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EXCHANGE RINGS HAVING STABLE RANGE ONE

HUANYIN CHEN

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ABSTRACT. We investigate the sufficient conditions and the necessary conditions on an exchange ring R under which R has stable range one. These give nontrivial generalizations of Theorem 3 of V. P. Camillo and H.-P. Yu (1995), Theorem 4.19 of K. R. Goodearl (1979, 1991), Theorem 2 of R. E. Hartwig (1982), and Theorem 9 of H.-P. Yu (1995).

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An associative ring R is called exchange if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\bigoplus_{i\in I} A_i')$. We know that regular rings, π-regular rings, strongly π-regular rings, semiperfect rings, left or right continuous rings, clean rings, unit C^*-algebras of real rank zero [2, Theorem 4.2], right semiartinian rings, and the ring of all $\omega\times\omega$ matrices over regular R, which are both row and colume-finite, are exchange rings.

We call R has stable range one provided that $aR+bR = R$ implies that $a+by \in U(R)$ for a $y \in R$. It is well known that an exchange ring R has stable range one if and only if $A \oplus B \cong A \oplus C$ implies $B \cong C$ for all finitely generated projective right R-modules A, B, and C. It has been realized that the class of rings having stable range one has good stability properties in a K-theoretic sense (cf. [8, 13]). Many authors have studied stable range one conditions over exchange rings such as [1, 2, 4, 5, 6, 7, 15, 16].

In this paper, we investigate stable range one conditions over exchange rings by virtue of Drazin inverses, nilpotent elements, and prime ideals. We showed that stable range conditions can be determined by Drazin inverses for exchange rings. Also, we see that these stable range conditions can be determined by regular elements out of any proper ideal of R. Moreover, we prove that an exchange ring R has stable range one if the set of nilpotents is closed under product. These extend the corresponding results of [4, Theorem 3], [9, Theorem 4.19], [12, Theorem 2], and [16, Theorem 9].

Throughout this paper, all rings are associative ring with identities and all right R-modules are unitary right R-modules. $M \lesssim^{\oplus} N$ means that right R-module M is isomorphic to a direct summand of right R-module N. The notation $x \approx y$ means that $x = uyu^{-1}$ for some $u \in U(R)$, where $U(R)$ denotes the set of all units of R. Call $a \in R$ is regular if $a = axa$ for some $x \in R$ and $a \in R$ is unit-regular if $a = aua$ for some $u \in U(R)$.

1. Drazin inverses. Recall that $a \in R$ is called strongly π-regular if there exist $n \geq 1$ and $x \in R$ such that $a^n = a^{n+1}x$, $ax = xa$, and $x = xax$. By [1, Theorem 3], we know that every strongly π-regular element of an exchange ring is unit-regular.

Clearly, the solution $x \in R$ is unique, and we say that x is the Drazin inverse a^d of a. Call $a \in R$ is pseudo-similar to $b \in R$ provided that there exist some $x, y \in R$ such that $a = xby$, $b = yax$, and $x = xyx$ and denote it by $a \eqcirc b$. It is well known that R is partially unit-regular is equivalent to the statement $a \eqcirc b$ if and only if $a \approx b$. Now we investigate Drazin inverses in view of pseudo-similarity and extend [12, Theorem 2] to exchange rings having stable range one.

Lemma 1.1. *Let R be an exchange ring. Then the following statements are equivalent:*
(1) *R has stable range one.*
(2) *Whenever $eR \cong fR$ with idempotents $e, f \in R$, $e \approx f$.*

Proof. (1)⇒(2). Given $eR \cong fR$ with $e = e^2$, $f = f^2 \in R$, then we can find some $a \in eRf$, $b \in fRe$ such that $e = ab$ and $f = ba$. Clearly, $e = afb$, $f = bea$, and $a = af = aba$. Thus $e \eqcirc f$. By virtue of [10, Corollary 1], we know that $e \approx f$.

(2)⇒(1). Given $eR \cong fR$ with idempotents $e, f \in R$, then we can find some $u \in U(R)$ such that $e = ufu^{-1}$. We easily check that $1-e = u(1-f)u^{-1} = ((1-e)u(1-f))((1-f)u^{-1}(1-e))$ and $1-f = (1-f)(1-u^{-1}eu)(1-f) = (1-f)u^{-1}(1-e)u(1-f) = ((1-f)u^{-1}(1-e))((1-e)u(1-f))$. Set $x = (1-e)u(1-f)$ and $y = (1-f)u^{-1}(1-e)$. Then $1-e = xy$ and $1-f = yx$ with $x \in (1-e)R(1-f)$, $1-f \in (1-f)R(1-e)$. Consequently, $(1-e)R \cong (1-f)R$. In view of [16, Theorem 9], we conclude that R has stable range one. □

Theorem 1.2. *Let R be an exchange ring. Then the following statements are equivalent:*
(1) *R has stable range one.*
(2) *If ab and ba are strongly π-regular, then $(ab)^d \approx (ba)^d$.*
(3) *If ab and ba are strongly π-regular, then $(ab)(ab)^d \approx (ba)(ba)^d$.*

Proof. (1)⇒(2). Since $ab \in R$ are strongly π-regular, we have $k \geq 1$ such that $(ab)^k = (ab)^{k+1}(ab)^d$, $(ab)(ab)^d = (ab)^d(ab)$, and $(ab)^d = (ab)^d(ab)(ab)^d$. Thus, we check that

$$\begin{aligned}
(ab)^{k+2}a(ba)^d(ba)^d b &= (ab)a(ba)^{k+1}(ba)^d(ba)^d b \\
&= (ab)a(ba)^k(ba)^d b = a(ba)^{k+1}(ba)^d b \\
&= a(ba)^k b = (ab)^{k+1}, \\
(ab)\big(a(ba)^d(ba)^d b\big) &= a(ba)^d(ba)^d(ba)b \\
&= a(ba)^d b = \big(a(ba)^d(ba)^d b\big)(ab), \\
\big(a(ba)^d(ba)^d b\big)(ab)\big(a(ba)^d(ba)^d b\big) &= a(ba)^d(ba)^d b.
\end{aligned} \tag{1.1}$$

Hence $(ab)^d = a(ba)^d(ba)^d b$. So $(ab)^d = a(ba)^d(ba)^d b$. Furthermore, we claim that $(ba)^d = (ba)^d(ba)(ba)^d = (ba)^d bab(ab)^d(ab)^d a = (ba)^d b(ab)^d a$. Clearly, $(ba)^d ba(ba)^d b = (ba)^d b$. Thus we show that $(ab)^d \eqcirc (ba)^d$. In view of [10, Corollary 1], we have $(ab)^d \approx (ba)^d$.

(2)⇒(1). Suppose $eR \cong fR$ with idempotents $e, f \in R$. Then we can find some $a \in eRf$, $b \in fRe$ such that $e = ab$ and $f = ba$. Obviously, e and f are both π-regular, hence $(ab)^d \approx (ba)^d$. Since $e^d = e$ and $f^d = f$, we claim that $e \approx f$. Therefore R has stable range one by Lemma 1.1.

(2)⇒(3). Similarly to the consideration in [12, Theorem 1], we see that $(ab)^d \approx (ba)^d$ implies $(ab)(ab)^d \approx (ba)(ba)^d$, as desired.

(3)⇒(1). Given $eR \cong fR$ with $e = e^2$, $f = f^2 \in R$, then we have $a \in eRf$, $b \in fRe$ such that $e = ab$ and $f = ba$. Similarly to the consideration in (2)⇒(1), we see that $(ab)(ab)^d \approx (ba)(ba)^d$. Hence $e \approx f$, as required. □

Corollary 1.3. *Let R be an exchange ring with artinian primitive factors. If ab and ba are strongly π-regular, then $(ab)^d \approx (ba)^d$.*

Proof. Using [16, Theorem 1], we know that R has stable range one. Thus we complete the proof by Theorem 1.2. □

Corollary 1.4. *Let R be a strongly π-regular ring. Then $(ab)^d \approx (ba)^d$ for all $a, b \in R$.*

Proof. Clearly, R is an exchange ring. By [1, Theorem 4], we see that R has stable range one. So the result follows from Theorem 1.2. □

2. Unit-regularity. Let M be a proper ideal of R. In [4, Theorem 3], V. P. Camillo and H.-P. Yu showed that an exchange ring R has stable range one if and only if every regular element of R is unit-regular. Now we extend Camillo and Yu's result and show that stable range one conditions over an exchange ring R can be determined only by regular elements out of M.

Lemma 2.1. *Let R be an exchange ring, M an ideal of R. Then R has stable range one if and only if*

(1) *R/M has stable range one.*

(2) *Whenever $(1-e)R \cong (1-f)R$ with idempotents $e, f \in M$, $eR \cong fR$.*

Proof. One direction is trivial by [16, Theorem 9].

Conversely, we assume (1) and (2) hold. Given $gR \cong hR$ with idempotents $g, h \in R$, then $\bar{g}(R/M) \cong \bar{h}(R/M)$. Since R/M is an exchange ring having stable range one, by [16, Theorem 9], we see that $(\bar{1}-\bar{g})(R/M) \cong (\bar{1}-\bar{h})(R/M)$. Analogously to the consideration in [2, Proposition 1.4], there are right R-module decompositions $(1-g)R = A_1 \oplus A_2$ and $(1-h)R = B_1 \oplus B_2$ with $A_1 \cong B_1$, $A_2 = A_2M$, and $B_2 = B_2M$. Obviously, there are idempotents $e, f \in R$ such that $eR = A_2$ and $fR = B_2$. Thus $e, f \in M$. From $R = (1-e)R \oplus eR = gR \oplus A_1 \oplus eR$, we have $(1-e)R \cong gR \oplus A_1$. Likewise, $(1-f)R \cong hR \oplus B_1$. So $(1-e)R \cong (1-f)R$ with $e, f \in M$, hence $A_2 \cong eR \cong fR \cong B_2$. Therefore, we conclude that $(1-g)R = A_1 \oplus A_2 \cong B_1 \oplus B_2 = (1-h)R$. According to [16, Theorem 9], we complete the proof. □

Theorem 2.2. *Let R be an exchange ring, and let M be an ideal of R. Then the following statements are equivalent:*

(1) *R has stable range one.*

(2) *Whenever $eR \cong fR$ with $e = e^2 \in R\backslash M$ and $f = f^2 \in R\backslash M$, $(1-e)R \cong (1-f)R$.*

(3) *Every regular element out of M is unit-regular.*

Proof. (1)⇒(2) is obvious from [16, Theorem 9].

(2)⇒(1). Suppose $(1-e)R \cong (1-f)R$ with idempotents $e, f \in M$. Since $1-e = (1-e)^2$, $1-f = (1-f)^2 \in R\backslash M$, we claim that $eR \cong fR$.

Given $\overline{ax}+\bar{b}=\bar{1}$ with $\bar{a},\bar{x},\bar{b}\in R/M$. Then we have a $c\in M$ such that $ax+(b+c)=1$ with $a,x,b+c\in R$. Since R is an exchange ring, by [14, Theorem 2.1], we can find an idempotent $e\in R$ such that $e=(b+c)s$ and $1-e=(1-b-c)t$ for $s,t\in R$. Hence $axt+e=1$, and then $(1-e)axt+e=1$. Clearly, $(1-e)a\in R$ is regular. Assume that $(1-e)a\in M$. Then $\overline{(1-e)a}=\bar{0}$ in R/M, so $\overline{(1-e)a}+\bar{e}=\bar{1}$. Thus we have $\bar{a}+\overline{bs}(\bar{1}-\bar{a})\in U(R/M)$. Assume that $(1-e)a\in R\backslash M$. Since $(1-e)a\in R$ is regular, there exists $d\in R$ such that $(1-e)a=(1-e)ad(1-e)a$. It is easy to verify that $\psi:(1-e)adR\cong d(1-e)aR$ given by $\psi((1-e)adr)=d(1-e)adr$ for any $r\in R$. Because $(1-e)ad=((1-e)ad)^2\in R\backslash M$ and $d(1-e)a=(d(1-e)a)^2\in R\backslash M$, we have an isomorphism $\psi:(1-(1-e)ad)R\cong(1-d(1-e)a)R$. Thus there is an isomorphism $\phi:(1-(1-e)ad)R\to(1-d(1-e)a)R$. Define $u:R\to R$ given by $u(z_1+z_2)=\psi(z_1)+\phi(z_2)$ for any $z_1\in(1-e)adR$ and $z_2\in(1-(1-e)ad)R$. It is easy to check that $(1-e)a=(1-e)au(1)(1-e)a$ with $u(1)\in U(R)$. Therefore $(1-e)a\in R$ is unit-regular. Assume $(1-e)a=gv$ with $g=g^2\in R$, $v\in U(R)$. Then $gvxt+e=(1-e)axt+e=1$. So it follows that $(1-e)a+e(1-g)v=gv+e(1-g)v=(1+gvxt(1-g))^{-1}v\in U(R)$. Thus, there is a $z\in R$ such that $\bar{a}+\overline{bs}(\bar{z}-\bar{a})=\overline{(1-e)a}+\overline{ez}\in U(R/M)$. Therefore R has stable range one by Lemma 2.1.

(1)⇒(3) is clear by [4, Theorem 3].

(3)⇒(2). Given $eR\cong fR$ with idempotents $e,f\in R\backslash M$, we have $x\in eRf$, $y\in fRe$ such that $e=xy$ and $y=yx$. Clearly, $e=xfy$, $f=yex$, and $x=xyx$. Since regular element $yxy\in R\backslash M$, $yxy=yxyuyxy$ for a $u\in U(R)$. Set $w=(1-xy-uyxy)u(1-yx-yxyu)$. Then we check that $w\in U(R)$ and $e=wfw^{-1}$. It is easy to check that $1-e=w(1-f)w^{-1}$, $1-f=w^{-1}(1-e)w$. Set $s=(1-e)w(1-f)$ and $t=(1-f)w^{-1}(1-e)$. Then $1-e=st$ and $1-f=ts$. Therefore $(1-e)R\cong(1-f)R$, as asserted. □

Corollary 2.3. *Let R be an exchange ring, and let M be an ideal of R. Then the following statements are equivalent:*

(1) *R has stable range one.*

(2) *Whenever $eR\cong fR$ with idempotents $e,f\in R\backslash M$, $e\approx f$.*

Proof. (1)⇒(2). Given $eR\cong fR$ with idempotents $e,f\in R\backslash M$, we have $x\in eRf$, $y\in fRe$ such that $e=xy$ and $y=yx$. Similarly to the consideration in Theorem 2.2, we know that $e\eqcirc f$. Hence $e\approx f$ by [10, Corollary 1].

(2)⇒(1). Given $eR\cong fR$ with idempotents $e,f\in R\backslash M$, then $e=ufu^{-1}$ for some $u\in U(R)$. We easily check that $1-e=((1-e)u(1-f))((1-f)u^{-1}(1-e))$, $1-f=((1-f)u^{-1}(1-e))((1-e)u(1-f))$. Hence $(1-e)R\cong(1-f)R$. By Theorem 2.2, we complete the proof. □

Corollary 2.4. *Let R be an exchange ring, and let M be an ideal of R. Then the following statements are equivalent:*

(1) *R has stable range one.*

(2) *If ab and ba are strongly π-regular for $a,b\in R\backslash M$, then $(ab)^d\approx(ba)^d$.*

Proof. (1)⇒(2) is trivial by Theorem 1.2.

(2)⇒(1). Given $eR\cong fR$ with idempotents $e,f\in R\backslash M$, then we can find $a\in eRf$,

$b \in fRe$ such that $e = ab$ and $f = ba$. Since $e, f \in R\backslash M$, we know that $a, b \in R\backslash M$. Hence $(ab)^d \approx (ba)^d$. That is, $e \approx f$. Therefore the result follows by Theorem 2.2. □

3. Nilpotent elements. Recall that $a \in R$ is said to be nilpotent if $a^n = 0$ for some positive integer n. In [16], H.-P. Yu proved that every exchange ring of bounded index of nilpotence has stable range one. Now we investigate stable range one conditions over exchange rings by virtue of nilpotent element. We see that an exchange ring R has stable range one provided that the set of all nilpotents of R is closed under product. First, we extend [11, Lemma 3.1 and Proposition 3.3] to exchange rings by a similar route as follows.

Lemma 3.1. *Let A, B be finitely generated projective right modules over an exchange ring R, and let k be a positive integer. Then $A \lesssim^{\oplus} kB$ if and only if there is a decomposition $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ such that $A_1 \lesssim^{\oplus} A_2 \lesssim^{\oplus} \cdots \lesssim^{\oplus} A_k \lesssim^{\oplus} B$.*

Proof. One direction is obvious. Conversely, it is trivial if $k = 1$. Now we assume that the result holds for $k-1$. Suppose that $A \lesssim^{\oplus} kB$. Then $A \oplus C \cong (k-1)B \oplus B$ for some right R-module C. By [2, Proposition 1.2], there are right R-modules U, W, E, and F such that $A = U \oplus W$, $C = E \oplus F$, $U \oplus E \cong (k-1)B$, and $W \oplus F \cong B$. So $U \lesssim^{\oplus} (k-1)B$. By the induction hypothesis, we have $U = U_1 \oplus U_2 \oplus \cdots \oplus U_{k-1}$ with $U_1 \lesssim^{\oplus} U_2 \lesssim^{\oplus} \cdots \lesssim^{\oplus} U_{k-1} \lesssim^{\oplus} B$. Clearly, there are right R-modules $D_k, D_{k-1}, \ldots, D_2$, $U_1 = D_1$ such that $B = D_1 \oplus D_2 \oplus \cdots \oplus D_k$, $U_{i-1} \oplus D_i \cong U_i$ $(2 \le i \le k)$. Hence $W \oplus F \cong D_1 \oplus D_2 \oplus \cdots \oplus D_k$. In view of [2, Proposition 1.2], we have right R-modules $X_1, X_2, \ldots, X_k$ such that $W = X_1 \oplus X_2 \oplus \cdots \oplus X_k$ with all $X_i \lesssim^{\oplus} D_i$. Consequently, $A = U \oplus W = (U_1 \oplus U_2 \oplus \cdots \oplus U_{k-1}) \oplus (X_1 \oplus X_2 \oplus \cdots \oplus X_k) = X_1 \oplus (X_2 \oplus U_1) \oplus (X_3 \oplus U_2) \oplus \cdots \oplus (X_k \oplus U_{k-1})$. Clearly, $X_1 \lesssim^{\oplus} X_2 \oplus U_1 \lesssim^{\oplus} X_3 \oplus U_2 \lesssim^{\oplus} \cdots \lesssim^{\oplus} X_k \oplus U_{k-1}$. Therefore we complete the proof. □

Lemma 3.2. *If $e = e^2$, $f = f^2 \in R$ such that $eR \lesssim^{\oplus} fR$, then there exist $x \in eRf$ and $y \in fRe$ such that $xy = e$.*

Proof. Since $eR \lesssim^{\oplus} fR$, we can find some right R-module D such that $\psi : eR \oplus D \cong fR$. Assume that $\psi(e) = fr$ for some $r \in R$. Then $e = \psi^{-1}(fr) = (e\psi^{-1}(f)f)(fre)$. Set $x = e\psi^{-1}(f)f$ and $y = fre$. Then $x \in eRf$ and $y \in fRe$ with $xy = e$, as asserted. □

Lemma 3.3. *Let R be an exchange ring. If $R(1-e)R = R$ with $e = e^2 \in R$, then e is the product of two nilpotents of R.*

Proof. Since $R(1-e)R = R$, we have $y_1, \ldots, y_n \in R$ such that $1 \in \sum y_i(1-e)R$. Hence $R = \sum y_i(1-e)R$. So we have an epimorphism $\psi : n(1-e)R \to \sum y_i(1-e)R = R$. Since R is a projective right R-module, we know that ψ is a split epimorphism. Thus $R \lesssim^{\oplus} n(1-e)R$.

Since $eR \lesssim^{\oplus} R \lesssim^{\oplus} n(1-e)R$, by virtue of Lemma 3.1, there is a decomposition $eR = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ such that $A_1 \lesssim^{\oplus} A_2 \lesssim^{\oplus} \cdots \lesssim^{\oplus} A_n \lesssim^{\oplus} (1-e)R$. Thus we have orthogonal idempotents $e_1, \ldots, e_n \in R$ such that $e = e_1 + \cdots + e_n$ with $e_1R \lesssim^{\oplus} e_2R \lesssim^{\oplus} \cdots \lesssim^{\oplus} e_nR \lesssim^{\oplus} (1-e)R$. Set $e_{n+1} = 1-e$. Because $e_iR \lesssim^{\oplus} e_{i+1}R$ for $i = 1, \ldots, n$, from Lemma 3.2, we can find $x_i \in e_iRe_{i+1}$ and $y_i \in e_{i+1}Re_i$ such that $x_iy_i = e_i$. Set

$x = x_1 + \cdots + x_n$ and $y = y_1 + \cdots + y_n$. It is easy to verify that $x^{n+1} = y^{n+1} = 0$ and $e = e_1 + \cdots + e_n = xy$. Thus e is the product of two nilpotents. □

Theorem 3.4. *Let R be an exchange ring. If the set of all nilpotents of R is closed under product, then R has stable range one.*

Proof. Assume $ax + b = 1$ in R. Since R is an exchange ring, by [14, Theorem 2.1], we have $e = e^2 \in R$ such that $e = bs$ and $1 - e = (1 - b)t$ for some $s, t \in R$. Thus $1 - e = axt$. It is easy to check that $(1-e)aR \subseteq (1-e)R = (1-e)axtR \subseteq (1-e)aR$, and then $(1-e)aR = (1-e)R$. Hence $R = (1-e)aR \oplus eR$, so there exist $u, v \in R$ such that $1 = (1-e)au + ev$. We easily check that $(1-e)((1-e)a+e)u + e((1-e)a+e)v = (1-e)au + ev = 1$, hence $R((1-e)a+e)R = R$. Set $y = s(1-a)$. Then $R(a+by)R = R(a+bs(1-a))R = R(a+e(1-a))R = R((1-e)a+e)R = R$. In view of [3, Corollary 2.2], we can find an idempotent $e \in (a+by)R$ such that $ReR = R$. Set $e = 1 - f$. Then $R(1-f)R = R$ with $f = f^2 \in R$. It follows from Lemma 3.3 that $1 - e = f \in R$ is the product of two nilpotents of R. Therefore $1 - e = (1-e)^2$ is a nilpotent. Hence $e = 1$. Consequently, we claim that $(a+by)r = 1$ for some $r \in R$.

From $r(a+by) + (1 - r(a+by)) = 1$, we can find $s \in R$ such that $(r + (1 - r(a+by))z)s = 1$ for some $z \in R$. Therefore $a + by = (a+by)(r + (1 - r(a+by))z)s = (a+by)rs = s$, hence $a + by \in U(R)$. That is, R has stable range one. □

Recall that an ideal P of R is called completely prime if R/P is a domain. As an immediate consequence of Theorem 3.4, we now derive the following corollary.

Corollary 3.5. *Every exchange ring with all minimal prime ideal completely prime has stable range one.*

Proof. Clearly, R is 2-prime. So we claim that all nilpotents in R belong to the prime radical of R. Thus the product of two nilpotents is nilpotent. By Theorem 3.4, the result follows. □

Proposition 3.6. *Let R be an exchange ring. If every regular element which is the sum of a unit and a nilpotent element is a unit. Then R has stable range one.*

Proof. Assume $ax + b = 1$ in R. Similar to the consideration in Theorem 3.4, we can find a $y \in R$ such that $R(a+by)R = R$. Thus, we have $e = e^2 \in (a+by)R$ such that $R(1-f)R = R$ with $f = 1 - e$. By Lemma 3.3, $1 - e = n_1 n_2$ is the product of two nilpotents n_1 and n_2. Hence $e = 1 - n_1 n_2 = (1-n_1) + n_1(1-n_2) = ((1-n_1)(1-n_2)^{-1} + n_1)(1-n_2)$. Since $(1-n_1)(1-n_2)^{-1} + n_1 \in U(R)$, we know that $e = e^2 \in U(R)$. Hence $(a+by)r = e = 1$ for some $r \in R$. Analogously to the consideration in Theorem 3.4, we conclude that R has stable range one. □

4. Prime ideals. In Section 3, we see that stable range one conditions over exchange rings can be studied by minimal prime ideal of R. Now we investigate such stable range condition by prime ideals. We denote the set of all finitely generated projective right R-modules by $\mathrm{FP}(R)$.

Lemma 4.1. *Let J and K be ideals of an exchange ring R such that $JK = 0$ and R/K has stable range one. Let $A, B \in \mathrm{FP}(R)$. If $A/AJ \cong B/BJ$ and $A/AK \cong B/BK$, then $A \cong B$.*

Proof. Suppose that $A/AJ \cong B/BJ$ and $A/AK \cong B/BK$. Similar to [2, Proposition 1.4], we have right R-module decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ with $A_1 \cong B_1$, $A_2 = A_2J$, and $B_2 = B_2J$. It is easy to verify that $A_1/A_1K \oplus A_2 \cong A/AK \cong B/BK \cong B_1/B_1K \oplus B_2$ because $A_2K = B_2K = 0$. Since R/K has stable range one, we have $A_2 \cong B_2$. Thus $A \cong A_1 \oplus B_1 \cong A_2 \oplus B_2 \cong B$, as required. □

Theorem 4.2. *Let R be an exchange ring having stable range one and $A, B \in \mathrm{FP}(R)$. If $A/AP \cong B/BP$ for all prime ideal of R, then $A \cong B$.*

Proof. Suppose that $A \ncong B$. Set $\Omega = \{M \mid J \text{ is an ideal of } R \text{ such that } A/AJ \ncong B/BJ\}$. Clearly, $\Omega \neq \varnothing$.

Given $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$ in Ω. Set $J = \bigcup_{1\le i<\infty} J_i$. Then J is an ideal of R. If $J \notin \Omega$, then $A/AJ \cong B/BJ$. This is equivalent to some equations modulo J. Thus we have some J_i such that $A/AJ_i \cong B/BJ_i$, a contradiction. Hence Ω is inductive. Therefore, we have some $Q \in \Omega$ such that it is maximal in Ω. Clearly, Q is not prime. So we have ideals K and L such that $K \supset Q$, $L \supseteq Q$, and $KL \subseteq Q$. Therefore, $(A/Q)/(A/Q)(K/Q) \cong AK \cong B/BK \cong (B/Q)/(B/Q)(K/Q)$. Likewise, $(A/Q)/(A/Q)(L/Q) \cong (B/Q)/(B/Q)(L/Q)$. Because $(K/Q)\bigcap(L/Q) = 0$, we conclude that $(A/Q)/(A/Q)(Q/Q) \cong (B/Q)/(B/Q)(Q/Q)$ by Lemma 4.1. Hence $A/Q \cong B/Q$, a contradiction. Therefore $A \cong B$, as asserted. □

Corollary 4.3. *Let R be an exchange ring with artinian primitive factors and $A, B \in \mathrm{FP}(R)$. If $A/AP \cong B/BP$ for all prime ideal of R, then $A \cong B$.*

Proof. Since R is an exchange ring with artinian factors, in view of [16, Theorem 1], it has stable range one. Thus we complete the proof by Theorem 4.2. □

Corollary 4.4. *Let R be a strongly π-regular ring and $A, B \in \mathrm{FP}(R)$. If $A/AP \cong B/BP$ for all prime ideal of R, then $A \cong B$.*

Proof. Since R is a strongly π-regular ring, by [1, Theorem 4], R has stable range one. Therefore, we conclude that $A \cong B$ by Theorem 4.2. □

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References

[1] P. Ara, *Strongly π-regular rings have stable range one*, Proc. Amer. Math. Soc. **124** (1996), no. 11, 3293–3298. MR 97a:16024. Zbl 865.16007.

[2] P. Ara, K. R. Goodearl, K. C. O'Meara, and E. Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. **105** (1998), 105–137. MR 99g:16006. Zbl 908.16002.

[3] P. Ara, K. R. Goodearl, K. C. O'Meara, and R. Raphael, *K_1 of separative exchange rings and C^*-algebras with real rank zero*, Pacific J. Math. **195** (2000), no. 2, 261–275. CMP 1 782 176.

[4] V. P. Camillo and H.-P. Yu, *Stable range one for rings with many idempotents*, Trans. Amer. Math. Soc. **347** (1995), no. 8, 3141–3147. MR 95j:16006. Zbl 848.16008.

[5] M. J. Canfell, *Completion of diagrams by automorphisms and Bass' first stable range condition*, J. Algebra **176** (1995), no. 2, 480–503. MR 97a:16004. Zbl 839.16007.

[6] H. Chen, *Rings with stable range conditions*, Comm. Algebra **26** (1998), no. 11, 3653–3668. MR 99m:16005. Zbl 914.16002.

[7] ———, *On m-fold stable exchange rings*, Comm. Algebra **27** (1999), no. 11, 5639–5647. MR 2000h:16007. Zbl 937.16008.

[8] H. Chen and F.-A. Li, *Whitehead groups of exchange rings with primitive factors artinian*, to appear in Int. J. Math. Math. Sci.

[9] K. R. Goodearl, *von Neumann Regular Rings*, Pitman, London, 1979. MR 80e:16011. Zbl 411.16007.

[10] R. Guralnick and C. Lanski, *Pseudosimilarity and cancellation of modules*, Linear Algebra Appl. **47** (1982), 111–115. MR 84a:16036. Zbl 493.15010.

[11] J. Hannah and K. C. O'Meara, *Depth of idempotent-generated subsemigroups of a regular ring*, Proc. London Math. Soc. (3) **59** (1989), no. 3, 464–482. MR 90h:16021. Zbl 653.16006.

[12] R. E. Hartwig, *On a theorem of Flanders*, Proc. Amer. Math. Soc. **85** (1982), no. 3, 310–312. MR 83h:15011. Zbl 508.15008.

[13] P. Menal and L. N. Vaserstein, *On the structure of* GL_2 *over stable range one rings*, J. Pure Appl. Algebra **64** (1990), no. 2, 149–162. MR 92a:20058. Zbl 702.20037.

[14] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278. MR 55#12757. Zbl 352.16006.

[15] H.-P. Yu, *On $\aleph_0$-quasi-continuous exchange rings*, Comm. Algebra **23** (1995), no. 6, 2187–2197. MR 96m:16049. Zbl 830.16009.

[16] ———, *Stable range one for exchange rings*, J. Pure Appl. Algebra **98** (1995), no. 1, 105–109. MR 96g:16006. Zbl 837.16009.

Huanyin Chen: Department of Mathematics, Hunan Normal University, Changsha 410006, China

E-mail address: chyzxl@sparc2.hunnu.edu.cn