IJMMS 28:4 (2001) 243–246
PII. S0161171201011589
http://ijmms.hindawi.com

ON THE ZEROS AND CRITICAL POINTS OF A RATIONAL MAP

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(Received 23 January 2001)

ABSTRACT. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree d. It is well known that f has d zeros and $2d-2$ critical points counted with multiplicities. In this note, we explain how those zeros and those critical points are related.

2000 Mathematics Subject Classification. 30C15.

In this note, $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map. We denote by $\{\alpha_i\}_{i\in I}$ the set of zeros of f, and by $\{\omega_j\}_{j\in J}$ the set of critical points of f which are not zeros of f (the sets I and J are finite). Moreover, we denote by n_i the multiplicity of α_i as a zero of f and by m_j the multiplicity of ω_j as a critical point of f. The local degree of f at α_i is n_i and the local degree of f at ω_j is $d_j = m_j + 1$. In particular, when $\omega_j \neq \infty$ and $f(\omega_j) \neq \infty$, the point ω_j is a zero of f' of order m_j.

Our goal is to understand the relations that exist between the points α_i and the points ω_j.

PROPOSITION 1. *Given a finite collection of distinct points $\alpha_i \in \mathbb{P}^1$ with multiplicities n_i and $\omega_j \in \mathbb{P}^1$ with multiplicities m_j, there exists a rational map f vanishing exactly at the points α_i with multiplicities n_i and having extra critical points exactly at the points ω_j with multiplicities m_j if and only if*

(i) $\sum(n_i+1) - \sum m_j = 2$, *and*

(ii) *for any k such that $\alpha_k \in \mathbb{C}$,*

$$\operatorname{res}\left(\frac{\prod_{\omega_j\in\mathbb{C}}(z-\omega_j)^{m_j}}{\prod_{\alpha_i\in\mathbb{C}}(z-\alpha_i)^{n_i+1}}dz, \alpha_k\right) = 0. \tag{1}$$

We will give a geometric interpretation of (ii) in the case where α_k is a simple zero of f: working in a coordinate where $\alpha_k = \infty$, the barycentre of the remaining zeros weighted with their multiplicities is equal to the barycentre of the critical points of f weighted with their multiplicities (see Proposition 3 below).

PROOF. The proof is elementary. It is based on the observation that the 1-forms $d(1/f)$ and

$$\phi = \frac{\prod_{\omega_j\in\mathbb{C}}(z-\omega_j)^{m_j}}{\prod_{\alpha_i\in\mathbb{C}}(z-\alpha_i)^{n_i+1}}dz \tag{2}$$

are proportional. The differential equation $d(1/f) = \phi$ has a rational solution if and only if ϕ is exact, if and only if the residues of ϕ at all finite poles are equal to zero.

Lemma 2. *Let f be a rational map. Denote by α_i its zeros and by n_i their multiplicities. Denote by ω_j the critical points of f which are not multiple zeros of f and by m_j their multiplicities. The zeros of the 1-form $d(1/f)$ are exactly the points ω_j with order m_j and its poles are exactly the points α_i with order n_i+1.*

Proof. A singularity of the 1-form $d(1/f) = -df/f^2$ is necessarily a zero or a pole of f, a zero of f', or ∞ (where ϕ is defined by analytic continuation). Considering the Laurent series of f at each of those points, one immediately gets the result. □

Now assume that there exists a rational map f with the required properties. Lemma 2 shows that the 1-forms ϕ and $d(1/f)$ have the same poles and the same zeros in $\mathbb{C}$, with the same multiplicities. Hence, their ratio is a rational function which does not vanish in $\mathbb{C}$. Thus, ϕ and $d(1/f)$ are proportional. In particular, ϕ has a singularity at ∞ if and only if $d(1/f)$ has a singularity at ∞ and the singularity is of the same kind for both 1-forms. Since the number of poles minus the number of zeros of any nonzero 1-form on $\mathbb{P}^1$ is equal to 2 (the Euler characteristic of $\mathbb{P}^1$), we see that $\sum(n_i+1) - \sum m_j = 2$ which is precisely condition (i) in Proposition 1. Besides, since ϕ is exact, it follows that the residues at all the poles α_k vanish and condition (ii) is satisfied.

Conversely, the 1-form ϕ has poles of order n_i+1 at the points $\alpha_i \in \mathbb{C}$ and zeros of order m_j at the points $\omega_j \in \mathbb{C}$. Condition (ii) implies that ϕ is exact, that is, there exists a rational map $g : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi = dg$. Since the number of poles of ϕ in $\mathbb{P}^1$ minus the number of zeros of ϕ in $\mathbb{P}^1$ is equal to 2, condition (i) implies that when ∞ is neither a point α_i nor a point ω_j, it is a regular point of ϕ, when $\infty = \alpha_{i_0}$, it is a pole of ϕ of order n_{i_0}, and when $\infty = \omega_{j_0}$, it is a zero of ϕ of order m_{j_0}. Finally, $\phi = d(1/f)$, with $f = 1/g$, and Lemma 2 shows that the rational map $f = 1/g$ vanishes exactly at the points α_i with multiplicities n_i and has extra critical points exactly at the points ω_j with multiplicities m_j. □

We will now give a geometric interpretation of (ii) when α_k is a simple zero of f. We first work in a coordinate where ∞ is neither one of the points α_i nor a point ω_j. Define

$$R(z) = \frac{\prod_j (z-\omega_j)^{m_j}}{\prod_{i\neq k}(z-\alpha_i)^{n_i+1}}. \tag{3}$$

Then,

$$\operatorname{res}\left(\frac{\prod_j (z-\omega_j)^{m_j}}{\prod_i (z-\alpha_i)^{n_i+1}}dz, \alpha_k\right) = \operatorname{res}\left(\frac{R(z)}{(z-\alpha_k)^2}dz, \alpha_k\right) = R'(\alpha_k). \tag{4}$$

Since $R(\alpha_k) \neq 0$, this residue vanishes if and only if

$$\frac{R'(\alpha_k)}{R(\alpha_k)} = \sum_j \frac{m_j}{\alpha_k - \omega_j} - \sum_{i\neq k} \frac{n_i+1}{\alpha_k - \alpha_i} = 0. \tag{5}$$

Let d be the number of zeros counted with multiplicities, that is, $d = \sum_i n_i$. The total number of critical points is $2d-2 = \sum_j m_j + \sum_i (n_i - 1)$ (the critical points of f are the points ω_j and the multiple zeros of f). Then, (5) can be rewritten as

$$\frac{1}{2d-2}\left(\sum_j \frac{m_j}{\alpha_k-\omega_j}+\sum_{i\neq k}\frac{n_i-1}{\alpha_k-\alpha_i}\right)=\frac{1}{d-1}\sum_{i\neq k}\frac{n_i}{\alpha_k-\alpha_i}. \tag{6}$$

This last equality can be interpreted in the following way.

Proposition 3. *Assume that f is a rational map having a simple zero at ∞. Then, the barycentre of the remaining zeros weighted with their multiplicities is equal to the barycentre of the critical points of f weighted with their multiplicities.*

Remark 4. One can prove this proposition directly. We may write $f = P/Q$, where

$$P=\sum_{k=0}^{d-1} a_k z^k, \qquad Q=\sum_{k=0}^{d} b_k z^k, \tag{7}$$

are co-prime polynomials with $\deg(Q) = \deg(P) + 1 = d$. Without loss of generality, we may assume that the barycentre of the zeros of f is equal to 0. In other words, we may assume that P is a centered polynomial, that is, $a_{d-2} = 0$. A simple calculation shows that

$$P'Q-Q'P=\sum_{k=0}^{2d-2} c_k z^k \tag{8}$$

is a polynomial of degree $2d-2$ and that $c_{2k-1} = 0$. Therefore, the barycentre of the zeros of $P'Q-Q'P$, that is, the barycentre of the critical points of f, is equal to 0.

Apply this geometric interpretation in order to re-prove two known results. The first corollary is related to the Sendov conjecture (cf. [1] and more particularly Section 4). This conjecture asserts that if a polynomial P has all its roots in the closed unit disk, then, for each zero α_i there exists a critical point ω (possibly a multiple zero) such that $|\alpha_i-\omega| \le 1$.

Corollary 5. *Let $P:\mathbb{C}\to\mathbb{C}$ be a polynomial. Assume the zeros of P are all contained in the closed unit disk and $\alpha_0 \in S^1$ is a zero of P. Then, the closed disk of diameter $[0,\alpha_0]$ contains at least one critical point of f.*

Proof. Denote by d the degree of P. If α_0 is a multiple zero of P, then the result is trivial. Thus, assume α_0 is a simple zero of P. We work in the coordinate $Z = \alpha_0/(\alpha_0-z)$. The rational map $f: Z \mapsto P(\alpha_0-\alpha_0/Z)$ has a simple zero at $Z=\infty$ and the remaining zeros are contained in the half-plane $\{Z\in\mathbb{P}^1 \mid \Re(Z)\ge 1/2\}$. Thus the barycentre β of those zeros satisfies $\Re(\beta)\ge 1/2$. Moreover, f has a critical point of multiplicity d at $Z=0$. Thus, the barycentre of the d remaining critical points is 2β. Since $\Re(2\beta)\ge 1$, we see that f has at least one critical point ω contained in the half plane $\{Z\in\mathbb{P}^1 \mid \Re(Z)\ge 1\}$. Then, $\alpha_0-\alpha_0/\omega$ is a critical point of P contained in the closed disk of diameter $[0,\alpha_0]$. □

The second corollary has been proved by Videnskii [2]. Our result provides an alternate proof.

COROLLARY 6. *Assume that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map and $\Delta \subset \mathbb{P}^1$ is a closed disk or a closed half-plane containing all the zeros of f. Then, Δ contains at least one critical point of f.*

PROOF. Without loss of generality, we may assume that the zeros are simple and that at least one zero, say α_0, is on the boundary of Δ. In a coordinate where $\alpha_0 = \infty$, Δ is a closed half-plane. The barycentre of the remaining zeros is contained in this half-plane. Consequently, the barycentre of the critical points is contained in Δ. Thus, Δ contains at least one critical point. □

Videnskii also proved that this result is optimal in the sense that there exist rational maps of arbitrary degrees with simple zeros contained in a disk Δ but only one critical point in Δ.

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