FINITE AG-GROUPOID WITH LEFT IDENTITY
AND LEFT ZERO

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ABSTRACT. A groupoid $G$ whose elements satisfy the left invertive law: $(ab)c = (cb)a$ is known as Abel-Grassman’s groupoid (AG-groupoid). It is a nonassociative algebraic structure midway between a groupoid and a commutative semigroup. In this note, we show that if $G$ is a finite AG-groupoid with a left zero then, under certain conditions, $G$ without the left zero element is a commutative group.

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1. Preliminaries. An Abel-Grassman’s groupoid [6], abbreviated as AG-groupoid, is a groupoid $G$ whose elements satisfy the left invertive law: $(ab)c = (cb)a$. It is also called a left almost semigroup [2, 3, 4, 5]. In [1], the same structure is called left invertive groupoid. In this note we call it AG-groupoid.

It is a nonassociative algebraic structure midway between a groupoid and a commutative semigroup. The structure is medial [5], that is, $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in G$. It has been shown in [5] that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. An element $a_0$ of an AG-groupoid $G$ is called a left (right) zero if $a_0a = a$ ($aa_0 = a$) for all $a \in G$.

Let $a, b, c,$ and $d$ belong to an AG-groupoid with left identity and $ab = cd$. Then it has been shown in [5] that $ba = dc$.

An element $a^{-1}$ of an AG-groupoid with left identity $e$ is called a left inverse if $a^{-1}a = e$. It has been shown in [5] that if $a^{-1}$ is a left inverse of $a$ then it is unique and is also the right inverse of $a$.

If for all $a, b, c$ in an AG-groupoid $G$, $ab = ac$ implies that $b = c$, then $G$ is known as left cancellative. Similarly, if $ba = ca$, implies that $b = c$, then $G$ is called right cancellative. It is known [5] that every left cancellative AG-groupoid is right cancellative but the converse is not true. However, every right cancellative AG-groupoid with left identity is left cancellative.

In this note, we show that if $G$ is a finite AG-groupoid with left identity and a left zero $a_0$, under certain conditions $G \setminus \{a_0\}$ is a commutative group without a left zero.

2. Results. We need the following theorem from [4] for our main result.

**Theorem 2.1** [4]. A cancellative AG-groupoid $G$ is a commutative semigroup if $a(bc) = (cb)a$ for all $a, b, c \in G$. 

We now state and prove our main result.

**Theorem 2.2.** Let \((G, \circ)\) be a finite AG-groupoid with at least two elements. Suppose that it contains a left identity and a left zero \(a_0\). Then \(G^0 = G \setminus \{a_0\}\) is a commutative group under the binary operation \((\circ)\) provided there is another binary operation \((\ast)\) such that

1. \((G, \ast)\) is an AG-groupoid with left identity and left inverses,
2. \(a_0 \ast a = a, \) for all \(a \in G,\)
3. \((a \ast b) \circ c = (a \circ c) \ast (b \circ c),\) for all \(a, b, c \in G,\)
4. \(a \circ b = a_0\) implies that either \(a = a_0\) or \(b = a_0\) for all \(a, b \in G,\)
5. \(a \circ (b \circ c) = (c \circ b) \ast a,\) for all \(a, b, c \in G.\)

**Proof.** Suppose that \(G = \{a_0, a_1, \ldots, a_m\}\), where \(m\) is a positive integer, is an AG-groupoid with left identity under the binary operation \((\circ)\). Let \(e\) be the identity element of \(G\). It is certainly different from \(a_0\) because of (ii) and because \(a_0\) is the left zero under \((\circ)\). The left invertive law together with (iv) implies that \((a \circ a_0) \circ e = (e \circ a_0) \circ a = a_0 \circ a = a_0,\) where \(e \neq a_0\). That is,

\[
a_0 \circ a = a \circ a_0 = a_0. \tag{2.1}
\]

Now consider the subset \(G^0\) of \(G\) which is obtained from it by deleting \(a_0\), so that \(G^0 = \{a_i : i = 1, 2, \ldots, m\}\). In view of the facts that \(a_0\) is a zero under the binary operation \((\circ)\) and it is the left identity under \((\ast)\) and that \((G, \circ)\) is a finite AG-groupoid with left identity. \((G^0, \circ)\) is also a finite AG-groupoid with left identity having the same \(e\) as the left identity in which all elements are distinct.

We now examine whether an element \(a\) of \(G^0\) has an inverse in \(G^0\) under \((\circ)\) or not. We construct a set \(H_k = \{a_k \circ a_1, a_k \circ a_2, \ldots, a_k \circ a_m\}\), where \(a_k \neq a_0\). If \(a_k = a_0\), then because \(a_0\) is a left zero in \(G\) under \((\circ)\) and the left identity under \((\ast)\), the ultimate form of the set \(H_k\) will be \(\{a_0\}\). Therefore it validates our supposition that \(a_k \neq a_0\).

We assert that \(H_k\) contains \(m\) elements. Suppose otherwise and let

\[
a_k \circ a_r = a_k \circ a_s, \tag{2.2}
\]

for some \(r, s = 1, 2, \ldots, m\) and \(r \neq s\). Since \(H_k\) is an AG-groupoid with left identity under \((\circ)\), therefore (2.2) implies that

\[
a_r \circ a_k = a_s \circ a_k, \tag{2.3}
\]

for some \(r, s = 1, 2, \ldots, m\) and \(r \neq s\). Consider now the element \((a_s \ast a_r^{-1}) \circ a_k\), which is certainly an element of \(G\), where \(a_r^{-1}\) is the left inverse of \(a_r\) in \(G\) with respect to \((\ast)\). Now,

\[
(a_s \ast a_r^{-1}) \circ a_k = (a_s \circ a_k) \ast (a_r^{-1} \circ a_k) = (a_r \circ a_k) \ast (a_r^{-1} \circ a_k) = (a_r \circ a_k) \ast (a_r^{-1} \circ a_k) = (a_r \circ a_k) \cdot a_k = a_0 \circ a_k = a_0. \tag{2.4}
\]

Because of (iii), equation (2.3) and the facts that \(a_r^{-1}\) is the inverse of \(a_r\) under \((\ast)\). Thus \((a_s \ast a_r^{-1}) \circ a_k = a_0\). Since \(a_k \neq a_0\), therefore because of (iv), \(a_s \ast a_r^{-1} = a_0\). Next \((a_s \ast a_r^{-1}) \circ a_r = a_0 \ast a_r\) implies that \((a_s \ast a_r^{-1}) \circ a_r = a_r\) because \(a_0\) is the left identity in \(G\) under \((\ast)\). Hence, \(a_r = (a_s \ast a_r^{-1}) \ast a_r = (a_r \ast a_r^{-1}) \ast a_r = a_0 \ast a_s = a_s\), that is, \(a_r = a_s\). Since \(|H_k| = m\), therefore the result \(a_r = a_s\) contradicts our assumption; thus
proving that $H_k$ contains distinct elements. Since $H_k$ is contained in $G^0$ and $|G^0| = m$ we have $H_k = G^0$.

Also, since $G^0$ is an AG-groupoid under $(\circ)$ with the left identity $e$, so is $H_k$ and hence $H_k$ contains the left identity $e$. So, $e$ will be of the form $a_i \circ a_j$, that is, $e = a_i \circ a_j$ implying that $a_i$ is the left inverse of $a_j$ under the binary operation $(\circ)$. But in an AG-groupoid with left identity, if it contains left inverses, every left inverse is a right inverse. Thus $a_j$ is the right inverse of $a_j$ under $(\circ)$.

Since $k = 1, 2, \ldots, m$ has been chosen arbitrarily, we have shown that $G^0$ is an AG-groupoid with left identity and inverses under the binary operation $(\circ)$.

If $a_i, a_j, a_k \in G^0$ such that $a_i \circ a_k = a_j \circ a_k$, then $(a_i \circ a_k) \circ a_k^{-1} = (a_j \circ a_k) \circ a_k^{-1}$ implies that $(a_k^{-1} \circ a_k) \circ a_i = (a_k^{-1} \circ a_k) \circ a_j$ and so $a_i = a_j$. Thus $G^0$ is right cancellative under $(\circ)$. But $G^0$ being right cancellative under $(\circ)$, is left cancellative also, therefore $G^0$ is cancellative. Since $G^0$ is cancellative whose elements satisfy condition (v), therefore by applying Theorem 2.1, we conclude that $G^0$ is a commutative group under $(\circ)$. ☐

**Corollary 2.3.** If $(G, \circ)$ is a finite AG-groupoid with left identity and a left zero $a_0$, then $(G \setminus \{a_0\}, \circ)$ is a cancellative AG-groupoid with left identity and inverses provided there is another binary operation $(\ast)$ such that

(i) $(G, \ast)$ is an AG-groupoid with left identity and left inverses,

(ii) $a_0 \ast a = a$, for all $a \in G$,

(iii) $(a \ast b) \circ c = (a \circ c) \ast (b \circ c)$, for all $a, b, c \in G$,

(iv) $a \circ b = a_0$ implies that either $a = a_0$ or $b = a_0$ for all $a, b \in G$.

**Proof.** The proof is analogous to the proof of Theorem 2.2. ☐

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**References**


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