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ON SEPARATION AXIOMS IN INTUITIONISTIC TOPOLOGICAL SPACES

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ABSTRACT. The purpose of this paper is to investigate several types of separation axioms in intuitionistic topological spaces, developed by Çoker (2000). After giving some characterizations of T_1 and T_2 separation axioms in intuitionistic topological spaces, we give interrelations between several types of separation axioms and some counterexamples.

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1. Introduction. After the introduction of the concept of a fuzzy set by Zadeh [15], Atanassov [1, 2] has introduced the concept of intuitionistic fuzzy set. Later Çoker et al. [4, 5, 8] have defined intuitionistic fuzzy topological spaces, intuitionistic sets, and intuitionistic topological spaces in [6, 9, 12].

2. Preliminaries. First we present the fundamental definitions (see Çoker [4]).

Definition 2.1 (see [4]). Let X be a nonempty fixed set. An intuitionistic fuzzy set (IS for short) A is an object having the form $A = \langle X, A_1, A_2\rangle$, where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \varnothing$. The set A_1 is called the set of members of A, while A_2 is called the set of nonmembers of A.

Definition 2.2 (see [4]). Let X be a nonempty set and let the IS's A and B be in the form $A = \langle X, A_1, A_2\rangle$, $B = \langle X, B_1, B_2\rangle$, respectively. Furthermore, let $\{A_i : i \in J\}$ be an arbitrary family of IS's in X, where $A_i = \langle X, A_i^{(1)}, A_i^{(2)}\rangle$. Then

(a) $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;
(c) $\bar{A} = \langle X, A_2, A_1\rangle$;
(d) $\cup A_i = \langle X, \cup A_i^{(1)}, \cap A_i^{(2)}\rangle$;
(e) $\cap A_i = \langle X, \cap A_i^{(1)}, \cup A_i^{(2)}\rangle$;
(f) $[\ ]A = \langle X, A_1, A_1^c\rangle$;
(g) $\langle\ \rangle A = \langle X, A_2^c, A_2\rangle$;
(h) $\underset{\sim}{\varnothing} = \langle X, \varnothing, X\rangle$; $\underset{\sim}{X} = \langle X, X, \varnothing\rangle$.

Let X be a nonempty set, $p \in X$ a fixed element in X, and let $A = \langle X, A_1, A_2\rangle$ be an IS. The IS $\underset{\sim}{p}$ defined by $\underset{\sim}{p} = \langle X, \{p\}, \{p\}^c\rangle$ is called an intuitionistic point (IP for short) in X. The IS $\underset{\approx}{p} = \langle \varnothing, \{p\}^c\rangle$ is called a vanishing intuitionistic point (VIP for short) in X. The IS $\underset{\sim}{p}$ is said to be contained in A ($\underset{\sim}{p} \in A$ for short) if and only if $p \in A_1$, and similarly, $\underset{\approx}{p}$ is said to be contained in A ($\underset{\approx}{p} \in A$ for short) if and only if $p \notin A_2$. For a given IS A in X, we may write

$$A = \left(\cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}\right) \cup \left(\cup\{\underset{\approx}{p} : \underset{\approx}{p} \in A\}\right), \tag{2.1}$$

(cf. [9]) and whenever A is not a proper IS (i.e., if A is not of the form $A = \langle X, A_1, A_2\rangle$, where $A_1 \cup A_2 \neq X$), then $A = \cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}$ follows. In general, any IS A in X can be written in the form $A = \underset{\sim}{A} \cup \underset{\approx}{A}$, where $\underset{\sim}{A} = \cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}$ and $\underset{\approx}{A} = \cup\{\underset{\approx}{p} : \underset{\approx}{p} \in A\}$. Furthermore it is easy to show that, if $A = \langle X, A_1, A_2\rangle$, then $\underset{\sim}{A} = \langle X, A_1, A_1^c\rangle$ and $\underset{\approx}{A} = \langle X, \emptyset, A_2\rangle$ (cf. [4, 7]).

Definition 2.3 (see [4]). Let X and Y be two nonempty sets and $f : X \to Y$ a function, $B = \langle Y, B_1, B_2\rangle$ an IS in Y and $A = \langle X, A_1, A_2\rangle$ an IS in X. Then the preimage of B under f, denoted by $f^{-1}(B)$, is the IS in X defined by $f^{-1}(B) = \langle X, f^{-1}(B_1), f^{-1}(B_2)\rangle$, and the image of A under f, denoted by $f(A)$, is the IS in Y defined by $f(A) = \langle Y, f(A_1), f_-(A_2)\rangle$ where $f_-(A_2) = (f(A_2^c))^c$.

You may find the fundamental properties of preimages and images in [4].

Definition 2.4 (see [6]). An intuitionistic topology (IT for short) on a nonempty set X is a family τ of IS's in X containing $\underset{\sim}{\emptyset}$, $\underset{\sim}{X}$ and closed under finite infima and arbitrary suprema. In this case the pair (X, τ) is called an intuitionistic topological space (ITS for short) and any IS in τ is known as an intuitionistic open set (IOS for short) in X. The complement $\bar{A}$ of an IOS A in an ITS (X, τ) is called an intuitionistic closed set (ICS for short) in X.

Let (X, τ) be an ITS on X. Then, we can also construct several other ITS's on X in the following way: $\tau_{0,1} = \{[\ ]G : G \in \tau\}$ and $\tau_{0,2} = \{\langle\ \rangle G : G \in \tau\}$. Furthermore,

$$\tau_1 = \{G_1 : G = \langle X, G_1, G_2\rangle \in \tau\}, \qquad \tau_2 = \{G_2^c : G = \langle X, G_1, G_2\rangle \in \tau\} \tag{2.2}$$

are topological spaces in X (cf. [6]).

Definition 2.5. Let A and B be two IS's on X and Y, respectively. Then the product intuitionistic set (PIS for short) of A and B on $X \times Y$ is defined by $U \times V = \langle (X, Y), A_1 \times B_1, (A_2^c \times B_2^c)^c\rangle$, where $A = \langle X, A_1, A_2\rangle$ and $B = \langle Y, B_1, B_2\rangle$.

If (X, τ) and (Y, Φ) are ITS's, then the product topology $\tau \times \Phi$ on $X \times Y$ is the IT generated by the base $\mathcal{B} = \{A \times B : A \in \tau, B \in \Phi\}$. This is so, because, if $A \times B, C \times D \in \mathcal{B}$, then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. Let $A \in \tau$, $B \in \Phi$, and $A = \langle X, A_1, A_2\rangle$, $B = \langle Y, B_1, B_2\rangle$. Then we have $\pi_1^{-1}(A) = \langle (x, y), A_1 \times Y, A_2 \times Y\rangle = A \times \underset{\sim}{Y}$, $\pi_2^{-1}(B) = \langle (X, Y), X \times B_1, X \times B_2\rangle = \underset{\sim}{X} \times B$, and

$$\begin{aligned} \pi_1^{-1}(A) \cap \pi_2^{-1}(B) &= (A \times \underset{\sim}{Y}) \cap (\underset{\sim}{X} \times B) \\ &= \langle (X, Y), (A_1 \times Y) \cap (X \times B_1), (A_2 \times Y) \cup (X \times B_2)\rangle \\ &= \langle (X, Y), A_1 \times B_1, (A_2 \times Y) \cup (X \times B_2)\rangle \\ &= \langle (X, Y), A_1 \times B_1, (A_2^c \times B_2^c)^c\rangle = A \times B. \end{aligned} \tag{2.3}$$

The definition of "neighborhoods" of IP's and VIP's can be found in Coşkun and Çoker [9] and "continuous function" between ITS's can be found in Çoker [6].

Lemma 2.6. *The projections $\pi_1 : X\times Y \to X$, $\pi_2 : X\times Y \to Y$, $\pi_1(x,y) = x$, $\pi_2(x,y) = y$ are continuous.*

Proof. Let $A \in \tau$, then $\pi_1^{-1}(A) = \langle (x,y), \pi_1^{-1}(A_1), \pi_1^{-1}(A_2)\rangle$. Thus we have $\pi_1^{-1}(A) = \langle (x,y), A_1\times Y, A_2\times Y\rangle = A\times \underset{\sim}{Y}$, that is, π_1 is continuous.

In other words, the product topology $\tau\times\Phi$ on $X\times Y$ is indeed the initial topology on $X\times Y$ with respect to the projections $\pi_1 : X\times Y \to X$ and $\pi_2 : X\times Y \to Y$. Here the subbase $\{\pi_1^{-1}(A), \pi_2^{-1}(B) : A \in \tau,\ B \in \Phi\}$ generates this product topology and the base $\mathcal{B}$ is given by

$$\mathcal{B} = \{\pi_1^{-1}(A)\cap\pi_2^{-1}(B) : A\in\tau,\ B\in\Phi\} = \{A\times B : A\in\tau,\ B\in\Phi\}. \tag{2.4}$$

□

Definition 2.7. Given the nonempty set X, we define the diagonal Δ_x as the following IS in $X\times X$:

$$\Delta_x = \langle (x_1,x_2), \{(x_1,x_2) : x_1 = x_2\}, \{(x_1,x_2) : x_1 \neq x_2\}\rangle. \tag{2.5}$$

Notice that, if X and Y are two nonempty sets and $(p,q) \in X\times Y$ a fixed element in $X\times Y$, then $(p,q)_\sim$ is contained in $U\times V((p,q)_\sim \in U\times V$ for short) if and only if $(p,q) \in U_1\times V_1$, and $(p,q)_\approx$ is contained in $U\times V((p,q)_\approx \in U\times V$ for short) if and only if $(p,q) \notin (U_2^c\times V_2^c)^c$, or equivalently $(p,q) \in U_2^c\times V_2^c$.

Definition 2.8. Let X, Y be two nonempty sets and $f : X \to Y$ a function. The graph of f, denoted by $\mathrm{GR}(f)$, is defined as the following IS in $X\times Y$:

$$\mathrm{GR}(f) = \langle (x,y), \{(x,f(x)) : x\in X\}, \{(x,f(x)) : x \in X\}^c\rangle. \tag{2.6}$$

3. Separation axioms in intuitionistic topological spaces.

In this section, we present T_1 and T_2 separation axioms in ITS's. The separation axioms T_1 and T_2 presented here have certain similarities to those in Bayhan and Çoker [3].

Definition 3.1. Let (X,τ) be an ITS, (X,τ) is said to be

(a) $T_1(i) \Leftrightarrow \forall x,y \in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y}\notin U$, and $\underset{\sim}{y}\in V$, $\underset{\sim}{x}\notin V$ (cf. [3, 14]);

(b) $T_1(ii) \Leftrightarrow \forall x,y \in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\approx}{x} \in U$, $\underset{\approx}{y}\notin U$, and $\underset{\approx}{y}\in V$, $\underset{\approx}{x}\notin \underset{\sim}{x}\in V$ (cf. [3, 14]);

(c) $T_1(iii) \Leftrightarrow \forall x,y \in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\sim}{x}\in U\subseteq \bar{\underset{\sim}{y}}$ and $\underset{\sim}{y}\in V\subseteq\bar{\underset{\sim}{x}}$ (cf. [3]);

(d) $T_1(iv) \Leftrightarrow \forall x,y \in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\approx}{x}\in U\subseteq \bar{\underset{\approx}{y}}$ and $\underset{\approx}{y}\in V\subseteq\bar{\underset{\approx}{x}}$ (cf. [3]);

(e) $T_1(v) \Leftrightarrow \forall x,y \in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\sim}{y}\notin U$ and $\underset{\sim}{x}\notin V$ (cf. [3]);

(f) $T_1(vi) \Leftrightarrow \forall x,y \in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\approx}{y}\notin U$ and $\underset{\approx}{x}\notin V$ (cf. [3]);

(g) $T_1(vii) \Leftrightarrow \forall x \in X$, $\underset{\sim}{x}$ is τ-closed;

(h) $T_1(viii) \Leftrightarrow \forall x \in X$, $\underset{\approx}{x}$ is τ-closed.

Theorem 3.2. *Let (X,τ) be an ITS, then the following implications are valid:*

$$\begin{array}{ccccc} T_1(v) & \longleftarrow & \longleftarrow & \longleftarrow & T_1(vi) \\ \uparrow & & & & \uparrow \\ T_1(i) & \longleftarrow & T_1(i)+T_1(ii) & \longrightarrow & T_1(ii) \\ & & \updownarrow & & \downarrow \\ T_1(vii) & \longleftarrow & T_1(iii) & & T_1(iv) \end{array} \tag{3.1}$$

Proof. The proof is obvious. □

Counterexample 3.3. Let $X=\{a,b,c\}$ and define the IT $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F,G\}$, where $A=\langle X,\{a,c\},\emptyset\rangle$, $B=\langle X,\{b\},\emptyset\rangle$, $C=\langle X,\{a\},\emptyset\rangle$, $D=\langle X,\{c\},\emptyset\rangle$, $E=\langle X,\{a,b\},\emptyset\rangle$, $F=\langle X,\{b,c\},\emptyset\rangle$, $G=\langle X,\emptyset,\emptyset\rangle$. Then (X,τ) is $T_1(i)$, but not $T_1(ii)$.

Counterexample 3.4. Let $X=\{a,b\}$ and define the IT $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B\}$ on X, where $A=\langle X,\emptyset,\{a\}\rangle$, $B=\langle X,\emptyset,\{b\}\rangle$. Then it is clear that (X,τ) is $T_1(v)$, but not $T_1(i)$.

Counterexample 3.5. Let $X=\{a,b,c\}$ and define the IT $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F\}$ on X, where $A=\langle X,\emptyset,\{a,b\}\rangle$, $B=\langle X,\{c\},\{a,b\}\rangle$, $C=\langle X,\emptyset,\{b,c\}\rangle$, $D=\langle X,\{c\},\{b\}\rangle$, $E=\langle X,\{a,c\},\{b\}\rangle$, $F=\langle X,\emptyset,\{b\}\rangle$. Then (X,τ) is $T_1(vi)$, but not $T_1(ii)$.

Counterexample 3.6. Let $X=\{a,b,c\}$ and define the IS's $A=\langle X,\{a\},\{c\}\rangle$, $B=\langle X,\{b\},\{a\}\rangle$, $C=\langle X,\{a\},\{b,c\}\rangle$, $D=\langle X,\emptyset,\{b\}\rangle$, $E=\langle X,\{a,b\},\emptyset\rangle$, $F=\langle X,\emptyset,\{a,c\}\rangle$, $G=\langle X,\emptyset,\{b,c\}\rangle$, $H=\langle X,\{a\},\emptyset\rangle$, $K=\langle X,\{a\},\{b\}\rangle$. Let τ denote the IT on X generated by the subbase $S=\{A,B,C,D,E,F,G,H,K\}$. Then (X,τ) is clearly $T_1(iv)$, but not $T_1(iii)$.

Counterexample 3.7. Let $X=\{a,b,c,d\}$ and consider the family $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F,G\}$, where $A=\langle X,\{a\},\emptyset\rangle$, $B=\langle X,\{b\},\{\emptyset\}\rangle$, $C=\langle X,\{c\},\emptyset\rangle$, $D=\langle X,\{a,b\},\emptyset\rangle$, $E=\langle X,\{b,c\},\emptyset\rangle$, $F=\langle X,\{a,b,c\},\emptyset\rangle$, $G=\langle X,\emptyset,\emptyset\rangle$. Then the ITS (X,τ) is $T_1(v)$, but not $T_1(vi)$.

Counterexample 3.8. Let $X=\{a,b,c\}$ and consider the family $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F,G,H,K\}$, where $A=\langle X,\{a\},\{c\}\rangle$, $B=\langle X,\{b\},\emptyset\rangle$, $C=\langle X,\{c\},\emptyset\rangle$, $D=\langle X,\{a,b\},\emptyset\rangle$, $E=\langle X,\{a,c\},\emptyset\rangle$, $F=\langle X,\{b,c\},\emptyset\rangle$, $G=\langle X,\emptyset,\{c\}\rangle$, $H=\langle X,\emptyset,\emptyset\rangle$, $K=\langle X,\{a\},\emptyset\rangle$. Then the ITS (X,τ) on X is $T_1(i)$, but not $T_1(iii)$.

Counterexample 3.9. Let $X=\{a,b,c\}$ and consider the family $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F,G\}$, where $A=\langle X,\{a,c\},\emptyset\rangle$, $B=\langle X,\{b,c\},\emptyset\rangle$, $C=\langle X,\{b\},\emptyset\rangle$, $D=\langle X,\{a,b\},\emptyset\rangle$, $E=\langle X,\{c\},\emptyset\rangle$, $F=\langle X,\{a\},\emptyset\rangle$, $G=\langle X,\emptyset,\emptyset\rangle$. Then the ITS (X,τ) on X is $T_1(iv)$, but not $T_1(ii)$.

Counterexample 3.10 (see [6]). Let $X = \mathbb{N}^+$ and consider the IS's A_n given below:

$$\begin{aligned} A_1 &= \langle X, \{2,3,4,\ldots\}, \varnothing\rangle, \\ A_2 &= \langle X, \{3,4,5,\ldots\}, \{1\}\rangle, \\ A_3 &= \langle X, \{4,5,6,\ldots\}, \{1,2\}\rangle, \\ A_n &= \langle X, \{n+1, n+2, n+3, \ldots\}, \{1,2,3,\ldots,n-1\}\rangle \quad (n \geq 2). \end{aligned} \tag{3.2}$$

Then $\tau = \{\underset{\sim}{\varnothing}, \underset{\sim}{X}\} \cup \{A_n : n = 1,2,3,\ldots\}$ is an IT on X. Clearly (X,τ) is $T_1(vi)$, but not $T_1(ii)$.

Proposition 3.11. *Let (X,τ) be an ITS. Then*
(a) *(X,τ) is $T_1(i)$ if and only if (X,τ_1) is T_1.*
(b) *(X,τ) is $T_1(ii)$ if and only if (X,τ_2) is T_1.*
(c) *(X,τ) is $T_1(i)$ if and only if $(X,\tau_{0,1})$ is $T_1(i)$.*
(d) *(X,τ) is $T_1(ii)$ if and only if $(X,\tau_{0,2})$ is $T_1(ii)$.*

Definition 3.12. Let (X,τ) be an ITS. (X,τ) is said to be
(a) $T_2(i) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$ (cf. [3, 13]);
(b) $T_2(ii) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\approx}{x} \in U$, $\underset{\approx}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$ (cf. [3, 13]);
(c) $T_2(iii) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \subseteq \bar{V}$ (cf. [3, 10]);
(d) $T_2(iv) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\approx}{x} \in U$, $\underset{\approx}{y} \in V$, and $U \subseteq \bar{V}$ (cf. [3, 10]);
(e) $T_2(v) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\sim}{x} \in U \subseteq \bar{\underset{\sim}{y}}$, $\underset{\sim}{y} \in V \subseteq \bar{\underset{\sim}{x}}$, and $U \cap V = \underset{\sim}{\varnothing}$ (cf. [3, 11]);
(f) $T_2(vi) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\approx}{x} \in U \subseteq \bar{\underset{\sim}{y}}$, $\underset{\sim}{y} \in V \subseteq \bar{\underset{\sim}{x}}$, and $U \cap V = \underset{\sim}{\varnothing}$ (cf. [3, 11]);
(g) $T_2(vii) \Leftrightarrow \Delta_X$ is an ICS in the product ITS $(X \times X, \tau_{X\times X})$.

Theorem 3.13. *Let (X,τ) be an ITS. Then the following implications are valid:*

$$\begin{array}{ccccc} & & T_2(v) & \longrightarrow & T_2(vi) \\ & & \downarrow & & \downarrow \\ T_2(vii) & \longleftarrow & T_2(i) & \longrightarrow & T_2(ii) \\ & & \downarrow & & \downarrow \\ & & T_2(iii) & \longrightarrow & T_2(iv) \end{array} \tag{3.3}$$

Proof. We prove only the case $T_2(i) \Rightarrow T_2(vii)$. We must see that $\bar{\Delta}_X$ is an IOS in $(X \times X, \tau_{X\times X})$. Let $(x,y)_\sim \in \bar{\Delta}_X$. This means that $(x,y) \in \{(x,y) : x \neq y\}$, that is, $x \neq y$. Since (X,τ) is $T_2(i)$, there exist $U,V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$. Now in this case we have $(x,y)_\sim \in U \times V \subseteq \bar{\Delta}_X$. Indeed, from $x \in U_1$ and $y \in V_1$ we get

$(x,y) \in U_1 \times V_1$, that is, $(x,y)_\sim \in U \times V$. We also know that $U \times V \subseteq \bar{\Delta}_X \Leftrightarrow U_1 \times V_1 \subseteq \{(x,y) : x \neq y\}$ and $(U_2^c \times V_2^c)^c \supseteq \{(x,y) : x = y\}$. If $(y_1,y_2) \in U_1 \times V_1$, then $y_1 \in U_1$, $y_2 \in V_1 \Rightarrow y_1 \neq y_2 \Rightarrow (y_1,y_2) \in \{(x,y) : x \neq y\}$ follows. Thus the first inclusion is true. For the second, $(y_1,y_2) \in U_2^c \times V_2^c \Rightarrow y_1 \in U_2^c$ and $y_2 \in V_2^c \Rightarrow y_1 \neq y_2$, that is, we have $U_2^c \times V_2^c \subseteq \{(x,y) : x \neq y\}$. Thus we see that $(y_1,y_2) \in \{(x,y) : x = y\}$. The second inclusion is true, too. Now since

$$\bar{\Delta}_X = \bigcup_{(y_1,y_2)_\sim \in \bar{\Delta}_X} (y_1,y_2)_\sim, \tag{3.4}$$

it follows from the fact that $\bar{\Delta}_X$ is not a proper IS, that $\bar{\Delta}_X$ is an IOS in $(X \times X)$, that is, (X,τ) is $T_2(vii)$. □

Counterexample 3.14. Let $X = \{a,b\}$ and consider the family $\tau = \{\underset{\sim}{\varnothing}, \underset{\sim}{X}, A, B\}$ on X, where $A = \langle X, \varnothing, \{b\}\rangle$, $B = \langle X, \varnothing, \{a\}\rangle$. Then the ITS (X,τ) on X is $T_2(ii)$, but not $T_2(i)$.

Counterexample 3.15. Let $X = \{a,b,c\}$ and define the IS's $A = \langle X, \varnothing, \{b,c\}\rangle$, $B = \langle X, \{b\}, \{a\}\rangle$, $C = \langle X, \{a\}, \{c\}\rangle$, and $D = \langle X, \varnothing, \{a,b\}\rangle$. Let τ denote the IT on X generated by the subbase $S = \{A,B,C,D\}$. Then (X,τ) is $T_2(iv)$, but not $T_2(iii)$

Counterexample 3.16. Let $X = \{a,b,c\}$ and consider the family $\tau = \{\underset{\sim}{\varnothing}, \underset{\sim}{X}, A, B, C, D, E, F, G, H, K, L, M\}$ on X, where $A = \langle X, \varnothing, \{b\}\rangle$, $B = \langle X, \varnothing, \{a,c\}\rangle$, $C = \langle X, \{a\}, \{b,c\}\rangle$, $D = \langle X, \varnothing, \{a\}\rangle$, $E = \langle X, \varnothing, \{a,b\}\rangle$, $F = \langle X, \varnothing, \{c\}\rangle$, $G = \langle X, \{a\}, \{c\}\rangle$, $H = \langle X, \{a\}, \varnothing\rangle$, $K = \langle X, \{a\}, \{b\}\rangle$, $L = \langle X, \varnothing, \{b,c\}\rangle$, and $M = \langle X, \varnothing, \varnothing\rangle$. Then the ITS (X,τ) on X is $T_2(vi)$, but not $T_2(v)$.

Counterexample 3.17. Let $X = \{a,b,c,d\}$ and define the IS's $A = \langle X, \{a\}, \{b\}\rangle$, $B = \langle X, \{b\}, \{a,d\}\rangle$, $C = \langle X, \{b\}, \{c\}\rangle$, $D = \langle X, \{c\}, \{a,b\}\rangle$, $E = \langle X, \{a\}, \{d\}\rangle$, $F = \langle X, \{d\}, \{a\}\rangle$, $G = \langle X, \{b\}, \{d\}\rangle$, $H = \langle X, \{d\}, \{b\}\rangle$, $K = \langle X, \{c\}, \{d\}\rangle$, $L = \langle X, \{d\}, \{c\}\rangle$, $M = \langle X, \{a\}, \{c\}\rangle$, and $N = \langle X, \{c\}, \{a\}\rangle$. Let τ denote the IT on X generated by the subbase $S = \{A,B,C,D,E,F,G,H,K,L,M,N\}$. Then (X,τ) is $T_2(iii)$, but not $T_2(i)$.

Counterexample 3.18. Let $X = \{a,b\}$ and consider the family $\tau = \{\underset{\sim}{\varnothing}, \underset{\sim}{X}, A, B\}$ on X, where $A = \langle X, \{b\}, \varnothing\rangle$, $B = \langle X, \varnothing, \{b\}\rangle$. Then the ITS (X,τ) on X is $T_2(iv)$, but not $T_2(ii)$.

Counterexample 3.19. We consider the IT on X as in Counterexample 3.15. (X,τ) is $T_2(iv)$, but not $T_2(i)$.

Counterexample 3.20. We consider the ITS on X as in Counterexample 3.14. (X,τ) is $T_2(ii)$, but not $T_2(v)$.

Proposition 3.21. *Let (X,τ) be an ITS. Then*

(a) (X,τ) is $T_2(i) \Rightarrow (X,\tau_1)$ is T_2.

(b) (X,τ) is $T_2(ii) \Rightarrow (X,\tau_2)$ is T_2.

Proposition 3.22. *Let (X,τ) be an ITS. Then*

(a) (X,τ) is $T_2(i) \Rightarrow (X,\tau_{0,1})$ is $T_2(i)$.

(b) (X,τ) is $T_2(ii) \Rightarrow (X,\tau_{0,2})$ is $T_2(ii)$.

Theorem 3.23. *Let (X,τ) be an ITS. Then the following implications are valid:*
(a) $T_2(i) \Rightarrow T_1(iii)$.
(b) $T_2(ii) \Rightarrow T_1(ii)$.
(c) $T_2(iii) \Rightarrow T_1(iii)$.
(d) $T_2(iv) \Rightarrow T_1(iv)$.
(e) $T_2(v) \Rightarrow T_1(iii)$.
(f) $T_2(vi) \Rightarrow T_1(vi)$.

Proof. The proof is obvious. □

Proposition 3.24. *Let (X,τ) be $T_2(i)$. Then every intuitionistic point $\underset{\sim}{x}$ is the intersection of all the intuitionistic closed neighborhoods of $\underset{\sim}{x}$.*

Proof. Let (X,τ) be $T_2(i)$ and $x \in X$. We denote the intersection of IC neighborhoods of $\underset{\sim}{x}$ by the IS $C = \langle X, C_1, C_2\rangle$. We assume the contrary and suppose that there exists a distinct IP $\underset{\sim}{y}$ in C, that is, $y \in C_1$.

Case 1. $\{x\} \subsetneq C_1$, then there exists $y \in C_1$ such that $x \neq y$. Since (X,τ) is $T_2(i)$, there exist IOS's U and V such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$ which implies that $U \subseteq \bar{V}$. Hence we have $\underset{\sim}{x} \in U \subseteq \bar{V}$. Thus $\bar{V}$ is a closed neighborhood of $\underset{\sim}{x}$. From our assumption, we get $\underset{\sim}{y} \in \bar{V}$. But it is a contradiction, since $V_1 \cap V_2 = \varnothing$. Thus our assumption is false. This means that C consists only of the IP $\underset{\sim}{x}$.

Case 2. $\{x\} \subsetneq C_2^c$ and $\{x\} = C_1$, then there exists $y \in C_2^c$ such that $y \neq x$. Since (X,τ) is $T_2(i)$, there exist IOS's $U,V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$ and the same result as in the previous assumption holds in this case, too. □

Proposition 3.25. *Let (X,τ) be an ITS, (Y,Φ) a $T_2(i)$ ITS and $f:(X,\tau)\to(Y,\Phi)$ a continuous function. Then the graph of f is an ICS in $X\times Y$.*

Proof. We must show that $\overline{\mathrm{GR}(f)}$ is an IOS in $X\times Y$. Let $(x,y)_\sim \in \overline{\mathrm{GR}(f)}$. Then $(x,y) \in \{(x,f(x)) : x \in X\}^c$ which implies that $y \neq f(x)$. Since (Y,Φ) is $T_2(i)$, there exist $U,V \in \Phi$ such that $\underset{\sim}{y} \in U$, $f(\underset{\sim}{x}) \in V$, and $U \cap V = \underset{\sim}{\varnothing}$. From the assumption that f is continuous, we see that $f^{-1}(V) = \langle X, f^{-1}(V_1), f^{-1}(V_2)\rangle$ is an open neighborhood of $\underset{\sim}{x}$. Also $f^{-1}(V)\times U$ is an open neighborhood of $(x,y)_\sim$. It can be shown easily that $f^{-1}(V)\times U \subseteq \overline{\mathrm{GR}(f)}$. Since $\overline{\mathrm{GR}(f)}$ is not a proper IS in $X\times Y$, our assumption holds, that is, $\overline{\mathrm{GR}(f)}$ is an IOS in $X\times Y$. □

Proposition 3.26. *Let (X,τ) be an ITS, (Y,Φ) a $T_2(i)$ ITS and $f:(X,\tau)\to(Y,\Phi)$ a continuous function. Then the IS $C = \langle (x_1,x_2), \{(x_1,x_2) : f(x_1) = f(x_2)\}, \{(x_1,x_2) : f(x_1) \neq f(x_2)\}\rangle$ in $X\times Y$ is an ICS in $X\times Y$.*

Proof. A similar argument as in the proof of Proposition 3.25 can be followed. □

Proposition 3.27. *Let (X,τ) and (Y,Φ) be two ITS's. Then*
(a) If (X,τ) and (Y,Φ) are $T_1(i)$, then so is $(X\times Y, \tau\times\Phi)$.
(b) If (X,τ) and (Y,Φ) are $T_1(ii)$, then so is $(X\times Y, \tau\times\Phi)$.

Proof. (a) Let (X,τ) and (Y,Φ) be $T_1(i)$. Let $(x_1,y_1),(x_2,y_2)\in X\times Y$ and $(x_1,y_1)\neq(x_2,y_2)$. Now suppose that $x_1\neq x_2$. Since (X,τ) is $T_1(i)$ then there exist $U,V\in\tau$ such that $\underset{\sim}{x_1}\in U$, $\underset{\sim}{x_2}\notin U$, and $\underset{\sim}{x_2}\in V$, $\underset{\sim}{x_1}\notin V$. Then we have IOS's $U\times\underset{\sim}{Y}=\langle(X,Y),U_1\times Y,(U_2^c\times\varnothing^c)^c\rangle$ and $V\times\underset{\sim}{Y}=\langle(X,Y),V_1\times Y,(V_2^c\times\varnothing^c)^c\rangle$ in $\tau\times\Phi$ having the properties $(x_1,y_1)_\sim\in U\times\underset{\sim}{Y}$, $(x_2,y_2)_\sim\notin U\times\underset{\sim}{Y}$, and $(x_2,y_2)_\sim\in V\times\underset{\sim}{Y}$, $(x_1,y_1)_\sim\notin V\times\underset{\sim}{Y}$. We can prove the case $y_1\neq y_2$ similarly. Thus we conclude that $(X\times Y,\tau\times\Phi)$ is $T_1(i)$.

(b) Similar to the previous one. □

Proposition 3.28. *Let (X,τ) and (y,Φ) be two ITS's. Then*

(a) If (X,τ) and (Y,Φ) are $T_2(i)$, then so is $(X\times Y,\tau\times\Phi)$.

(b) If (X,τ) and (Y,Φ) are $T_2(ii)$, then so is $(X\times Y,\tau\times\Phi)$.

(c) If (X,τ) and (Y,Φ) are $T_2(iii)$, then so is $(X\times Y,\tau\times\Phi)$.

(d) If (X,τ) and (Y,Φ) are $T_2(vii)$, then so is $(X\times Y,\tau\times\Phi)$.

Proof. (a) Let (X,τ), (Y,Φ) be $T_2(i)$. Let (x_1,y_1), $(x_2,y_2)\in X\times Y$, and $(x_1,y_1)\neq(x_2,y_2)$ and suppose that $x_1\neq x_2$. Since (X,τ) is $T_2(i)$ then there exist $U,V\in\tau$ such that $\underset{\sim}{x_1}\in U$, $\underset{\sim}{x_2}\in V$, and $U\cap V=\underset{\sim}{\varnothing}$. Then we can form the IOS's $U\times\underset{\sim}{Y}=\langle(X,Y),U_1\times Y,(U_2^c\times\varnothing^c)^c\rangle$ and $V\times\underset{\sim}{Y}=\langle(X,Y),V_1\times Y,(V_2^c\times\varnothing^c)^c\rangle$ in $\tau\times\Phi$ which contains $(x_1,y_1)_\sim$ and $(x_2,y_2)_\sim$, respectively. Now we must see that $(U\times\underset{\sim}{Y})\cap(V\times\underset{\sim}{Y})=\underset{\sim}{\varnothing}$. Indeed,

$$\begin{aligned}(U\times\underset{\sim}{Y})\cap(V\times\underset{\sim}{Y})&=\langle(X,Y),(U_1\times Y)\cap(V_1\times Y),(U_2^c\times\varnothing^c)^c\cup(V_2^c\times\varnothing^c)^c\rangle\\&=\langle(X,Y),(U_1\cap V_1)\times(Y\cap Y),[(U_2^c\times Y)\cap(V_2^c\times Y)]^c\rangle\\&=\Big\langle(X,Y),\varnothing\times Y,[(U_2^c)\cap(V_2^c)\times(Y\cap Y)]^c\Big\rangle\\&=\langle(X,Y),\varnothing,X\times Y\rangle=\underset{\sim}{\varnothing}.\end{aligned}\tag{3.5}$$

Thus $(X\times Y,\tau\times\Phi)$ is $T_2(i)$.

(b) Similar to previous one.

(c) Assume that (X,τ) and (Y,Φ) are $T_2(iii)$. Let (x_1,y_1), $(x_2,y_2)\in X\times Y$ and $(x_1,y_1)\neq(x_2,y_2)$. Suppose that $x_1\neq x_2$. Since (X,τ) is $T_2(iii)$, then there exist $U,V\in\tau$ such that $\underset{\sim}{x_1}\in U$, $\underset{\sim}{x_2}\in V$, and $U\subseteq\bar{V}$. Then we have IOS's $U\times\underset{\sim}{Y}=\langle(X,Y),U_1\times Y,(U_2^c\times\varnothing^c)^c\rangle$ and $V\times\underset{\sim}{Y}=\langle(X,Y),V_1\times Y,(V_2^c\times\varnothing^c)^c\rangle$ in $\tau\times\Phi$ containing $(x_1,y_1)_\sim$ and $(x_2,y_2)_\sim$, respectively. Now, it is easy to see that $U\times\underset{\sim}{Y}\subseteq\overline{V\times\underset{\sim}{Y}}$ holds, which is identical to $U_1\times Y\subseteq(V_2^c\times Y)^c$ and $V_1\times Y\subseteq(U_2^c\times Y)^c$. A similar argument holds if $y_1\neq y_2$. Thus we conclude that $(X\times Y,\tau\times\Phi)$ is $T_2(iii)$.

(d) We are to show that $\Delta_{X\times Y}$ is an ICS, that is, $\bar{\Delta}_{X\times Y}$ is an IOS. Since $\bar{\Delta}_{X\times Y}$ is not a proper IS in $X\times Y$, it is sufficient to show that for every $((p_1,q_1),(p_2,q_2))_\sim\in\bar{\Delta}_{X\times Y}$, there exists an IOS S in $(X\times Y)\times(X\times Y)$ such that $((p_1,q_1),(p_2,q_2))_\sim\in S\subseteq\bar{\Delta}_{X\times Y}$. Since $((p_1,q_1),(p_2,q_2))_\sim\in\bar{\Delta}_{X\times Y}$, we get $((p_1,q_1)\neq(p_2,q_2))_\sim$, that is, $p_1\neq p_2$ or $q_1\neq q_2$. Here come three possible cases:

(1) $p_1\neq p_2$, $q_1=q_2$;

(2) $p_1=p_2$, $q_1\neq q_2$;

(3) $p_1\neq p_2$, $q_1\neq q_2$.

Here we show only case (3). Other cases can be proved similarly. Let $p_1\neq p_2$, $q_1\neq q_2$. Since $(p_1,p_2)_\sim\in\bar{\Delta}_X$, $(q_1,q_2)_\sim\in\bar{\Delta}_Y$ and $\bar{\Delta}_X$, $\bar{\Delta}_Y$ are IOS's, $\exists U_1,U_2\in\tau$ and V_1,

$V_2 \in \Phi$ such that $(p_1,p_2)_\sim \in U_1 \times U_2 \subseteq \bar{\Delta}_X$ and $(q_1,q_2)_\sim \in V_1 \times V_2 \subseteq \bar{\Delta}_Y$. We prove that $((p_1,q_1),(p_2,q_2))_\sim \in (U_1 \times V_1) \times (U_2 \times V_2) \subseteq \bar{\Delta}_{X\times Y}$. This can be shown in two steps.

Step 1. The expression $((p_1,q_1),(p_2,q_2))_\sim \in (U_1 \times V_1) \times (U_2 \times V_2)$ is equivalent to $((p_1,q_1),(p_2,q_2)) \in (U_1 \times V_1)^{(1)} \times (U_2 \times V_2)^{(1)} \Leftrightarrow ((p_1,q_1),(p_2,q_2)) \in (U_1^{(1)} \times V_1^{(1)}) \times (U_2^{(1)} \times V_2^{(1)})$. This means that $(p_1,q_1) \in U_1^{(1)} \times V_1^{(1)}$ and $(p_2,q_2) \in U_2^{(1)} \times V_2^{(1)}$ which are true, since $p_1 \in U_1^{(1)}$, $p_2 \in U_2^{(1)}$, $q_1 \in V_1^{(1)}$, $q_2 \in V_2^{(1)}$.

Step 2. We show the inclusion $(U_1 \times V_1) \times (U_2 \times V_2) \subseteq \bar{\Delta}_{X\times Y}$. For this purpose we must first show that $(U_1 \times V_1)^{(1)} \times (U_2 \times V_2)^{(1)} \subseteq \{((u_1,v_1),(u_2,v_2)) : (u_1,v_1) \neq (u_2,v_2)\}$ or equivalently, $(U_1^{(1)} \times V_1^{(1)}) \times (U_2^{(1)} \times V_2^{(1)}) \subseteq \{((u_1,v_1),(u_2,v_2)) : (u_1,v_1) \neq (u_2,v_2)\}$. This is true since $U_1 \times U_2 \subseteq \bar{\Delta}_X$ and $V_1 \times V_2 \subseteq \bar{\Delta}_Y$, we have $U_1^{(1)} \times U_2^{(1)} \subseteq \{(u_1,u_2) : u_1 \neq u_2\}$ and $V_1^{(1)} \times V_2^{(1)} \subseteq \{(v_1,v_2) : v_1 \neq v_2\}$, respectively. Thus the first inclusion is true. The second inclusion can be proved similarly. Hence $\bar{\Delta}_{X\times Y}$ is an IOS, that is, $\bar{\Delta}_{X\times Y}$ is an ICS, which means that $(X,Y,\tau \times \Phi)$ is $T_2(vii)$. □

Remark 3.29. Let (X,τ) and (Y,Φ) be $T_2(iv)$. Then $(X\times Y,\tau\times\Phi)$ may not be $T_2(iv)$.

Here come the reverse implications.

Proposition 3.30. *Let (X,τ) and (Y,Φ) be two ITS's. Then*
(a) If $(X \times Y,\tau\times\Phi)$ is $T_2(i)$, then so are (X,τ) and (Y,Φ).
(b) If $(X \times Y,\tau\times\Phi)$ is $T_2(ii)$, then so are (X,τ) and (Y,Φ).
(c) If $(X \times Y,\tau\times\Phi)$ is $T_2(iii)$, then so are (X,τ) and (Y,Φ).

Proof. The proofs of (a) and (b) are easy. (c) Let $(X \times Y, \tau \times \Phi)$ be $T_2(iii)$, and $x_1 \neq x_2$ $(x_1,x_2 \in X)$. We take a fixed $y \in Y$. Then, since $(x_1,y) \neq (x_2,y)$ and $X \times Y$ is $T_2(iii)$, there exist $U \times Z$ and $V \times T$ where $U,V \in \tau$ and $Z,T \in \Phi$ such that $(x_1,y)_\sim \in U \times Z$, $(x_2,y)_\sim \in V \times T$, and $U \times Z \subseteq \overline{V \times T}$. Thus we get $(x_1,y) \in U_1 \times Z_1$, $(x_2,y) \in V_1 \times T_1$, and $U_1 \times Z_1 \subseteq (V_2^c \times T_2^c)^c$, $V_1 \times T_1 \subseteq (U_2^c \times Z_2^c)^c$; in other words $x_1 \in U_1$, $y \in Z_1$, $x_2 \in V_1$, $y \in T_1$, and $(U_1 \times Z_1) \cap (V_2^c \times T_2^c) = \varnothing, (V_1 \times T_1) \cap (U_2^c \times Z_2^c) = \varnothing$. From the last intersection we get $(U_1^c \times V_2^c) \times (Z_1 \cap T_2^c) = \varnothing$ and $(V_1 \cap U_2^c) \times (T_1 \cap Z_2^c) = \varnothing$, respectively. $y \in Z_1$ and $y \in T_1$ implies that $Z_1 \cap T_2^c \neq \varnothing$ and $U_1 \cap V_2^c = \varnothing$ from which $U_1 \subseteq V_2$ follows. Similarly $y \in T_1 \cap Z_2^c$ and $V_1 \cap U_2^c = \varnothing$ meaning that $V_1 \subseteq U_2$. Thus $\underset{\sim}{x_1} \in U$, $\underset{\sim}{x_2} \in V$, and $U \subseteq \bar{V}$, that is, (X,τ) is $T_2(iii)$. Similarly (Y,Φ) is $T_2(iii)$, too. □

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