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NEW INVERSION FORMULAS FOR THE KRÄTZEL TRANSFORMATION

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Abstract. We study in distributional sense by means of the kernel method an integral transform introduced by Krätzel. It is well known that the cited transform generalizes to the Laplace and Meijer transformation. Properties of analyticity, boundedness, and inversion theorems are established for the generalized transformation.

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1. Introduction. In this paper, we study the following integral transform,

$$(K_\nu^\rho f)(x) = \int_0^\infty Z_\rho^\nu(xy) f(y)\,dy, \quad x > 0, \tag{1.1}$$

where $Z_\rho^\nu(x)$ denotes the function

$$Z_\rho^\nu(x) = \int_0^\infty t^{\nu-1} e^{-t^\rho - x/t} dt \tag{1.2}$$

with $\rho > 0$, $\rho \in \mathbb{N}$, $\nu \in \mathbb{C}$. The K_ν^ρ transform is reduced to the Meijer transform when $\rho = 1$ and to the Laplace transform when $\rho = 1$ and $\nu = \pm 1/2$. Zemanian realized a wide study of the Laplace and Meijer transformations, in distribution spaces (cf. [12, 13]). Later, Krätzel, introduced the K_ν^ρ transformation, which generalizes the Meijer transform, and in a series of papers, investigated it in the classical sense (see [5, 6]). In [8], Rao and Debnath investigated the K_ν^ρ transformation on certain spaces of distributions by means of the kernel method. Recently, the cited transformation is studied in [1] by the adjoint method on the McBride's spaces.

In [3], the study of Krätzel is continued obtaining Abelian, Tauberian theorems, and some inversion formulas in the classical sense. Moreover, we can emphasize in [2] the work developed by the authors studying the K_ν^ρ transform on weighted L^p spaces, improving a result of [4].

Motivated by the cited papers, we accomplish a study of the K_ν^ρ transform by means of the kernel method on the space of distributions of compact support.

By $\mathfrak{E}(I)$ we denote the infinitely differentiable functions $\phi(t)$, $t \in I = (0,\infty)$, such that for all compact K we have

$$\gamma_k(\phi) = \sup_{t \in K} \left| \frac{d^k}{dt^k} \phi(t) \right| < \infty, \tag{1.3}$$

for every $k \in \mathbb{N}$. By $\mathfrak{E}'(I)$ we denote the dual space of $\mathfrak{E}(I)$. Moreover, by $\mathfrak{D}(I)$, $\mathfrak{D}'(I)$ denote the spaces of functions and distributions that can be found in [9, 11].

Consider the following useful properties.

In [1] we see that

$$\frac{d^k}{dx^k} Z_\rho^\nu(x) = (-1)^k Z_\rho^{\nu-k}(x). \tag{1.4}$$

By (1.4) and the asymptotic behaviour of $Z_\rho^\nu(x)$, we deduce that for certain positive constant α_1 and $k \in \mathbb{N}$, we have

$$\left| \frac{d^k}{dx^k} \left(Z_\rho^\nu(x) \right) \right| \le \alpha_1 \cdot x^{(2\nu-\rho)/(2\rho+2)} e^{-x^{\rho/(\rho+1)}}, \tag{1.5}$$

for all $x \in K$, K compact and $\nu \in \mathbb{C}$.

Moreover, by [5] we have that if $\rho \in \mathbb{N}$,

$$A_{\rho,\nu} Z_\rho^\nu(x) = (-1)^\rho Z_\rho^\nu(x), \tag{1.6}$$

where

$$A_{\rho,\nu} = -\rho^{-1} \cdot x^{\nu-\rho+1} \frac{d}{dx} x^{\rho-\nu} \frac{d^\rho}{dx^\rho}. \tag{1.7}$$

On the other hand, $B_{\rho,\nu,k}$ denotes $B_{\rho,\nu,k} = C_1 \cdot y^{-\rho k-1} B_{\rho,\nu}^k$, where

$$\begin{aligned} B_{\rho,\nu} &= x^{-\nu+\rho+1} \frac{d}{dx} x^{\rho+\nu} \frac{d^\rho}{dx^\rho}, \\ C_1 &= \frac{(\rho k)^{\rho k-(2\nu-\rho)/(2\rho+2)-1/(\rho+1)+1} \rho}{\Gamma\left(\rho k - \frac{2\nu-\rho}{2\rho+2} - \frac{1}{\rho+1} + 1\right) \Gamma\left(\frac{\rho k-(2\nu-\rho)/(2\rho+2)-1/(\rho+1)+1+\nu}{\rho}\right)}. \end{aligned} \tag{1.8}$$

The Mellin transform is defined by $\mathfrak{M}\{f\}(s) = \int_0^\infty x^{s-1} f(x)\,dx$ and the Mellin transform of the kernel is given by

$$\mathfrak{M}\left(Z_\rho^\nu(x)\right)(s) = \frac{1}{\rho} \Gamma(s) \Gamma\left(\frac{s+\nu}{\rho}\right), \tag{1.9}$$

if $\operatorname{Re} s + \min(0, \operatorname{Re} \nu) > 0$.

2. The generalized K_ν^ρ-transform on $\mathfrak{E}'(I)$. Let $\nu \in \mathbb{C}$ and $\rho \in \mathbb{N}$. For every $f \in \mathfrak{E}'(I)$ we define the generalized K_ν^ρ transform by the relation

$$F(x) = \left(K_\nu^\rho f\right)(x) = \langle f(t), Z_\rho^\nu(xt) \rangle. \tag{2.1}$$

It is easy to see that $Z_\rho^\nu(xt) \in \mathfrak{E}(I)$ for x fixed, $x > 0$. Furthermore, if f is a locally absolutely integrable function, then the generalized transform of f is reduced to the classic K_ν^ρ transform.

Proposition 2.1. *The operators $A_{\rho,\nu}$ and $B_{\rho,\nu}$ define a continuous linear mapping from $\mathfrak{E}(I)$ into itself.*

Proof. It is established without difficulty that

$$\gamma_k(A_{\rho,\nu}\phi) \le C \cdot \sum_{j=1}^{\rho+1} \gamma_{k+j}(\phi), \qquad \gamma_k(B_{\rho,\nu}\phi) \le C \cdot \sum_{j=1}^{\rho+1} \gamma_{k+j}(\phi), \tag{2.2}$$

for every $\phi \in \mathfrak{E}(I)$.

We define the generalized operator $A^*_{\rho,\nu}$ on $\mathcal{E}'(I)$ as the adjoint operator of $A_{\rho,\nu}$, that is,

$$\langle A^*_{\rho,\nu} f, \phi\rangle = \langle f, A_{\rho,\nu}\phi\rangle, \tag{2.3}$$

with $f \in \mathcal{E}'(I)$ and $\phi \in \mathcal{E}(I)$. Moreover, by Proposition 2.1, $A^*_{\rho,\nu}$ is a continuous linear mapping from $\mathcal{E}'(I)$ into itself. Note that the same occurs for the operator $B_{\rho,\nu}$. □

Next are established some properties of the generalized K^ρ_ν transformation.

Proposition 2.2. *Let $f \in \mathcal{E}'(I)$. If $F(x)$ denotes the generalized K^ρ_ν transform of f, then $F(x)$ is infinitely differentiable on $(0,\infty)$ and*

$$\frac{d^r}{dx^r}F(x) = \left\langle f(t), \frac{\partial^r}{\partial x^r} Z^\nu_\rho(xt)\right\rangle, \quad x > 0,\ r \in \mathbb{N}. \tag{2.4}$$

Proof. Consider h an arbitrary increment in $x > 0$. Assume, without loss of generality, that $0 < |h| < (x/2)^{\rho/(\rho+1)}$.

It is easy to see that

$$\frac{F(x+h)-F(h)}{h} = \left\langle f(t), \frac{1}{h}\left(Z^\nu_\rho(t(x+h)) - Z^\nu_\rho(tx)\right)\right\rangle. \tag{2.5}$$

We must see that

$$\varphi_h(x,t) = \frac{1}{h}\left(Z^\nu_\rho(t(x+h)) - Z^\nu_\rho(tx)\right) - \frac{\partial}{\partial x}Z^\nu_\rho(xt) \longrightarrow 0 \tag{2.6}$$

as $h \to 0$, in the sense of the convergence in $\mathcal{E}(I)$, since the result is obtained for $k = 1$, by (2.5) and the continuity of $f(t)$.

For every $r \in \mathbb{N}$ we can write

$$\begin{aligned}
\frac{\partial^r}{\partial t^r}\varphi_h(x,t) &= \frac{1}{h}\int_x^{x+h}\int_x^u \frac{\partial^2}{\partial y^2}\left(\frac{\partial^r}{\partial t^r}Z^\nu_\rho(ty)\right)dy\,du\\
&= \frac{1}{h}\int_x^{x+h}\int_x^u \frac{\partial^2}{\partial y^2}\left(y^r\frac{d^r}{d(ty)^r}Z^\nu_\rho(ty)\right)dy\,du\\
&= \frac{1}{h}\int_x^{x+h}\int_x^u \left(r(r-1)y^{r-2}\frac{d^r}{d(ty)^r}Z^\nu_\rho(ty) + 2ry^{r-1}t\frac{d^{r+1}}{d(ty)^{r+1}}Z^\nu_\rho(ty)\right.\\
&\qquad\left. + y^r t^2\frac{d^{r+2}}{d(ty)^{r+2}}Z^\nu_\rho(ty)\right)dy\,du.
\end{aligned} \tag{2.7}$$

By virtue of (1.5), given $t \in K$, K compact, $y > x/2$, it follows that there exists a positive constant C such that

$$\left|\frac{d^{r+j}}{d(ty)^{r+j}}\left(Z^\nu_\rho(ty)\right)\right| \le C(yt)^{(2\nu-\rho)/(2\rho+2)}\cdot e^{-(tx/2)^{\rho/(\rho+1)}}, \quad \text{for } j \in \mathbb{N}. \tag{2.8}$$

Therefore

$$\begin{aligned}
\left|\frac{\partial^r}{\partial t^r}\varphi_h(x,t)\right| &\le C\cdot t^{(2\nu-\rho)/(2\rho+2)}\cdot e^{-(tx/2)^{\rho/(\rho+1)}}\\
&\qquad\cdot\frac{1}{h}\int_x^{x+h}\int_x^u\left(y^{r-2}+y^{r-1}t+y^r t^2\right)dy\,du\\
&\le C\cdot e^{-\alpha\cdot(tx)^{\rho/(\rho+1)}}
\end{aligned} \tag{2.9}$$

for all $t \in K$ and $0 < |h| < (x/2)^{\rho/(\rho+1)}$, being α a suitable constant.

Then given $\epsilon > 0$, there exists $t_0 \in K = [a,b]$ such that

$$\left| \frac{\partial^r}{\partial t^r} \varphi_h(x,t) \right| < \epsilon \tag{2.10}$$

for $t > t_0$ and $0 < |h| < x^{\rho/(\rho+1)}/2$.

Furthermore, for each $t \in [a,t_0]$ we have

$$\left| \frac{\partial^r}{\partial t^r} \varphi_h(x,t) \right| \le C \cdot \left| \frac{1}{h} \int_x^{x+h} \int_x^u (y^{r-2} + y^{r-1} t + y^r t^2)\, dy\, du \right|. \tag{2.11}$$

Then $(\partial^r/\partial t^r)\varphi_h(x,t) \to 0$ as $h \to 0$ uniformly in $t \in [a,t_0]$. Therefore it is concluded that $\gamma_r(\varphi_h(x,t)) \to 0$ as $h \to 0$.

By proceeding inductively the result follows. □

Proposition 2.3. *Let $f \in \mathcal{E}'(I)$ and $F(x) = (K_\nu^\rho f)(x)$ for $x > 0$, then*

$$|F(x)| \le C \cdot x^{r+(2\nu-\rho)/(2\rho+2)} \cdot e^{-\alpha x^{\rho/(\rho+1)}}, \quad x > 0, \tag{2.12}$$

being C and α suitable constants and $r \in \mathbb{N}$.

Proof. We know by [13, Theorem 1.8-1, pages 18–19] that there exists $r \in \mathbb{N}$ such that

$$|F(x)| \le C \max_{0 \le k \le r} \gamma_k(Z_\rho^\nu(xt)) \quad \forall x > 0. \tag{2.13}$$

By the asymptotic behaviour we have for every K compact,

$$\begin{aligned} |F(x)| &\le C \max_{0\le k\le r} \sup_{t\in K} \left| x^k \frac{d^k}{d(xt)^k} (Z_\rho^\nu(xt)) \right| \\ &\le C \max_{0\le k\le r} \sup_{t\in K} \left| x^k (xt)^{(2\nu-\rho)/(2\rho+2)} e^{-(xt)^{\rho/(\rho+1)}} \right| \\ &\le C \cdot x^{r+(2\nu-\rho)/(2\rho+2)} \cdot \max_{0\le k\le r} \sup_{t\in K} \left| t^{(2\nu-\rho)/(2\rho+2)} e^{-(xt)^{\rho/(\rho+1)}} \right| \\ &\le C \cdot x^{r+(2\nu-\rho)/(2\rho+2)} e^{-\alpha x^{\rho/(\rho+1)}}. \end{aligned} \tag{2.14}$$

□

With the following proposition we obtain an operational formula for the generalized K_ν^ρ transform, that includes the operator $A_{\rho,\nu}^*$, $\rho \in \mathbb{N}$.

Proposition 2.4. *Let P be a polynomial, if $f \in \mathcal{E}'(I)$, then*

$$(K_\nu^\rho P(A_{\rho,\nu}^*) f)(x) = P((-x)^\rho)(K_\nu^\rho f)(x) \tag{2.15}$$

for $x > 0$ and $\rho \in \mathbb{N}$.

Proof. By (1.6) and according to Proposition 2.1 it follows that

$$\begin{aligned} (K_\nu^\rho P(A_{\rho,\nu}^*) f)(x) &= \langle P(A_{\rho,\nu}^*) f(t), Z_\rho^\nu(xt) \rangle \\ &= \langle f(t), P(A_{\rho,\nu}) Z_\rho^\nu(xt) \rangle \\ &= \langle f(t), P((-x)^\rho) Z_\rho^\nu(xt) \rangle \\ &= P((-x)^\rho)(K_\nu^\rho f)(x). \end{aligned} \tag{2.16}$$

□

Now we establish an inversion theorem for the K_ν^ρ- transform using a similar procedure to employ by Malgonde and Saxena [7].

Lemma 2.5. *Let $f \in \mathcal{E}'(I)$. Then*

$$\int_0^\infty x^{-s}\langle f(t), Z_\rho^\nu(xt)\rangle dx = \left\langle f(t), \int_0^\infty x^{-s} Z_\rho^\nu(xt)dx\right\rangle. \tag{2.17}$$

Proof. Let $N \in \mathbb{N}$. First, we prove that

$$\int_0^N x^{-s}\langle f(t), Z_\rho^\nu(xt)\rangle dx = \left\langle f(t), \int_0^N x^{-s} Z_\rho^\nu(xt)dx\right\rangle. \tag{2.18}$$

We take $\{x_{r,l}\}_{r=0}^l$ a partition of the interval $[0,N]$ such that $d_l = x_{r,l} - x_{r-1,l}$ for each $r = 1,2,\dots,l$. Then we can write

$$\begin{aligned} \int_0^N x^{-s}\langle f(t), Z_\rho^\nu(xt)\rangle dx &= \lim_{l\to\infty} d_l \cdot \sum_{r=0}^l x_{r,l}^{-s}\langle f(t), Z_\rho^\nu(x_{r,l}t)\rangle, \\ d_l \cdot \sum_{r=0}^l x_{r,l}^{-s}\langle f(t), Z_\rho^\nu(x_{r,l}t)\rangle &= \left\langle f(t), d_l \sum_{r=0}^l x_{r,l}^{-s} Z_\rho^\nu(x_{r,l}t)\right\rangle. \end{aligned} \tag{2.19}$$

Then (2.18) is established if we demonstrated

$$\lim_{l\to\infty} d_l \sum_{r=0}^l x_{r,l}^{-s} \cdot Z_\rho^\nu(x_{r,l}t) = \int_0^N x^{-s} Z_\rho^\nu(xt)\,dx \tag{2.20}$$

in the sense of the convergence in $\mathcal{E}(I)$.

Then, since the function

$$g(t,x) = \begin{cases} x^{-s+k}\dfrac{d^k}{d(xt)^k}\left(Z_\rho^\nu(xt)\right) & \text{if } x \in [0,N], \\ 0 & \text{if } x = 0, \end{cases} \tag{2.21}$$

with $t \in [a,b]$, $0 < a < b < \infty$, is uniformly continuous on $(t,x) \in [a,b]\times[0,N]$, we have that if $\epsilon > 0$ and $m \in \mathbb{N}$, there exists $l_0 \in \mathbb{N}$ such that

$$\sup_{t\in K}\left|\frac{d^k}{dt^k}\left[d_l \sum_{r=0}^l x_{r,l}^{-s} Z_\rho^\nu(x_{r,l}t) - \int_0^N x^{-s} Z_\rho^\nu(xt)\,dx\right]\right| < \epsilon \tag{2.22}$$

for $l > l_0$. Therefore we obtain (2.20).

Using (1.5), for $m,k \in \mathbb{N}$, we have that

$$\left|\frac{d^k}{dt^k}\int_N^\infty x^{-s} Z_\rho^\nu(xt)dx\right| \le C\cdot t^{(2\nu-\rho)/(2\rho+2)}\int_N^\infty x^{-\operatorname{Re} s+(2\nu-\rho)/(2\rho+2)} e^{-(xt)^{\rho/(\rho+1)}} dx \longrightarrow 0 \tag{2.23}$$

uniformly in $t \in K$, as $N \to \infty$.

Moreover, by Proposition 2.3 we have

$$\left|\int_N^\infty x^{-s}\langle f(t), Z_\rho^\nu(xt)\rangle dx\right| \le C\int_N^\infty x^{-s+r+(2\nu-\rho)/(2\rho+2)} e^{-\alpha x^{\rho/(\rho+1)}} dx, \tag{2.24}$$

being α a suitable constant.

Then

$$\lim_{N\to\infty}\int_N^\infty x^{-s}\langle f(t),Z_\rho^\nu(xt)\rangle\,dx=0. \tag{2.25}$$

Hence by (2.18), (2.23), and (2.25) we obtain (2.17). □

The following lemma is obtained from [7, Lemmas 2 and 3] with a slight modification.

Lemma 2.6. *Let $\phi\in\mathfrak{D}(I)$, we denote*

$$\psi(s)=\int_0^\infty y^{-s}\phi(y)\,dy. \tag{2.26}$$

Then

(i) $\int_{-R}^{R}\langle f(u),u^{\sigma+iw-1}\rangle\psi(\sigma+iw)\,dw=\langle f(u),\int_{-R}^{R}u^{\sigma+iw-1}\psi(\sigma+iw)\,dw\rangle$ *with* $\sigma>0$ *and* $R>0$,

(ii) $(1/\pi)\int_0^\infty\phi(y)(u/y)^\sigma(\sin(R\log(u/y))/u\log(u/y))\,dy\to\phi(u)$ *as* $R\to\infty$, *in the sense of the convergence in* $\mathfrak{E}(I)$, *with* $\sigma>0$.

Now we demonstrate the first inversion theorem.

Theorem 2.7. *Let $f\in\mathfrak{E}'(I)$, $\phi\in\mathfrak{D}(I)$, $\nu\in\mathbb{C}$ and $\sigma<1+\min(0,\operatorname{Re}\nu)$. Then*

$$\lim_{R\to\infty}\left\langle\frac{1}{2\pi i}\int_{\sigma-iR}^{\sigma+iR}\frac{1}{K(s)}y^{-s}\int_0^\infty x^{-s}F(x)\,dx\,ds,\phi(y)\right\rangle=\langle f(t),\phi(t)\rangle, \tag{2.27}$$

where $F(x)=(K_\nu^\rho f)(x)$, for $x>0$, and

$$K(s)=\frac{1}{\rho}\Gamma(1-s)\Gamma\left(\frac{1-s+\nu}{\rho}\right). \tag{2.28}$$

Proof. Given $f\in\mathfrak{E}'(I)$, we denote $F(x)=\langle f(t),Z_\rho^\nu(xt)\rangle$, for $x>0$. It is easy to see that the function

$$\varphi_R(y)=\frac{1}{2\pi i}\int_{\sigma-iR}^{\sigma+iR}\frac{1}{K(s)}y^{-s}\int_0^\infty x^{-s}F(x)\,dx\,ds \tag{2.29}$$

is continuous for $y>0$, $R>0$. Therefore, $\varphi_R(y)$ defines a regular distribution in $\mathfrak{D}'(I)$ being

$$\langle\varphi_R(y),\phi(y)\rangle=\frac{1}{2\pi i}\int_0^\infty\phi(y)\int_{\sigma-iR}^{\sigma+iR}\frac{1}{K(s)}y^{-s}\int_0^\infty x^{-s}F(x)\,dx\,ds\,dy \tag{2.30}$$

for all $\phi\in\mathfrak{D}(I)$.

By the Fubini's theorem we can interchange the order of integration and we can write

$$\langle\varphi_R(y),\phi(y)\rangle=\frac{1}{2\pi i}\int_{\sigma-iR}^{\sigma+iR}\frac{1}{K(s)}\left[\int_0^\infty x^{-s}\langle f(t),Z_\rho^\nu(xt)\rangle dx\right]\int_0^\infty y^{-s}\phi(y)\,dy\,ds \tag{2.31}$$

By Lemma 2.5, we get

$$\begin{aligned}\langle\varphi_R(y),\phi(y)\rangle &= \frac{1}{2\pi i}\int_{\sigma-iR}^{\sigma+iR}\frac{1}{K(s)}\left\langle f(t),t^{s-1}\int_0^\infty u^{-s}Z_\rho^\nu(u)\,du\right\rangle\int_0^\infty y^{-s}\phi(y)dy\,ds\\ &= \frac{1}{2\pi i}\int_{\sigma-iR}^{\sigma+iR}\langle f(t),t^{s-1}\rangle\int_0^\infty y^{-s}\phi(y)\,dy\,ds.\end{aligned} \tag{2.32}$$

Then Lemma 2.6(i) permits us to obtain

$$\langle\varphi_R(y),\phi(y)\rangle = \left\langle f(t),\frac{1}{2\pi i}\int_{\sigma-iR}^{\sigma+iR}t^{s-1}\int_0^\infty y^{-s}\phi(y)\,dy\,ds\right\rangle. \tag{2.33}$$

Finally, interchanging the order of integration and using Lemma 2.6(ii) we achieve

$$\langle\varphi_R(y),\phi(y)\rangle = \left\langle f(t),\frac{1}{\pi}\int_0^\infty\left(\frac{u}{y}\right)^\sigma\phi(y)\frac{\sin\left(R\log(u/y)\right)}{u\log\left(u/y\right)}dy\right\rangle \longrightarrow \langle f(t),\phi(t)\rangle\Big\rangle \tag{2.34}$$

as $R\to\infty$. With this, Theorem 2.7 is demonstrated. □

By Theorem 2.7 the following uniqueness theorem is deduced.

Theorem 2.8. *Let $f,g\in\mathfrak{E}'(I)$ and $\nu\in\mathbb{C}$. If $(K_\nu^\rho f)(x)=(K_\nu^\rho g)(x)$, for $x>0$, then $f=g$ in the sense of an equality in $\mathfrak{D}'(I)$.*

Proof. It is sufficient to see that for each $\phi\in\mathfrak{D}(I)$, $\langle f(t)-g(t),\phi(t)\rangle=0$.

This fact is established immediately since,

$$\begin{aligned}&\langle f(t)-g(t),\phi(t)\rangle\\ &\quad=\left\langle\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\sigma-iR}^{\sigma+iR}\frac{1}{K(s)}y^{-s}\int_0^\infty x^{-s}\big((K_\nu^\rho f)(x)-(K_\nu^\rho g)(x)\big)\,dx\,ds,\phi(y)\right\rangle=0,\end{aligned} \tag{2.35}$$

where σ and $K(s)$ are as in Theorem 2.7. □

In the following theorem we obtain another inversion formula, in which appears a generalization of an operator of Post-Widder type, being obtained as a particular case an inversion formula for the Laplace and Meijer transformation.

Theorem 2.9. *Let $f\in\mathfrak{E}'(I)$ and $\operatorname{Re}\nu\geq\rho/2-1$. If $F(x)$ denotes the K_ν^ρ generalized transform of f, then*

$$\lim_{k\to\infty}\left\langle B_{\rho,\nu,k}^{*}F\left(\frac{\rho k}{y}\right),\phi(y)\right\rangle=\langle f(t),\phi(t)\rangle \tag{2.36}$$

for each $\phi\in\mathfrak{D}(I)$, where $B_{\rho,\nu,k}^{*}$ is the adjoint operator of $B_{\rho,\nu,k}$.

Proof. Let $f\in\mathfrak{E}'(I)$ and $F(x)=\langle f(t),Z_\rho^\nu(xt)\rangle$, for $x>0$.

By Proposition 2.1 and (1.6) we have

$$\begin{aligned}B_{\rho,\nu,k}^{*}F\left(\frac{\rho k}{y}\right)&=C_1y^{-\rho k-1}\left\langle f(t),B_{\nu,\rho}^{k}Z_\rho^\nu\left(\frac{\rho kt}{y}\right)\right\rangle\\ &=C_1y^{-\rho k-1}\left\langle f(t),\rho^k(\rho kt)^{\rho k}Z_\rho^\nu\left(\frac{\rho kt}{y}\right)\right\rangle,\end{aligned} \tag{2.37}$$

C_1 being introduced in Section 1.

Moreover, $B^*_{\rho,\nu,k}F(\rho k/y)$ defines a regular distribution in $\mathfrak{D}'(I)$ by

$$\left\langle B^*_{\rho,\nu,k}F\left(\frac{\rho k}{y}\right),\phi(y)\right\rangle = \int_0^\infty B^*_{\rho,\nu,k}F\left(\frac{\rho k}{y}\right)\phi(y)\,dy, \quad \forall\phi\in\mathfrak{D}(I). \tag{2.38}$$

We must see that

$$\left\langle B^*_{\rho,\nu,k}F\left(\frac{\rho k}{y}\right),\phi(y)\right\rangle = \left\langle f(t),C_1t^{\rho k}\int_0^\infty y^{-\rho k-1}Z^\nu_\rho\left(\frac{\rho kt}{y}\right)\phi(y)\,dy\right\rangle \tag{2.39}$$

for every $\phi\in\mathfrak{D}(I)$.

Consider $0<a<b<\infty$ such that $\phi(x)=0$, $x\notin[a,b]$, $\phi\in\mathfrak{D}(I)$.

We denote by $\{y_{m,l}\}_{m=0}^{l}$ a partition of $[a,b]$ with $d_l=y_{m,l}-y_{m,l-1}$, $m=1,2,\dots,l$, then we achieve

$$\begin{aligned}\int_0^\infty & B^*_{\rho,\nu,k}F\left(\frac{\rho k}{y}\right)\phi(y)\,dy\\ &=\lim_{l\to\infty}d_l\sum_{m=0}^{l}B^*_{\rho,\nu,k}F\left(\frac{\rho k}{y_{m,l}}\right)\phi(x_{m,l})\\ &=\lim_{l\to\infty}\left\langle f(t),t^{\rho k}C_1\,\rho^k(\rho k)^{\rho k}d_l\sum_{m=0}^{l}y_{m,l}^{-\rho k-1}Z^\nu_\rho\left(\frac{\rho kt}{y_{m,l}}\right)\phi(x_{m,l})\right\rangle.\end{aligned} \tag{2.40}$$

Therefore we must see that

$$\lim_{l\to\infty}d_l\sum_{m=0}^{l}y_{m,l}^{-\rho k-1}Z^\nu_\rho\left(\frac{\rho kt}{y_{m,l}}\right)\phi(y_{m,l}) = \int_a^b y^{-\rho k-1}Z^\nu_\rho\left(\frac{\rho kt}{y}\right)\phi(y)\,dy \tag{2.41}$$

in the sense of the convergence in $\mathfrak{E}(I)$.

Consider $K\subset(0,\infty)$ compact and $r\in\mathbb{N}$.

Then we have

$$\begin{aligned}&\frac{d^r}{dt^r}\left(d_l\sum_{m=0}^{l}y_{m,l}^{-\rho k-1}Z^\nu_\rho\left(\frac{\rho kt}{y_{m,l}}\right)\phi(y_{m,l})-\int_a^b y^{-\rho k-1}Z^\nu_\rho\left(\frac{\rho kt}{y}\right)\phi(y)\,dy\right)\\ &\quad=d_l\sum_{m=0}^{l}y_{m,l}^{-\rho k-1}\frac{d^r}{dt^r}\left(Z^\nu_\rho\left(\frac{\rho kt}{y_{m,l}}\right)\right)\phi(y_{m,l})-\int_a^b y^{-\rho k-1}\frac{d^r}{dt^r}\left(Z^\nu_\rho\left(\frac{\rho kt}{y}\right)\right)\phi(y)\,dy.\end{aligned} \tag{2.42}$$

Hence, since the function $y^{-\rho k-1}(d^r/dt^r)(Z^\nu_\rho(\rho kt/y))\phi(y)$ is uniformly continuous for $(x,y)\in[a,b]\times K$, then

$$\lim_{l\to\infty}d_l\sum_{m=0}^{l}y_{m,l}^{-\rho k-1}Z^\nu_\rho\left(\frac{\rho kt}{y_{m,l}}\right)\phi(x_{m,l}) = \int_a^b y^{-\rho k-1}Z^\nu_\rho\left(\frac{\rho kt}{y}\right)\phi(y)\,dy \tag{2.43}$$

uniformly in $y\in K$. Thus (2.39) is demonstrated.

On the other hand, making a change of variables we obtain

$$\int_0^\infty B^*_{\rho,\nu,k}F\left(\frac{\rho k}{y}\right)\phi(y)\,dy = \left\langle g(t), C_1 t^{-\rho k-1}\int_0^\infty u^{\rho k} Z^\nu_\rho\left(\frac{\rho k u}{t}\right)\psi(u)\,du\right\rangle \tag{2.44}$$

with $g(t) = (1/t)f(1/t)$ and $\psi(u) = (1/u)\phi(1/u)$.

To complete the demonstration we see that

$$\lim_{k\to\infty} C_1 t^{-\rho k-1}\int_0^\infty u^{\rho k} Z^\nu_\rho\left(\frac{\rho k u}{t}\right)\psi(u)\,du = \psi(t) \tag{2.45}$$

in the sense of the convergence in $\mathfrak{E}(I)$.

Now, according to (1.9) we have

$$\begin{aligned}
&\int_0^\infty u^{\rho k-((2\nu-\rho)/(2\rho+2))-(1/(\rho+1))} Z^\nu_\rho\left(\frac{\rho k u}{t}\right)du \\
&\quad= \left(\frac{t}{\rho k}\right)^{(\rho+1)k-((2\nu-\rho)/(2\rho+2))-(1/(\rho+1))+1}\rho^{-1}\Gamma\left(\rho k-\frac{2\nu-\rho}{2\rho+2}-\frac{1}{\rho+1}+1\right) \\
&\qquad\cdot\Gamma\left(\frac{\rho k-((2\nu-\rho)/(2\rho+2))-(1/(\rho+1))+1+\nu}{\rho}\right).
\end{aligned} \tag{2.46}$$

Then, we can deduce that

$$\begin{aligned}
&\frac{d^r}{dt^r}\left(C_1 t^{-\rho k-1}\int_0^\infty u^{\rho k} Z^\nu_\rho\left(\frac{\rho k u}{t}\right)\psi(u)\,du-\psi(t)\right) \\
&\quad= C_1\, t^{-\rho k-1}\int_0^\infty u^{\rho k-((2\nu-\rho)/(2\rho+2))-(1/(\rho+1))} Z^\nu_\rho\left(\frac{\rho k u}{t}\right) \\
&\qquad\cdot\left(u^{((2\nu-\rho)/(2\rho+2))+(1/(\rho+1))}\left(\frac{u}{t}\frac{d}{du}\right)^r\psi(u)-t^{((2\nu-\rho)/(2\rho+2))+(1/(\rho+1))}\frac{d^r}{dt^r}\psi(t)\right)du
\end{aligned} \tag{2.47}$$

for every $r\in\mathbb{N}$.

Therefore by (1.5) we have that for all $t\in K$, $K\subset(0,\infty)$, K compact, there exists a constant $C>0$ such that

$$\begin{aligned}
&\left|\frac{d^r}{dt^r}\left(C\cdot t^{-(\rho+1)k-1}\int_0^\infty Z^\nu_\rho\left(\frac{\rho k t}{y}\right)z^{(\rho+1)k}\psi(z)dz-\psi(t)\right)\right| \\
&\quad\le C\frac{(\rho k)^{\rho k-k+1-1/(\rho+1)}}{((\rho+1)k)!}t^{-\rho k-\rho/(\rho+1)}\frac{\rho}{\rho+1}\int_0^\infty z^{(\rho+1)k}e^{-\alpha_2(\rho k z/t)} \\
&\qquad\cdot\left|z^{(2\nu-\rho)/(2\rho+2)}\left(\frac{z}{t}\frac{d}{dz}\right)^r\psi(z)-t^{(2\nu-\rho)/(2\rho+2)}\frac{d^r}{dt^r}\psi(t)\right|dz.
\end{aligned} \tag{2.48}$$

We divided the last integral in this way

$$\begin{aligned}
C_1\int_0^\infty &= C_1\left(\int_0^{((\rho+1)k)^{1/\rho}(1-\eta)^{(\rho+1)/\rho}t(\rho k)^{-1}}+\int_{((\rho+1)k)^{1/\rho}(1-\eta)^{(\rho+1)/\rho}t(\rho k)^{-1}}^{((\rho+1)k)^{1/\rho}(1+\eta)^{(\rho+1)/\rho}t(\rho k)^{-1}}\right. \\
&\qquad\left.+\int_{((\rho+1)k)^{1/\rho}(1+\eta)^{(\rho+1)/\rho}t(\rho k)^{-1}}^\infty\right) = I_1(t,k)+I_2(t,k)+I_3(t,k)
\end{aligned} \tag{2.49}$$

with $\eta>0$ and $C_1 = C((\rho k)^{\rho k-k+1-1/(\rho+1)}/((\rho+1)k)!)t^{-\rho k-\rho/(\rho+1)}\rho/(\rho+1)$.

For each $t \in K$, we obtain

$$\begin{aligned}
&|I_1(t,k)| \\
&\le C\frac{(\rho k)^{\rho k-k+1-1/(\rho+1)}}{((\rho+1)k)!}t^{-\rho k-\rho/(\rho+1)}\frac{\rho}{\rho+1} \\
&\quad\cdot\left(\int_0^{((\rho+1)k)^{1/\rho}(1-\eta)^{(\rho+1)/\rho}t(\rho k)^{-1}} u^{\rho k+(2\nu-\rho)/(2\rho+2)}e^{-(\rho k u/t)^{\rho/(\rho+1)}}\left|\left(\frac{u}{t}\frac{d}{du}\right)^r\psi(u)\right|du\right. \\
&\qquad+\int_0^{((\rho+1)k)^{1/\rho}(1-\eta)^{(\rho+1)/\rho}t(\rho k)^{-1}} u^{\rho k-1/(\rho+1)}e^{-(\rho k u/t)^{\rho/(\rho+1)}} \\
&\qquad\qquad\left.\cdot\, t^{(2\nu-\rho)/(2\rho+2)+1/(\rho+1)}\left|\frac{d^r}{dt^r}\psi(t)\right|du\right)
\end{aligned} \tag{2.50}$$

and making an appropriate change of variables, we obtain

$$|I_1(t,k)| \le C\frac{((\rho+1)k)^{(\rho+1)k+1}}{((\rho+1)k)!}\int_0^{1-\eta}x^{(\rho+1)k}e^{-(\rho+1)kx}\,dx, \tag{2.51}$$

C being a suitable constant and $\operatorname{Re}\nu \ge \rho/2-1$.

Now, using [10, Theorem (5b), page 288] we achieve

$$I_1(t,k) \longrightarrow 0, \tag{2.52}$$

as $k \to \infty$ uniformly in $t \in K$.

Proceeding in a similar way we obtain that

$$I_3(t,k) \longrightarrow 0, \tag{2.53}$$

as $k \to \infty$ uniformly in $t \in K$.

Then, it remains $I_2(t,k)$, for this, using the mean value theorem we can write

$$\left|u^{(2\nu-\rho)/(2\rho+2)+1/(\rho+1)}\left(\frac{u}{t}\frac{d}{du}\right)^r\psi(u) - t^{(2\nu-\rho)/(2\rho+2)+1/(\rho+1)}\frac{d^r}{dt^r}\psi(t)\right| \le C_2|t-u| \tag{2.54}$$

for $u,t \in (0,\infty)$ and C_2 a suitable constant.

Then if $u \in (((\rho+1)k)^{1/\rho}(1-\eta)^{(\rho+1)/\rho}t(\rho k)^{-1},\ ((\rho+1)k)^{1/\rho}(1+\eta)^{(\rho+1)/\rho}t(\rho k)^{-1})$ we have

$$|I_2(t,k)| \le C_3\frac{((\rho+1)k)^{(\rho+1)k+1}}{((\rho+1)k)!}\int_{1-\eta}^{1+\eta}x^{(\rho+1)k}e^{-(\rho+1)kx}\,dx, \tag{2.55}$$

applying [10, Theorem (5b), page 287] we obtain the desired result.

Therefore we can deduce that

$$\lim_{k\to\infty}\left\langle C^*_{\rho,\nu,k}F\left(\frac{\rho k}{y}\right),\phi(y)\right\rangle = \langle g(t),\psi(t)\rangle = \langle f(t),\phi(t)\rangle \quad \forall\phi\in\mathfrak{D}(I). \tag{2.56}$$

□

References

[1] B. Bonilla, M. Rivero, J. Rodríguez, J. Trujillo, and A. A. Kilbas, *Bessel-type functions and Bessel-type integral transforms on spaces $\mathfrak{F}_{p,\mu}$ and $\mathfrak{F}'_{p,\mu}$*, Integral Transform. Spec. Funct. **8** (1999), no. 1-2, 13–30. CMP 1 730 610. Zbl 937.33004.

[2] D. I. Cruz and J. Rodríguez, *A weighted norm inequality for a generalization of the K transform*, J. Inst. Math. Comput. Sci. Comput. Sci. Ser. **9** (1998), no. 1, 25–29. CMP 1 702 625.

[3] ———, *Abelian and Tauberian theorems for the Krätzel transform*, Pure Appl. Math. Sci. **49** (1999), no. 1-2, 45–60. MR 2000d:44003.

[4] S. A. Emara and H. P. Heinig, *Weighted norm inequalities for the Hankel- and K-transformations*, Proc. Roy. Soc. Edinburgh Sect. A **103** (1986), no. 3-4, 325–333. MR 88f:44002. Zbl 622.44004.

[5] E. Krätzel, *Integral transformations of Bessel-type*, Generalized Functions and Operational Calculus (Proc. Conf., Varna, 1975), Bulgar. Acad. Sci., Sofia, 1979, pp. 148–155. MR 80m:44015. Zbl 403.44003.

[6] E. Krätzel and H. Menzer, *Verallgemeinerte Hankel-funktionen*, Publ. Math. Debrecen **18** (1971), 139–147 (1972) (German). MR 46#9410. Zbl 247.33014.

[7] S. P. Malgonde and R. K. Saxena, *An inversion formula for the distributional H-transformation*, Math. Ann. **258** (1981/82), no. 4, 409–417. MR 83e:44002. Zbl 486.46032.

[8] G. L. N. Rao and L. Debnath, *A generalized Meijer transformation*, Int. J. Math. Math. Sci. **8** (1985), no. 2, 359–365. MR 87k:46091. Zbl 597.46036.

[9] L. Schwartz, *Théorie des Distributions*, Publications de l'Institut de Mathematique de l'Universite de Strasbourg, no. 9-10. Nouvelle edition, entierement corrigee, refondue et augmentee, Hermann, Paris, 1966 (French). MR 35#730. Zbl 149.09501.

[10] D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series, vol. 6, Princeton University Press, Princeton, NJ, 1941. MR 3,232d. Zbl 063.08245.

[11] A. H. Zemanian, *Distribution Theory and Transform Analysis. An Introduction to Generalized Functions, with Applications*, McGraw-Hill Book Co., New York, Toronto, London, Sydney, 1965. MR 31#1556. Zbl 127.07201.

[12] ———, *A distributional K transformation*, SIAM J. Appl. Math. 14 (1966), 1350–1365; errata, ibid. **15** (1966), 765. MR 35#2094. Zbl 154.13902.

[13] ———, *Generalized Integral Transformations*, Pure and Applied Mathematics, vol. 18, Interscience Publishers [John Wiley & Sons, Inc.], New York, London, Sydney, 1968. MR 54#10991. Zbl 181.12701.

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