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# ON $Q$-ALGEBRAS

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ABSTRACT. We introduce a new notion, called a $Q$-algebra, which is a generalization of the idea of $BCH/BCI/BCK$-algebras and we generalize some theorems discussed in $BCI$-algebras. Moreover, we introduce the notion of "quadratic" $Q$-algebra, and show that every quadratic $Q$-algebra $(X;*,e)$, $e \in X$, has a product of the form $x * y = x - y + e$, where $x, y \in X$ when $X$ is a field with $|X| \geq 3$.

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**1. Introduction.** Imai and Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras (see [4, 5]). It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [2, 3] Hu and Li introduced a wide class of abstract algebras: $BCH$-algebras. They have shown that the class of $BCI$-algebras is a proper subclass of the class of $BCH$-algebras. Neggers and Kim (see [8]) introduced the notion of $d$-algebras, that is, (I) $x * x = e$; (IX) $e * x = e$; (VI) $x * y = e$ and $y * x = e$ imply $x = y$, which is another useful generalization of $BCK$-algebras, after which they investigated several relations between $d$-algebras and $BCK$-algebras, as well as other relations between $d$-algebras and oriented digraphs. At the same time, Jun, Roh, and Kim [6] introduced a new notion, called a $BH$-algebra, that is, (I) $x * x = e$; (II) $x * e = x$; (VI) $x * y = e$ and $y * x = e$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras, and they showed that there is a maximal ideal in bounded $BH$-algebras. We introduce a new notion, called a $Q$-algebra, which is a generalization of $BCH/BCI/BCK$-algebras and generalize some theorems from the theory of $BCI$-algebras. Moreover, we introduce the notion of "quadratic" $Q$-algebra, and obtain the result that every quadratic $Q$-algebra $(X;*,e)$, $e \in X$, is of the form $x * y = x - y + e$, where $x, y \in X$ and $X$ is a field with $|X| \geq 3$, that is, the product is linear in a special way.

**2. $Q$-algebras.** A *$Q$-algebra* is a nonempty set $X$ with a constant 0 and a binary operation "$*$" satisfying axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

For brevity we also call $X$ a *$Q$-algebra*. In $X$ we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$. Recently, Ahn and Kim [1] introduced the notion of $QS$-algebras. A $Q$-algebra $X$ is said to be a *$QS$-algebra* if it satisfies the additional relation:

(IV) $(x * y) * (x * z) = z * y$, for any $x, y, z \in X$.

**Example 2.1.** Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} := \{nz \mid z \in \mathbb{Z}\}$ where $n \in \mathbb{Z}$. Then $(\mathbb{Z}; -, 0)$ and $(n\mathbb{Z}; -, 0)$ are $Q$-algebras, where "$-$" is the usual subtraction of integers.

**Example 2.2.** Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 0 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 |

Then $(X; *, 0)$ is a $Q$-algebra, which is not a $BCH/BCI/BCK$-algebra.

Neggers and Kim [7] introduced the related notion of $B$-algebra, that is, algebras $(X; *, 0)$ which satisfy (I) $x * x = 0$; (II) $x * 0 = x$; (V) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$. It is easy to see that $B$-algebras and $Q$-algebras are different notions. For example, Example 2.2 is a $Q$-algebra, but not a $B$-algebra, since $(3 * 2) * 1 = 0 \neq 3 = 3 * (1 * (0 * 2))$. Consider the following example. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 3 | 4 | 5 |
| 1 | 1 | 0 | 2 | 4 | 5 | 3 |
| 2 | 2 | 1 | 0 | 5 | 3 | 4 |
| 3 | 3 | 4 | 5 | 0 | 2 | 1 |
| 4 | 4 | 5 | 3 | 1 | 0 | 2 |
| 5 | 5 | 3 | 4 | 2 | 1 | 0 |

Then $(X; *, 0)$ is a $B$-algebra (see [7]), but not a $Q$-algebra, since $(5 * 3) * 4 = 3 \neq 4 = (5 * 4) * 3$.

**Proposition 2.3.** *If $(X; *, 0)$ is a $Q$-algebra, then*
(VII) $(x * (x * y)) * y = 0$, *for any $x, y \in X$.*

**Proof.** By (I) and (III), $(x * (x * y)) * y = (x * y) * (x * y) = 0$. □

We now investigate some relations between $Q$-algebras and $BCH$-algebras (also $BCK/BCI$-algebras). The following theorems are easily proven, and we omit their proofs.

**Theorem 2.4.** *Every BCH-algebra $X$ is a $Q$-algebra. Every $Q$-algebra $X$ satisfying condition (VI) is a BCH-algebra.*

**Theorem 2.5.** *Every $Q$-algebra satisfying condition (IV) and (VI) is a BCI-algebra.*

**Theorem 2.6.** *Every $Q$-algebra $X$ satisfying conditions (V), (VI), and*
(VIII) $(x*y)*x=0$ *for any $x,y\in X$, is a BCK-algebra.*

**Theorem 2.7.** *Every $Q$-algebra $X$ satisfying $x*(x*y)=x*y$ for all $x,y,z\in X$, is a trivial algebra.*

**Proof.** Putting $x=y$ in the equation $x*(x*y)=x*y$, we obtain $x*0=0$. By (II) $x=0$. Hence $X$ is a trivial algebra. □

The following example shows that a $Q$-algebra may not satisfy the associative law.

**Example 2.8.** (a) Let $X:=\{0,1,2\}$ with the table as follows:

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 2 | 1 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 1 | 0 |

Then $X$ is a $Q$-algebra, but associativity does not hold, since $(0*1)*2=0\neq 1=0*(1*2)$.

(b) Let $\mathbb{Z}$ and $\mathbb{R}$ be the set of all integers and real numbers, respectively. Then $(\mathbb{Z};-,0)$ and $(\mathbb{R};\div,1)$ are nonassociative $Q$-algebras where "$-$" is the usual subtraction and "$\div$" is the usual division.

**Theorem 2.9.** *Every $Q$-algebra $(X;*,0)$ satisfying the associative law is a group under the operation "$*$".*

**Proof.** Putting $x=y=z$ in the associative law $(x*y)*z=x*(y*z)$ and using (I) and (II), we obtain $0*x=x*0=x$. This means that 0 is the zero element of $X$. By (I), every element $x$ of $X$ has as its inverse the element $x$ itself. Therefore $(X;*)$ is a group. □

## 3. The $G$-part of $Q$-algebras.

In this section, we investigate the properties of the $G$-part in $Q$-algebras.

**Lemma 3.1.** *If $(X;*,0)$ is a $Q$-algebra and $a*b=a*c$, $a,b,c\in X$, then $0*b=0*c$.*

**Proof.** By (I) and (II) $(a*b)*a=(a*a)*b=0*b$ and $(a*c)*a=(a*a)*c=0*c$. Since $a*b=a*c$, $0*b=0*c$. □

**Definition 3.2.** Let $(X;*,0)$ be a $Q$-algebra. For any nonempty subset $S$ of $X$, we define

$$G(S):=\{x\in S\mid 0*x=x\}. \tag{3.1}$$

In particular, if $S=X$ then we say that $G(X)$ is the *$G$-part* of $X$.

**Corollary 3.3.** *A left cancellation law holds in $G(X)$.*

**Proof.** Let $a,b,c\in G(X)$ with $a*b=a*c$. By Lemma 3.1, $0*b=0*c$. Since $b,c\in G(X)$, we obtain $b=c$. □

**Proposition 3.4.** *Let $(X;*,0)$ be a $Q$-algebra. Then $x \in G(X)$ if and only if $0*x \in G(X)$.*

**Proof.** If $x \in G(X)$, then $0*x = x$ and $0*(0*x) = 0*x$. Hence $0*x \in G(X)$.

Conversely, if $0*x \in G(x)$, then $0*(0*x) = 0*x$. By applying Corollary 3.3, we obtain $0*x = x$. Therefore $x \in G(X)$. □

For any $Q$-algebra $(X;*,0)$, the set

$$B(X) := \{x \in X \mid 0*x = 0\} \tag{3.2}$$

is called the *$p$-radical* of $X$. If $B(X) = \{0\}$, then we say that $X$ is a *$p$-semisimple* $Q$-algebra. The following property is obvious.

(IX) $G(X) \cap B(X) = \{0\}$.

**Proposition 3.5.** *If $(X;*,0)$ is a $Q$-algebra and $x,y \in X$, then*

$$y \in B(X) \Longleftrightarrow (x*y)*x = 0. \tag{3.3}$$

**Proof.** By (I) and (III) $(x*y)*x = (x*x)*y = 0*y = 0$ if and only if $y \in B(X)$. □

**Definition 3.6.** Let $(X;*,0)$ be a $Q$-algebra and $I(\neq \emptyset) \subseteq X$. The set $I$ is called an *ideal* of $X$ if for any $x,y,z \in X$,

(1) $0 \in I$,

(2) $x*y \in I$ and $y \in I$ imply $x \in I$.

Obviously, $\{0\}$ and $X$ are ideals of $X$. We call $\{0\}$ and $X$ the *zero ideal* and the *trivial ideal* of $X$, respectively. An ideal $I$ is said to be *proper* if $I \neq X$.

In Example 2.2 the set $I := \{0,1,2\}$ is an ideal of $X$.

**Proposition 3.7.** *Let $(X;*,0)$ be a $Q$-algebra. Then $B(X)$ is an ideal of $X$.*

**Proof.** Since $(0*0)*0 = 0$, by Proposition 3.5, $0 \in B(X)$. Let $x*y \in B(X)$ and $y \in B(X)$. Then by Proposition 3.5, $((x*y)*x)*(x*y) = 0$. By (III), $((x*y)*(x*y))*x = 0*x = 0$. Hence $x \in B(X)$. Therefore $B(X)$ is an ideal of $X$. □

**Proposition 3.8.** *If $S$ is a subalgebra of a $Q$-algebra $(X;*,0)$, then $G(X) \cap S = G(S)$.*

**Proof.** It is obvious that $G(X) \cap S \subseteq G(S)$. If $x \in G(S)$, then $0*x = x$ and $x \in S \subseteq X$. Then $x \in G(X)$ and so $x \in G(X) \cap S$, which proves the proposition. □

**Theorem 3.9.** *Let $(X;*,0)$ be a $Q$-algebra. If $G(X) = X$, then $X$ is $p$-semisimple.*

**Proof.** Assume that $G(X) = X$. By (X), $\{0\} = G(X) \cap B(X) = X \cap B(X) = B(X)$. Hence $X$ is $p$-semisimple. □

**Theorem 3.10.** *If $(X;*,0)$ is a $Q$-algebra of order 3, then $|G(X)| \neq 3$, that is, $G(X) \neq X$.*

**Proof.** For the sake of convenience, let $X = \{0,a,b\}$ be a $Q$-algebra. Assume that $|G(X)| = 3$, that is, $G(X) = X$. Then $0*0 = 0$, $0*a = a$, and $0*b = b$. From $x*x = 0$ and $x*0 = x$, it follows that $a*a = 0$, $b*b = 0$, $a*0 = a$, and $b*0 = b$. Now let $a*b = 0$. Then 0, $a$, and $b$ are candidates of the computation. If $b*a = 0$, then

$a*b=0=b*a$ and so $(a*b)*a=(b*a)*a$. By (III), $(a*a)*b=(b*a)*a$. Hence $0*b=0*a$. By the cancellation law in $G(X)$, $b=a$, a contradiction. If $b*a=a$, then $a=b*a=(0*b)*a=(0*a)*b=a*b=0$, a contradiction. For the case $b*a=b$, we have $b=b*a=(0*b)*a=(0*a)*b=a*b=0$, which is also a contradiction. Next, if $a*b=a$, then $(a*(a*b))*b=(a*a)*b=0*b=b\neq 0$. This leads to the conclusion that Proposition 2.3 does not hold, a contradiction. Finally, let $a*b=b$. If $b*a=0$, then $b=a*b=(0*a)*b=(0*b)*a=b*a=0$, a contradiction. If $b*a=a$, $b=a*b=(0*a)*b=(0*b)*a=b*a=0$, a contradiction. For the case $b*a=b$, we have $a=0*a=(b*b)*a=(b*a)*b=b*b=0$, which is again a contradiction. This completes the proof. □

**Proposition 3.11.** *If $(X;*,0)$ is a $Q$-algebra of order 2, then in every case the $G$-part $G(X)$ of $X$ is an ideal of $X$.*

**Proof.** Let $|X|=2$. Then either $G(X)=\{0\}$ or $G(X)=X$. In either case, $G(X)$ is an ideal of $X$. □

**Theorem 3.12.** *Let $(X;*,0)$ be a $Q$-algebra of order 3. Then $G(X)$ is an ideal of $X$ if and only if $|G(X)|=1$.*

**Proof.** Let $X:=\{0,a,b\}$ be a $Q$-algebra. If $|G(X)|=1$, then $G(X)=\{0\}$ is the trivial ideal of $X$.

Conversely, assume that $G(X)$ is an ideal of $X$. By Theorem 3.10, we know that either $|G(X)|=1$ or $|G(X)|=2$. Suppose that $|G(X)|=2$. Then either $G(X)=\{0,a\}$ or $G(X)=\{0,b\}$. If $G(X)=\{0,a\}$, then $b*a\notin G(X)$ because $G(X)$ is an ideal of $X$. Hence $b*a=b$. Then $a=0*a=(b*b)*a=(b*a)*b=b*b=0$, which is a contradiction. Similarly, $G(X)=\{0,b\}$ leads to a contradiction. Therefore $|G(X)|\neq 2$ and so $|G(X)|=1$. □

**Definition 3.13.** An ideal $I$ of a $Q$-algebra $(X;*,0)$ is said to be *implicative* if $(x*y)*z\in I$ and $y*z\in I$, then $x*z\in I$, for any $x,y,z\in X$.

**Theorem 3.14.** *Let $(X;*,0)$ be a $Q$-algebra and let $I$ be an implicative ideal of $X$. Then $I$ contains the $G$-part $G(X)$ of $X$.*

**Proof.** If $x\in G(X)$, then $(0*x)*x=x*x=0\in I$ and $x*x=0\in I$. Since $I$ is implicative, it follows that $x=0*x\in I$. Hence $G(X)\subseteq I$. □

**Definition 3.15.** Let $X$ and $Y$ be $Q$-algebras. A mapping $f:X\to Y$ is called a *homomorphism* if

$$f(x*y)=f(x)*f(y),\quad \forall x,y\in X. \tag{3.4}$$

A homomorphism $f$ is called a *monomorphism* (resp., *epimorphism*) if it is injective (resp., surjective). A bijective homomorphism is called an *isomorphism*. Two $Q$-algebras $X$ and $Y$ are said to be *isomorphic*, written by $X\cong Y$, if there exists an isomorphism $f:X\to Y$. For any homomorphism $f:X\to Y$, the set $\{x\in X\mid f(x)=0\}$ is called the *kernel* of $f$, denoted by $\operatorname{Ker}(f)$ and the set $\{f(x)\mid x\in X\}$ is called the *image* of $f$, denoted by $\operatorname{Im}(f)$. We denote by $\operatorname{Hom}(X,Y)$ the set of all homomorphisms of $Q$-algebras from $X$ to $Y$.

**Proposition 3.16.** *Suppose that $f : X \to X'$ is a homomorphism of $Q$-algebras. Then*
(1) $f(0) = 0'$,
(2) *$f$ is isotone, that is, if $x * y = 0$, $x, y \in X$, then $f(x) * f(y) = 0'$.*

**Proof.** Since $f(0) = f(0 * 0) = f(0) * f(0) = 0'$, (1) holds. If $x, y \in X$ and $x \le y$, that is, $x * y = 0$, then by (1), $f(x) * f(y) = f(x * y) = f(0) = 0'$. Hence $f(x) \le f(y)$, proving (2). □

**Theorem 3.17.** *Let $(X; *, 0)$ and $(X; *', 0')$ be $Q$-algebras and let $B$ be an ideal of $Y$. Then for any $f \in \mathrm{Hom}(X, Y)$, $f^{-1}(B)$ is an ideal of $X$.*

**Proof.** By Proposition 3.16(1), $0 \in f^{-1}(B)$. Assume that $x * y \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $f(x) * f(y) = f(x * y) \in B$. It follows from the fact that $B$ is an ideal of $Y$ that $f(x) \in B$, that is, $x \in f^{-1}(B)$. This means that $f^{-1}(B)$ is an ideal of $X$. The proof is complete. □

Since $\{0'\}$ is an ideal of $X'$, $\mathrm{Ker}(f) = f^{-1}(\{0'\})$ for any $f \in \mathrm{Hom}(X, Y)$. Hence we obtain the following corollary.

**Corollary 3.18.** *The kernel* $\mathrm{Ker}(f)$ *is an ideal of $X$.*

**4. The quadratic $Q$-algebras.** Let $X$ be a field with $|X| \ge 3$. An algebra $(X; *)$ is said to be *quadratic* if $x * y$ is defined by $x * y := a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6$, where $a_1, \dots, a_6 \in X$, for any $x, y \in X$. A quadratic algebra $(X; *)$ is said to be *quadratic $Q$-algebra* (resp., *$QS$-algebra*) if it satisfies conditions (I), (II), and (III) (resp., (IV)).

**Theorem 4.1.** *Let $X$ be a field with $|X| \ge 3$. Then every quadratic $Q$-algebra $(X; *, e)$, $e \in X$, has the form $x * y = x - y + e$ where $x, y \in X$.*

**Proof.** Define

$$x * y := Ax^2 + Bxy + Cy^2 + Dx + Ey + F. \tag{4.1}$$

Consider (I).

$$e = x * x = (A + B + C)x^2 + (D + E)x + F. \tag{4.2}$$

Let $x := 0$ in (4.2). Then we obtain $F = e$. Hence (4.1) turns out to be

$$x * y = Ax^2 + Bxy + Cy^2 + Dx + Ey + e. \tag{4.3}$$

If $y := x$ in (4.3), then

$$e = x * x = (A + B + C)x^2 + (D + E)x + e, \tag{4.4}$$

for any $x \in X$, and hence we obtain $A + B + C = 0 = D + E$, that is, $E = -D$ and $B = -A - C$. Hence (4.3) turns out to be

$$x * y = (x - y)(Ax - Cy + D) + e. \tag{4.5}$$

Let $y := e$ in (4.5). Then by (II) we have

$$x = x * e = (x - e)(Ax - Ce + D) + e, \tag{4.6}$$

that is, $(Ax - Ce + D - 1)(x - e) = 0$. Since $X$ is a field, either $x - e = 0$ or $Ax - Ce + D - 1 = 0$. Since $|X| \geq 3$, we have $Ax - Ce + D - 1 = 0$, for any $x \neq e$ in $X$. This means that $A = 0, 1 - D + Ce = 0$. Thus (4.5) turns out to be

$$x * y = (x - y) + C(x - y)(e - y) + e. \tag{4.7}$$

To satisfy condition (III) we consider $(x * y) * z$ and $(x * z) * y$.

$$\begin{aligned}(x * y) * z &= (x * y - z) + C(x * y - z)(e - z) + e \\ &= (x - y - z) + C(x - y)(e - z) + 2e \\ &\quad + C\big[(x - y) + C(x - y)(e - y) + (e - z)\big](e - z) \\ &= (x - y - z) + C(x - y)(2e - y - z) + 2e \\ &\quad + C^2(x - y)(e - y)(e - z) + C(e - z)^2.\end{aligned} \tag{4.8}$$

Interchange $y$ with $z$ in (4.8). Then

$$\begin{aligned}(x * z) * y = (x - z - y) &+ C(x - z)(2e - z - y) + 2e \\ &+ C^2(x - z)(e - z)(e - y) + C(e - y)^2.\end{aligned} \tag{4.9}$$

By (4.8) and (4.9) we obtain

$$0 = (x * y) * z - (x * z) * y = C^2(e - y)(e - z)(z - y). \tag{4.10}$$

Since $X$ is a field with $|X| \geq 3$, we obtain $C = 0$. This means that every quadratic $Q$-algebra $(X; *, e)$, has the form $x * y = x - y + e$ where $x, y \in X$, completing the proof. □

**Example 4.2.** Let $\mathbb{R}$ be the set of all real numbers. Define $x * y := x - y + \sqrt{2}$. Then $(\mathbb{R}; *, \sqrt{2})$ is a quadratic $Q$-algebra.

**Example 4.3.** Let $\mathcal{K} := \mathrm{GF}(p^n)$ be a Galois field. Define $x * y := x - y + e$, $e \in \mathcal{K}$. Then $(\mathcal{K}; *, e)$ is a quadratic $Q$-algebra.

**Theorem 4.4.** *Let $X$ be a field with $|X| \geq 3$. Then every quadratic $Q$-algebra on $X$ is a (quadratic) $QS$-algebra.*

**Proof.** Let $(X; *, e)$ be a quadratic $Q$-algebra. Then $x * y = x - y + e$ for any $x, y \in X$, and hence

$$\begin{aligned}(x * y) * (x * z) &= (x - y + e) * (x - z + e) \\ &= (x - y + e) - (x - z + e) + e \\ &= z - y + e = z * y,\end{aligned} \tag{4.11}$$

completing the proof. □

**Remark 4.5.** Usually a nonquadratic $Q$-algebra need not be a $QS$-algebra. See the following example.

**Example 4.6.** Consider the $Q$-algebra $(X;*,0)$ in Example 2.2. This algebra is not a $QS$-algebra, since $(3*1)*(3*2)=3\neq 0=2*1$.

**Corollary 4.7.** *Let $X$ be a field with $|X|\geq 3$. Then every quadratic $Q$-algebra on $X$ is a BCI-algebra.*

**Proof.** It is an immediate consequences of Theorems 2.5 and 4.4. □

**Theorem 4.8.** *Let $X$ be a field with $|X|\geq 3$. Then every quadratic $Q$-algebra $(X;*,e)$ is $p$-semisimple. Furthermore, if* $\operatorname{char}(X)\neq 2$, *then $G(X)=B(X)$.*

**Proof.** Notice that $B(X)=\{x\in X\mid e*x=e\}=\{x\in X\mid e-x+e=e\}=\{x\in X\mid e-x=0\}=\{e\}$, that is, $(X;*,e)$ is $p$-semisimple. Also, if $\operatorname{char}(X)\neq 2$, then 2 is invertible in $X$ and $G(X)=\{x\in X\mid e*x=x\}=\{x\in X\mid e-x+e=x\}=\{x\in X\mid 2e=2x\}=\{x\in X\mid e=x\}=\{e\}$. Of course, if $\operatorname{char}(X)=2$, then $2e=2x=0$ for all $x\in X$, whence $G(X)=X$. □

This shows that there is a large class of examples of $p$-semisimple $QS$-algebras obtained as quadratic $Q$-algebras.

**Theorem 4.9.** *Let $X$ be a field with $|X|\geq 3$. Then every quadratic $Q$-algebra on $X$ is isomorphic to every other such algebra defined on $X$.*

**Proof.** Let $x*y:=x-y+e_1$ and $x*'y:=x-y+e_2$, where $e_1,e_2\in X$. Let $\pi(x):=x+(e_2-e_1)$, for all $x\in X$. Then $\pi(x*y)=[(x-y)+e_1]+(e_2-e_1)=(x-y)+e_2=(x+(e_2-e_1))+(y+(e_2-e_1))+e_2=\pi(x)*'\pi(y)$, whence the fact that $\pi^{-1}(x)=x+(e_1-e_2)$ yields the conclusion that $\pi$ is an isomorphism of $Q$-algebras. □

**Theorem 4.10.** *Let $X$ be a field with $|X|\geq 3$. Then every quadratic $Q$-algebra $(X;*,e)$ determines the abelian group $(X,+)$ via the definition $x+y=x*(e-y)$.*

**Proof.** Note that $x*(e-y)=x-(e-y)+e=x+y$ returns the additive operation of the field $X$, which is an abelian group. □

Not every quadratic $Q$-algebra $(X;*,e)$, $e\in X$, on a field $X$ with $|X|\geq 3$ need be a $BCK$-algebra, since $((x*y)*(x*z))*(z*y)=e+(y-z)\neq e$ in general.

**Problem 4.11.** Construct a cubic $Q$-algebra which is not quadratic. Verify that among such cubic $Q$-algebras there are examples which are not $QS$-algebras. Furthermore, the question whether there are non-$p$-semisimple cubic $Q$-algebras is also of interest.

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