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DEGREE OF APPROXIMATION OF CONJUGATE OF A FUNCTION BELONGING TO $\mathrm{Lip}(\xi(t),p)$ CLASS BY MATRIX SUMMABILITY MEANS OF CONJUGATE FOURIER SERIES

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ABSTRACT. We determine the degree of approximation of conjugate of a function belonging to $\mathrm{Lip}(\xi(t),p)$ class by matrix summability means of a conjugate series of a Fourier series.

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1. Introduction. Bernstein [2], Alexits [1], Sahney and Goel [14], and Chandra [4] have determined the degree of approximation of a function belonging to $\mathrm{Lip}\,\alpha$ by $(C,1)$, (C,δ), (N,p_n), and $(\overline{N},p_n)$ means of its Fourier series. Working in the same direction Sahney and Rao [15] and Khan [6] have studied the degree of approximation of functions belonging to $\mathrm{Lip}(\alpha,p)$ by (N,p_n) and (N,p,q) means, respectively. The (N,p,q) summability reduces to (N,p_n) summability for $q_n=1$ for all n, and to $(\overline{N},q_n)$ means when $p_n=1$ for all n. After quite a good amount of work on degree of approximation of function by different summability means of its Fourier series, for the first time in 1981, Qureshi [12, 13] discussed the degree of approximation of conjugate of a function belonging to $\mathrm{Lip}\,\alpha$ and $\mathrm{Lip}(\alpha,p)$ by (N,p_n) means of conjugate Fourier series. But nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function belonging to $\mathrm{Lip}(\xi(t),p)$ class by matrix means of conjugate Fourier series. The $\mathrm{Lip}(\xi(t),p)$ class is a generalization of $\mathrm{Lip}\,\alpha$ and $\mathrm{Lip}(\alpha,p)$. Matrix means includes as special cases the method of $(C,1)$, (C,δ), (N,p_n), $(\overline{N},p_n)$, and (N,p,q) means. In an attempt to make an advance study in this direction, we, in this paper, establish a theorem on degree of approximation of conjugate of a function of $\mathrm{Lip}(\xi(t),p)$ class by matrix summability means of conjugate series of a Fourier series then both the results of Qureshi [12, 13] come out as particular cases of our theorem.

2. Definitions and notations. Let f be periodic with period 2π and integrable in the Lebesgue sense. Let its Fourier series be given by

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{2.1}$$

The conjugate series of (2.1) is given by

$$\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)=\sum_{n=1}^{\infty}B_n(x). \tag{2.2}$$

Let $\{p_n\}$ be a nonnegative nonincreasing generating sequence for (N,p_n) method such that

$$P_n = P(n) = p_0 + p_1 + p_2 + \cdots + p_n \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \tag{2.3}$$

Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Silverman Toeplitz [16], that is,

$$\begin{gathered} \sum_{k=0}^{n} a_{n,k} \longrightarrow 1, \quad \text{as } n \longrightarrow \infty, \qquad a_{n,k} = 0, \quad \text{for } k > n, \\ \sum_{k=0}^{n} |a_{n,k}| \le M, \quad \text{a finite constant.} \end{gathered} \tag{2.4}$$

Let $\sum_{m=0}^{\infty} u_m$ be an infinite series such that

$$s_k = u_0 + u_1 + u_2 + \cdots + u_k = \sum_{m=0}^{k} u_m, \tag{2.5}$$

that is, s_k denotes the kth partial sum of the series $\sum_{m=0}^{\infty} u_m$.

The sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{n} a_{n,k} s_k = \sum_{k=0}^{n} a_{n,n-k} s_{n-k} \tag{2.6}$$

defines the sequence $\{t_n\}$ of matrix means of the sequence $\{s_n\}$ generated by the sequence of the coefficients $(a_{n,k})$. The series $\sum u_n$ is said to be summable to the sum "S" by matrix method if $\lim_{n\to\infty} t_n$ exists and equal to S (see Zygmund [17]) and we write

$$t_n \longrightarrow S(T), \quad \text{as } n \longrightarrow \infty. \tag{2.7}$$

2.1. Particular cases. Seven important cases of matrix means are

(1) $(C,1)$ means when $a_{n,k} = 1/(n+1)$.

(2) Harmonic means when $a_{n,k} = 1/(n-k+1)\log n$.

(3) (C,δ) means when $a_{n,k} = \binom{n-k+\delta-1}{\delta-1} / \binom{n+\delta}{\delta}$.

(4) (H,p) means when $a_{n,k} = 1/\log^{p-1}(n+1)\prod_{q=0}^{p-1}\log^q(k+1)$.

(5) Nörlund means [11] when $a_{n,k} = p_{n-k}/P_n$ where $P_n = \sum_{k=0}^{n} p_k$, $q_n = 1$ for all n.

(6) Riesz means $(\overline{N},p_n)$ [5] when $a_{n,k} = p_k/P_n$, $q_n = 1$ for all n.

(7) Generalized Nörlund mean (N,p,q) [3] when $a_{n,k} = p_{n-k}q_k/R_n$ where

$$R_n = \sum_{k=0}^{n} p_k q_{n-k}. \tag{2.8}$$

In particular cases (5), (6), and (7), $\{p_n\}$ and $\{q_n\}$ are two nonnegative monotonic nonincreasing sequences of real constants.

We define the norm

$$\|f\|_p = \left\{ \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p}, \quad p \ge 1 \tag{2.9}$$

and let the degree of approximation be given by (see Zygmund [17])

$$E_n(f) = \operatorname*{Min}_{T_n} \|f - T_n\|_p, \tag{2.10}$$

where $T_n(x)$ is some nth degree trigonometric polynomial.

A function $f \in \operatorname{Lip}\alpha$ if

$$f(x+t) - f(x) = O(t^\alpha), \quad \text{for } 0 < \alpha \le 1 \tag{2.11}$$

and $f \in \operatorname{Lip}(\alpha, p)$ if

$$\left\{ \int_0^{2\pi} |f(x+t) - f(x)|^p \, dx \right\}^{1/p} = O(t^\alpha), \quad 0 < \alpha \le 1, \ p \ge 1 \tag{2.12}$$

(see [10, Definition 5.38]).

Given a positive increasing function $\xi(t)$ and an integer $p > 1$, then $f(x) \in \operatorname{Lip}(\xi(t), p)$ if

$$\left\{ \int_0^{2\pi} |f(x+t) - f(x)|^p \, dx \right\}^{1/p} = O(\xi(t)), \quad \text{(see [8])}. \tag{2.13}$$

In case $\xi(t) = t^\alpha$, we notice that $\operatorname{Lip}(\xi(t), p)$ class coincides with known $\operatorname{Lip}(\alpha, p)$ class [10].

We use the following notations:

$$\begin{gathered} \psi(t) = f(x+t) - f(x-t), \\ A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}, \\ \tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right], \\ \overline{K}_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin t/2}. \end{gathered} \tag{2.14}$$

3. Known theorems. Qureshi [12] has proved the following theorem.

Theorem 3.1. *If the sequence $\{p_n\}$ satisfies the following conditions:*

$$n|p_n| < C|P_n|, \qquad \sum_{k=1}^{n} k|p_k - p_{k-1}| < C|P_n|, \tag{3.1}$$

then the degree of approximation of a function $\tilde{f}(x)$, conjugate to a periodic function f with period 2π and belonging to the class $\operatorname{Lip}\alpha$, $0 < \alpha < 1$ *by Nörlund means of its conjugate series, is given by*

$$|\tilde{f}(x) - \tilde{t}_n(x)| = O\left(\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}} \right), \tag{3.2}$$

where $\tilde{t}_n(x)$ are the (N, p_n) means of series (2.2).

Qureshi [13] has proved another theorem in the following form.

Theorem 3.2. *If $f(x)$ is periodic and belongs to the class* $\mathrm{Lip}(\alpha,p)$ *for* $0<\alpha\le 1$, *and if the sequence* $\{p_n\}$ *is as defined in (2.3) with other requirements therein and if*

$$\int_1^n \left(\frac{(p(y)^q)}{y^{q\alpha+2-\delta q-q}}\right) = O\left(\frac{p(n)}{n^{\alpha-1/q-\delta-1}}\right), \tag{3.3}$$

then

$$\|\tilde{t}_n-\tilde{f}\|_p = O\left(\frac{1}{n^{\alpha-1/p}}\right), \tag{3.4}$$

where $\tilde{t}_n$ *are the* (N,p_n) *means of the series (2.2) and* $1/p+1/q=1$ *such that* $1\le p\le\infty$.

4. Main theorem. Our object of this paper is to prove the following theorem.

Theorem 4.1. *If* $T=(a_{n,k})$ *is an infinite regular triangular matrix such that the elements* $a_{n,k}$ *is nonnegative and nondecreasing with* k, *then the degree of approximation of a function* $\tilde{f}(x)$, *conjugate to a* 2π*-periodic function* f *belonging to* $\mathrm{Lip}(\xi(t),p)$ *class by matrix summability means of its conjugate series is given by*

$$\|\tilde{t}_n(x)-\tilde{f}(x)\| = O\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right) \tag{4.1}$$

provided $\xi(t)$ *satisfies the following conditions:*

$$\left\{\int_0^{1/n}\left(\frac{t\,|\psi(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O\left(\frac{1}{n}\right), \tag{4.2}$$

$$\left\{\int_{1/n}^{\pi}\left(\frac{t^{-\delta}\psi(t)}{\xi(t)}\right)^p dt\right\}^{1/p} = O(n^{\delta}), \tag{4.3}$$

where δ *is an arbitrary number such that* $q(1-\delta)-1>0$, *conditions (4.2) and (4.3) hold uniformly in* x,

$$\tilde{t}_n(x) = \sum_{k=0}^{n} a_{n,n-k}\overline{s}_{n-k}(x), \tag{4.4}$$

that is, matrix means of conjugate Fourier series (2.2), $1/p+1/q=1$, *such that* $1\le p\le\infty$ *and*

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{1}{2}t\,dt. \tag{4.5}$$

5. Lemmas. For the proof of our theorem the following lemmas are required.

Lemma 5.1 [9]. *If* $a_{n,k}$ *is nonnegative and nonincreasing with* k, *then for* $0\le a\le b\le\infty$, $a\le t\le\pi$ *and any* n,

$$\left|\sum_{k=a}^{b} a_{n,n-k}e^{i(n-k)t}\right| = O(A_{n,\tau}). \tag{5.1}$$

Lemma 5.2. *Under the conditions of Theorem 4.1 on* $(a_{n,k})$ *for* $0 < 1/n \le t \le \pi$,

$$\overline{K}_n(t) = O\left(\frac{A_{n,\tau}}{t}\right). \tag{5.2}$$

Proof. Since for $0 < 1/n \le t \le \pi$, $\sin(t/2) < t$, therefore for $t > 0$ and $\tau \le n$, we have,

$$\begin{aligned}
|\overline{K}(t)| &= \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\cos(n-k-1/2)t}{\sin(t/2)}\right| \\
&\le \left|\frac{1}{2\pi}\operatorname{Re}\sum_{k=0}^{n}\frac{a_{n,n-k}e^{i(n-k-1/2)t}}{\sin(t/2)}\right| \\
&= O\left(\frac{1}{t}\left|\sum_{k=0}^{n} a_{n,n-k}e^{i(n-k)t}\right| |e^{-it/2}|\right) \\
&= O\left(\frac{1}{t}\left|\sum_{k=0}^{n} a_{n,n-k}e^{i(n-k)t}\right|\right) \\
&= O\left(\frac{A_{n,\tau}}{t}\right)
\end{aligned} \tag{5.3}$$

by Lemma 5.1. □

6. Proof of the main theorem. Let $\overline{s}_n(x)$ denote the nth partial sum of series (2.2), then, following [7], we have

$$\overline{s}_n(x) - \left(-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{1}{2}t\,dt\right) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos(n+1/2)t}{\sin t/2}\,dt. \tag{6.1}$$

Now

$$\begin{aligned}
&\sum_{k=0}^{n} a_{n,n-k}\left\{\overline{s}_{n-k}(x) - \left(-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{1}{2}t\,dt\right)\right\} \\
&\qquad = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\sum_{k=0}^{n} a_{n,n-k}\frac{\cos(n-k-1/2)t}{\sin(t/2)}\,dt
\end{aligned} \tag{6.2}$$

or

$$\begin{aligned}
\tilde{t}_n(x) - \tilde{f}(x) &= \frac{1}{2\pi}\int_0^{\pi}\psi(t)\sum_{k=0}^{n} a_{n,n-k}\frac{\cos(n-k-1/2)t}{\sin(t/2)}\,dt \\
&= \int_0^{\pi}\psi(t)\overline{K}_n(t)\,dt \\
&= \int_0^{1/n}\psi(t)\overline{K}_n(t)\,dt + \int_{1/n}^{\pi}\psi(t)\overline{K}_n(t)\,dt \\
&= I_1 + I_2.
\end{aligned} \tag{6.3}$$

Applying Hölder's inequality and the fact that $\psi(t) = W(\operatorname{Lip}\xi(t), p)$, we get

$$
\begin{aligned}
I_1 &= \int_0^{1/n} \psi(t)\overline{K}_n(t)\,dt \\
&\le O\left[\int_0^{1/n}\left\{\frac{t\,|\psi(t)|}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_0^{1/n}\left\{\frac{\overline{K}_n(t)\xi(t)}{t}\right\}^q dt\right]^{1/q} \\
&= o\left(\frac{1}{n}\right)\left[\int_0^{1/n}\left\{\frac{\xi(t)}{t}\frac{1}{2\pi}\left|\sum_{k=0}^{n} a_{n,n-k}\frac{\cos(n-k-1/2)t}{\sin(t/2)}\right|\right\}^q dt\right]^{1/q} \quad \text{by (4.2)} \\
&= O\left(\frac{1}{n}\right)\left[\int_0^{1/n}\left\{\frac{\xi(t)}{t}\sum_{k=0}^{n}\frac{a_{n,n-k}}{t}\right\}^q\right]^{1/q} \\
&= O\left(\frac{1}{n}\right)\left[\int_0^{1/n}\left(\frac{\xi(t)}{t^2}\right)^q dt\right]^{1/q} \\
&= O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{1}{n}\right)\right)\left[\int_1^{1/n}\frac{dt}{t^{2q}}\right]^{1/q} \quad \text{by mean value theorem} \\
&= O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{1}{n}\right)\right)\left[\left\{\frac{t^{-2q+1}}{-2q+1}\right\}_1^{1/n}\right]^{1/q} \\
&= O\left(\frac{\xi(1/n)}{n}\right)O\left(n^{2-1/q}\right) \\
&= O\left(\xi\left(\frac{1}{n}\right)n^{1-1/q}\right), \\
I_1 &= O\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right) \quad \left(\text{since } \frac{1}{p}+\frac{1}{q}=1\right).
\end{aligned}
\tag{6.4}
$$

Consider I_2

$$
\begin{aligned}
I_2 &= \left[\int_{1/n}^{\pi}\left\{\frac{|t^{-\delta}\psi(t)|}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_{1/n}^{\pi}\left\{\frac{\overline{K}_n(t)\xi(t)}{t^{-\delta}}\right\}^q dt\right]^{1/q} \\
&= O\left[\int_{1/n}^{\pi}\left\{\frac{t^{-\delta}\,|\psi(t)|}{\xi(t)}\right\}^p dt\right]^{1/p} O\left[\int_{1/n}^{\pi}\left\{\frac{\xi(t)A_{n,\tau}}{t^{-\delta+1}}\right\}^q dt\right]^{1/q} \quad \text{by Lemma 5.2} \\
&= O(n^{\delta})\cdot O\left[\int_{1/n}^{\pi}\left\{\frac{\xi(t)A_{n,\tau}}{t^{-\delta+1}}\right\}^q dt\right]^{1/q} \quad \text{by condition (4.3)} \\
&= O(n^{\delta})\cdot O\left[\int_{1/n}^{n}\left\{\frac{\xi(1/y)A_{n,[y]}}{y^{\delta-1}}\right\}^q \frac{dy}{y^2}\right]^{1/q} \\
&= O(n^{\delta})\cdot O\left(\xi\left(\frac{1}{n}\right)A_{n,n}\right)\left[\int_1^{n}\left\{\frac{dy}{y^{q(\delta-1)+2}}\right\}\right]^{1/q} \quad \text{by mean value theorem}
\end{aligned}
$$

$$
\begin{aligned}
&= O\left(n^{\delta}\xi\left(\frac{1}{n}\right)\right)\left[\left\{\frac{y^{-q(\delta-1)-1}}{-q(\delta-1)-1}\right\}_1^n\right]^{1/q} \\
&= O\left(n^{\delta}\xi\left(\frac{1}{n}\right)\right)O(n^{-\delta+1-1/q}) \\
&= O\left(\xi\left(\frac{1}{n}\right)n^{1-1/q}\right) \\
I_2 &= O\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right) \quad \left(\text{since } \frac{1}{p}+\frac{1}{q}=1\right).
\end{aligned}
\tag{6.5}
$$

By combining (6.3), (6.4), and (6.5) we have

$$
|\tilde{t}_n(x)-\tilde{f}(x)| = O\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right), \tag{6.6}
$$

therefore

$$
\begin{aligned}
\|\tilde{t}_n(x)-\tilde{f}(x)\|_p &= O\left[\left\{\int_0^{2\pi}\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right)^p dx\right\}^{1/p}\right] \\
&= O\left[\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right)\left(\int_0^{2\pi} dx\right)^{1/p}\right] \\
&= O\left[\xi\left(\frac{1}{n}\right)n^{1/p}\right].
\end{aligned}
\tag{6.7}
$$

This completes the proof of the theorem. □

7. Applications. The following corollaries can be derived from the main theorem.

Corollary 7.1. *If $\xi(t)=t^{\alpha}$, $0<\alpha\le 1$, then the degree of approximation of a function $\tilde{f}(x)$, conjugate to 2π-periodic function f belonging to the class* $\mathrm{Lip}(\alpha,p)$ *is given by*

$$
|\tilde{t}_n(x)-\tilde{f}(x)| = O\left(\frac{1}{n^{\alpha-1/p}}\right). \tag{7.1}
$$

Proof. We have

$$
\|\tilde{t}_n(x)-\tilde{f}(x)\|_p = O\left\{\int_0^{2\pi}|\tilde{t}_n(x)-\tilde{f}(x)|^p dx\right\}^{1/p} \tag{7.2}
$$

or

$$
\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right)^p = O\left\{\int_0^{2\pi}|\tilde{t}_n(x)-\tilde{f}(x)|^p dx\right\}^{1/p} \tag{7.3}
$$

or

$$
O(1) = O\left\{\int_0^{2\pi}|\tilde{t}_n(x)-\tilde{f}(x)|^p dx\right\}^{1/p} O\left(\frac{1}{\xi(1/n)n^{1/p}}\right). \tag{7.4}
$$

Hence

$$
|\tilde{t}_n(x)-\tilde{f}(x)| = O\left[\xi\left(\frac{1}{n}\right)n^{1/p}\right] \tag{7.5}
$$

for if not the right-hand side will be $O(1)$, therefore

$$|\tilde{t}_n(x)-\tilde{f}(x)| = O\left[\left(\frac{1}{n}\right)^{\alpha} n^{1/p}\right] = O\left(\frac{1}{n^{\alpha-1/p}}\right). \tag{7.6}$$

This completes the proof. □

Corollary 7.2. *If $p\to\infty$ in Corollary 7.1, then for $0<\alpha<1$,*

$$|\tilde{t}_n(x)-\tilde{f}(x)|_p = O\left(\frac{1}{n^{\alpha}}\right). \tag{7.7}$$

Remark 7.3. An independent proof of Corollary 7.1 can be derived along the same lines as the theorem.

8. Particular cases. (1) If $a_{n,k}=p_{n-k}/p_n$, $\xi(t)=t^{\alpha}$, $0<\alpha<1$, $p\to\infty$ and using $1/n^{\alpha}\le 1/p_n\sum_{k=1}^{n}p_k/k^{\alpha+1}$ (see [14, Lemma 1]), then the result of Qureshi [12] becomes the particular case of the main theorem.

(2) The result of Qureshi [13] becomes the particular case of our theorem if $(a_{n,k})$ is defined as in case (1) and $\xi(t)=t^{\alpha}$, $0<\alpha\le 1$.

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