ON CHARACTERIZATIONS OF FIXED POINTS

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ABSTRACT. We give some necessary and sufficient conditions for the existence of fixed points of a family of self mappings of a metric space and we establish an equivalent condition for the existence of fixed points of a continuous compact mapping of a metric space.

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1. Introduction. Jungck [2] first gave a necessary and sufficient condition for the existence of fixed points of a continuous self mapping of a complete metric space. Afterwards, Park [8], Leader [4], and Khan and Fisher [3] established a few theorems similar to that of Jungck. Janos [1] and Park [9] proved fixed point theorems for compact self mappings of a metric space. Recently, Liu [5] established criteria for the existence of fixed points of a family of self mappings of a metric space. The aim of this paper is to offer some characterizations for the existence of fixed points of a family of self mappings and a continuous compact mappings of metric spaces, respectively. We also establish a fixed point theorem for two compact mappings, which extends properly the results of Janos [1] and Park [9].

Let ω and N denote the sets of nonnegative and positive integers, respectively. Suppose that (X, d) is a metric space. For $x, y \in X$, define

$$\begin{aligned} C_f &= \{g \mid g : X \rightarrow X \text{ and } fg = gf\}, \\ H_f &= \{g \mid g : X \rightarrow X \text{ and } g \cap_{n \in \omega} f^n X \subseteq \cap_{n \in \omega} f^n X\}, \\ H_f(x) &= \{hx \mid h \in H_f\}, \quad H_f(x, y) = H_f(x) \cup H_f(y), \\ O(x, f) &= \{f^n x \mid n \in \omega\}, \quad O(x, y, f) = O(x, f) \cup O(y, f). \end{aligned} \tag{1.1}$$

Obviously, $C_f \subseteq H_f$. Let Φ be a family of self mappings of X . A point $x \in X$ is said to be a *fixed point* of Φ if $fx = x$ for all $f \in \Phi$. Let $F : X \times X \rightarrow [0, +\infty)$ be continuous and $F(x, y) = 0$ if and only if $x = y$. For $A, B \subset X$, define

$$\delta(A, B) = \sup \{F(x, y) \mid x \in A, y \in B\} \tag{1.2}$$

and $\delta(A) = \delta(A, A)$. Particularly, $d(A) = \sup \{d(x, y) \mid x, y \in A\}$. Let $M(X)$ denote the set of all metrics on X that are topologically equivalent to d for a given metric space (X, d) . A self mapping f of a metric space (X, d) is said to be *compact* if there exists a compact set Y satisfying $fX \subseteq Y \subseteq X$.

In order to prove our main results, we need the following lemmas.

LEMMA 1.1 (see [6]). *Let f be a continuous compact self mapping of a metric space (X, d) . If $A = \bigcap_{n \in \omega} f^n X$, then*

- (a1) A is compact,
- (a2) $fA = A \neq \emptyset$,
- (a3) $d(f^n X) \rightarrow d(A)$ as $n \rightarrow \infty$,
- (a4) $\{f^n \mid n \in \omega\} \subseteq H_f$.

LEMMA 1.2 (see [7]). *Let f be a continuous self mappings of a metric space (X, d) with the following properties:*

- (a5) f has a unique fixed point w in X ,
- (a6) for every $x \in X$, the sequence of iterations $\{f^n x\}_{n=0}^\infty$ converges to w ,
- (a7) there exists an open neighborhood U of w with the property that given any open set V containing w , there exists $k \in \mathbb{N}$ such that $n \geq k$ implies $f^n U \subset V$.

Then for each $\alpha \in (0, 1)$, there exists a metric $d' \in M(X)$ relative to which f is a contraction with Lipschitz constant α .

2. Main results

THEOREM 2.1. *Let Φ be a family of self mappings of a metric space (X, d) . Then the following statements are equivalent:*

- (b1) Φ has a fixed point;
- (b2) there exist $m, n \in \mathbb{N}$ and continuous compact self mappings f, g of (X, d) such that either $\Phi \subseteq C_f$ or $\Phi \subseteq C_g$ and

$$F(f^m x, g^n y) < \delta(H_f(x), H_g(y)), \quad (2.1)$$

for all $x, y \in X$ with $f^m x \neq g^n y$;

- (b3) there exist $m, n \in \mathbb{N}$ and continuous self mappings f, g of (X, d) such that fg is compact, $f \in C_g$, $\Phi \subseteq C_{fg}$, and

$$F(f^m x, g^n y) < \delta(H_{fg}(x, y)), \quad (2.2)$$

for all $x, y \in X$ with $f^m x \neq g^n y$,

- (b4) there exists a continuous compact self mapping of (X, d) with $\Phi \subseteq C_f$ such that

$$F(fx, fy) < \max \left\{ F(x, y), F(x, fx), F(y, fy), \frac{F(x, fx)F(y, fy)}{F(x, y)}, \right. \\ \left. \frac{F(fx, fy)F(x, fx)}{F(x, y)}, \frac{F(x, fy)F(fx, y)}{F(x, y)} \right\}, \quad (2.3)$$

for all $x, y \in X$ with $x \neq y$.

Moreover, if (b2) holds, then f, g , and Φ have a unique common fixed point; if (b3) holds, then fg and Φ have a unique common fixed point; if (b4) holds, then f and Φ have a unique common fixed point.

PROOF. Let (b1) hold and w be a fixed point of Φ . Define $f, g : X \rightarrow X$ by $fx = gx = w$ for all $x \in X$. It is easy to show that (b2), (b3), and (b4) hold.

Assume that (b2) holds. Let $A = \cap_{n \in \omega} f^n X$, $B = \cap_{n \in \omega} g^n X$. Since f and g are continuous compact self mappings of (X, d) , it follows from (a1) and (a2) that A and B are compact and $fA = A$, $gB = B$. Consequently, $f^m A = A$, $g^n B = B$. Suppose that $\delta(A, B) > 0$. Then there exist $a \in A$, $b \in B$ with $\delta(A, B) = F(a, b)$ because F is continuous and $A \times B$ is compact. Since $f^m A = A$, $g^n B = B$, there exist $x \in A$, $y \in B$ such that $f^m x = a$, $g^n y = b$. In view of (2.1), we have

$$\delta(A, B) = F(a, b) = F(f^m x, g^n y) < \delta(H_f(x), H_g(y)) < \delta(A, B), \quad (2.4)$$

which is impossible, and hence $\delta(A, B) = 0$. That is, $A = B = \{w\}$ for some $w \in X$ so $fw = gw = w$. If v is another fixed point of f , then $v \in \cap_{n \in \omega} f^n X = \{w\}$, that is, $v = w$. Hence w is the only fixed point of f . Similarly, w is also the only fixed point of g .

Without loss of generality, we assume that $\Phi \subseteq C_f$. It follows from $C_f \subseteq H_f$ that $hA \subseteq A$ for all $h \in \Phi$. That is, $hw = w$ for all $h \in \Phi$. Thus w is the only common fixed point of f , g , and Φ . Therefore (b1) holds.

Assume that (b3) holds. Put $A = \cap_{n \in \omega} (fg)^n X$. Then A is compact and $fgA = A$. Since f is continuous and $f \in C_g$, we infer that

$$fA = f \cap_{n \in \omega} (fg)^n X \subseteq \cap_{n \in \omega} (fg)^n fX \subseteq \cap_{n \in \omega} (fg)^n X = A. \quad (2.5)$$

Similarly, we have

$$gA \subseteq A. \quad (2.6)$$

It follows from $fgA = A$, (2.5), and (2.6) that

$$fA \subseteq A = fgA \subseteq fA. \quad (2.7)$$

That is, $fA = A$. Similarly, we have $gA = A$. Suppose that $\delta(A) > 0$. Because F is continuous and A is compact, then there exist $a, b \in A$ such that $\delta(A) = F(a, b)$. Since $f^m A = g^n A = A$, there exist $x, y \in A$ with $f^m x = a$, $g^n y = b$. Using (2.2), we have

$$\delta(A) = F(a, b) = (f^m x, g^n y) < \delta(H_{FG}(x, y)) \leq \delta(A), \quad (2.8)$$

which is a contradiction. Hence $\delta(A) = 0$. That is, $A = \{w\}$ for some $w \in X$. This implies that $fw = gw = fgw = w$. As in the proof of above, we can prove that w is the only fixed point of fg , and w is the unique common fixed point of fg and Φ . So (b1) holds.

Assume that (b4) holds. As above we infer that $A = \cap_{n \in \omega} f^n X$ is compact and $fA = A$. Since F is continuous, the function $\phi(x)$ defined by $\phi(x) = F(x, fx)$ for $x \in A$ is continuous and attains its minimum value at some $w \in A$. Suppose that $w \neq fw$. By

virtue of (2.3), we get

$$\begin{aligned}
 \phi(fw) &= F(fw, ffw) \\
 &< \max \left\{ F(w, fw), F(w, fw), F(fw, ffw), \frac{F(w, fw)F(fw, ffw)}{F(w, fw)}, \right. \\
 &\quad \left. \frac{F(fw, ffw)F(w, fw)}{F(w, fw)}, \frac{F(w, fw)F(fw, fw)}{F(w, fw)} \right\} \\
 &= F(w, fw) \\
 &= \phi(w).
 \end{aligned} \tag{2.9}$$

This is a contradiction to the definition of w . So w is a fixed point of f . If f has a second distinct fixed point v , by (2.3), we obtain that

$$\begin{aligned}
 F(w, v) &= F(fw, fv) \\
 &< \max \left\{ F(w, v), F(w, w), F(v, v), \frac{F(w, w)F(v, v)}{F(w, v)}, \right. \\
 &\quad \left. \frac{F(w, v)F(w, w)}{F(w, v)}, \frac{F(w, v)F(w, v)}{F(w, v)} \right\} \\
 &= F(w, v),
 \end{aligned} \tag{2.10}$$

which is a contradiction. Therefore, w is the only fixed point of f . It is a simple matter to show that w is the unique common fixed point of f and Φ . Thus (b1) holds. This completes the proof. $\square$

Next, we give a theorem about the equivalent condition for the existence of fixed points of a continuous compact self mapping on a metric space.

THEOREM 2.2. *Let s be a continuous compact self mapping of a metric space (X, d) . Then s has a fixed point if and only if there exists a continuous self mapping f of X such that $f \in C_s$ and*

$$\begin{aligned}
 F(fx, fy) &< \max \left\{ F(sx, sy), F(sx, fx), F(sy, fy), \frac{F(sx, fx)F(sy, fy)}{F(sx, sy)}, \right. \\
 &\quad \left. \frac{F(fx, fy)F(sx, fx)}{F(sx, sy)}, \frac{F(sx, fy)F(fx, sy)}{F(sx, sy)} \right\},
 \end{aligned} \tag{2.11}$$

for all $x, y \in X$ with $sx \neq sy$. Indeed, f and s have a unique common fixed point.

PROOF. To see that the stated conditions is necessary, suppose that s has a fixed point $w \in X$. Define $f : X \rightarrow X$ by $fx = w$ for all $X \in X$. Then $fsx = w = sw = sfx$ for all $x \in X$, that is, $f \in C_s$. Clearly, (2.11) holds.

On the other hand, suppose that there exists a continuous self mapping f of X such that $f \in C_s$ and (2.11) holds. Let $A = \cap_{n \in \omega} S^n X$. From Lemma 1.1, we infer that A is compact and $sA = A$. Since f is continuous and $s \in C_f$, we have

$$fA = f \cap_{n \in \omega} S^n X \subseteq \cap_{n \in \omega} S^n fX \subseteq \cap_{n \in \omega} S^n X = A = sA. \tag{2.12}$$

Define the function $\phi(x)$ by $\phi(x) = F(sx, fx)$ for all $x \in A$. It is clear that $\phi(x)$ is continuous on A and attains its minimum value at some $w \in A$. We claim that $sw = fw$. If not, from (2.12), there exists $p \in A$ satisfying $fw = sp$. Using (2.11), we conclude that

$$\begin{aligned}
 \phi(p) &= F(sp, fp) = F(fw, fp) \\
 &< \max \left\{ F(sw, sp), F(sw, fw), F(sp, fp), \frac{F(sw, fw)F(sp, fp)}{F(sw, sp)}, \right. \\
 &\quad \left. \frac{F(fw, fp)F(sw, fw)}{F(sw, sp)}, \frac{F(sw, fp)F(fw, sp)}{F(sw, sp)} \right\} \\
 &= \max \{F(sw, fw), F(sp, fp)\} \\
 &= F(sw, fw) \\
 &= \phi(w),
 \end{aligned} \tag{2.13}$$

which is a contradiction to the choice of w . So $sw = fw$. By virtue of $f \in C_s$, we have

$$fsw = sfw = ss w. \tag{2.14}$$

Now suppose that $ssw \neq sw$. By (2.11) and (2.14), we get

$$\begin{aligned}
 F(ssw, sw) &= F(fsw, fw) \\
 &< \max \left\{ F(ssw, sw), F(ssw, fsw), F(sw, fw), \frac{F(ssw, fsw)F(sw, fw)}{F(ssw, sw)}, \right. \\
 &\quad \left. \frac{F(fsw, fw)F(ssw, fsw)}{F(ssw, sw)}, \frac{F(ssw, fw)F(fsw, sw)}{F(ssw, sw)} \right\} \\
 &= F(ssw, sw),
 \end{aligned} \tag{2.15}$$

which is a contradiction. Thus $ssw = sw$, that is, sw is a fixed point of s . Therefore, the set M of fixed points of s is not empty. Now s is continuous, so M is closed. Since $M \subseteq A$ and A is compact, M is compact. Moreover, since f and s commute, $f(M) \subseteq M$. Note also that (2.11) restricted to M reduces to (2.3). We can therefore apply Theorem 2.1(b4) to $f_M : M \rightarrow M$ to obtain a unique common fixed point u of f and s in M . Since M contains all the fixed points of s , u is a unique common fixed point f and s . This completes the proof. $\square$

THEOREM 2.3. *Let f, g be continuous compact self mappings of a metric space (X, d) satisfying (2.1). Then f and g have a unique fixed point, respectively, and furthermore, for any $\alpha \in (0, 1)$, there exist metrics d' and $d'' \in M(X)$ relative to which f and g satisfy, respectively,*

$$d'(fx, fy) \leq \alpha d'(x, y), \quad d''(gx, gy) \leq \alpha d''(x, y), \tag{2.16}$$

for all $x, y \in X$.

PROOF. Let $A = \bigcap_{n \in \mathbb{N}} f^n X$, $B = \bigcap_{n \in \mathbb{N}} g^n X$, and $U = X$. As in the proof of Theorem 2.1, we have $A = B = \{w\}$. Lemma 1.1 ensures that (a5) and (a6) hold. Note that

$d(f^n X), d(g^n X) \rightarrow 0$ as $n \rightarrow \infty$. Thus $f^n X$ and $g^n X$ squeeze into any neighborhood of w . That is (a7) is fulfilled. Thus [Theorem 2.3](#) follows from [Lemma 1.2](#). This completes the proof. $\square$

COROLLARY 2.4. *Let f be a continuous compact self mapping of a metric space (X, d) satisfying*

$$F(fx, fy) < \delta(H_f(x), H_f(y)), \quad (2.17)$$

for all $x, y \in X$ with $x \neq y$. Then f has a unique fixed point.

Furthermore, for any $\alpha \in (0, 1)$, there exists a metric $d' \in M(X)$ relative to which f satisfies

$$d'(fx, fy) \leq \alpha d'(x, y), \quad (2.18)$$

for all $x, y \in X$.

The following simple example reveals that [Corollary 2.4](#) extends properly [Theorem 1.1](#) of Janos [1] and [Theorem 1](#) of Park [9].

EXAMPLE 2.5. Let $X = \{0, 2, 4, 6, 9\}$ with the usual metric. Define a mapping $f : X \rightarrow X$ by $f0 = f4 = f6 = 6$, $f2 = 0$, and $f9 = 2$. Then f is a continuous compact self mapping of X . It is easy to check that

$$d(fx, fy) \leq 6 < 9 = \delta(H_f(x), H_f(y)), \quad (2.19)$$

for all $x, y \in X$ with $x \neq y$. So the conditions of [Corollary 2.4](#) are satisfied. But [Theorem 1.1](#) of Janos [1] and [Theorem 1](#) of Park [9] are not applicable since

$$\begin{aligned} d(f2, f4) &= 6 > 2 = \frac{1}{2}[d(2, f2) + d(4, f4)], \\ d(f2, f4) &= 6 = \delta(O(2, 4, f)). \end{aligned} \quad (2.20)$$

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