a contradiction since $\sigma(T_1) = K_1$ has no interior, and thus $\sigma(T_1) = \sigma_{\mathrm{le}}(T_1)$. In a similar way one shows that $\sigma(T_1) = \sigma_{\mathrm{re}}(T_1)$. Moreover, $T_1 \in \mathcal{BQT}(\mathcal{H})$ according to theorem of Apostol, Foiaş, and Voiculescu [2] since $\sigma_e(T_1)$ contains no pseudoholes or holes associated with Fredholm index different from 0. Thus, by Theorem 3, the operator T_1 can be written $T_1 = N + K$, where N is a normal operator and K is a compact operator. Moreover, since T_1 is pure, $T_1 = X_1 + iY_1$, where X_1, and Y_1 are absolutely continuous selfadjoint operators according to Proposition 5.

Thus we have shown that the operator T_1 has properties (a)–(f) of Proposition 10. Whether T_1 has properties (g) and (h) of this proposition the author is unable to conclude. However, techniques and results of Stampfli [15] may be applied to obtain the following theorem.

Theorem 15. *Any hyponormal operator T in $\mathcal{L}(\mathcal{H})$ such that $\sigma(T)$ is the set K_1 of Example 13 has a nontrivial hyperinvariant subspace.*

Proof. Consider the operator $T-(1/2+i)$, which has spectrum $K_1-(1/2+i)$. Define $K_+ := K_1 \cap D_1^-$, where $D_1 := \{z \in \mathbb{C} : |z-4| < 4\}$ and $K_- := K_1 \cap D_2^-$, where $D_2 = \{z \in \mathbb{C} : |z+4| < 4\}$. Then $\sigma(T) = K_+ \cup K_-$, $K_+ \cap K_- = \{(0,0)\}$, and $\partial D_1 \cap K_1 = \partial D_2 \cap K_1 = \{(0,0)\}$. Choosing $f_1(z) = f_2(z) = z^2$, we may follow [15, Example 1] and observe that the operators $A_i := \int_{\partial D_i} f_i(z)(z-T)^{-1}dz$, $i = 1,2$, commute with any operator S with which T commutes to conclude that T has a nontrivial hyperinvariant subspace. □

Remark 16. We note that it is quite easy to modify the construction of Example 13 to produce compact sets satisfying properties (b) and (c) of Proposition 10 such that the techniques of [15] of "integrating through the spectrum" are no longer available to produce nontrivial invariant subspaces for the corresponding operator T.

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