IJMMS 26:3 (2001) 167–172
PII. S0161171201005257
http://ijmms.hindawi.com

SUBMANIFOLDS OF F-STRUCTURE MANIFOLD SATISFYING $F^K + (-)^{K+1}F = 0$

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(Received 2 May 2000)

ABSTRACT. The purpose of this paper is to study invariant submanifolds of an n-dimensional manifold M endowed with an F-structure satisfying $F^K + (-)^{K+1}F = 0$ and $F^W + (-)^{W+1}F \neq 0$ for $1 < W < K$, where K is a fixed positive integer greater than 2. The case when K is odd (≥ 3) has been considered in this paper. We show that an invariant submanifold $\tilde{M}$, embedded in an F-structure manifold M in such a way that the complementary distribution D_m is never tangential to the invariant submanifold $\Psi(\tilde{M})$, is an almost complex manifold with the induced $\tilde{F}$-structure. Some theorems regarding the integrability conditions of induced $\tilde{F}$-structure are proved.

2000 Mathematics Subject Classification. 53C15, 53C40, 53D10.

1. Introduction. Invariant submanifolds have been studied by Blair et al. [1], Kubo [4], Yano and Okumura [7, 8], and among others. Yano and Ishihara [6] have studied and shown that any invariant submanifold of codimension 2 in a contact Riemannian manifold is also a contact Riemannian manifold. We consider an F-structure manifold M and study its invariant submanifolds. Let F be a nonzero tensor field of the type $(1,1)$ and of class C^∞ on an n-dimensional manifold M such that (see [3])

$$F^K + (-)^{K+1}F = 0, \quad F^W + (-)^{W+1}F \neq 0, \quad \text{for } 1 < W < K, \tag{1.1}$$

where K is a fixed positive integer greater than 2. Such a structure on M is called an F-structure of rank r and of degree K. If the rank of F is constant and $r = r(F)$, then M is called an F-structure manifold of degree $K(\geq 3)$.

Let the operator on M be defined as follows (see [3])

$$\ell = (-)^K F^{K-1}, \qquad m = I + (-)^{K+1} F^{K-1}, \tag{1.2}$$

where I denotes the identity operator on M. For the operators defined by (1.2), we have

$$\ell + m = I, \quad \ell^2 = \ell; \quad m^2 = m. \tag{1.3}$$

For F satisfying (1.1), there exist complementary distribution D_ℓ and D_m corresponding to the projection operators ℓ and m, respectively. If $\operatorname{rank}(F) = $ constant on M, then $\dim D_\ell = r$ and $\dim D_m = (n-r)$. We have the following results (see [3]).

$$F\ell = \ell F = F, \qquad Fm = mF = 0, \tag{1.4a}$$

$$F^{K-1} = (-)^K \ell, \qquad F^{K-1}\ell = -\ell, \qquad F^{K-1}m = 0. \tag{1.4b}$$

Thus F^{K-1} acts on D_ℓ as an almost complex structure and on D_m as a null operator.

2. Invariant submanifolds of F-structure manifold. Let $\tilde{M}$ be a differentiable manifold embedded differentially as a submanifold in an n-dimensional C^∞ Riemannian manifold M with an F-structure and we denote its embedding by $\Psi : \tilde{M} \to M$. Denote by $B : T(\tilde{M}) \to T(M)$ the differential mapping of Ψ, where $d\Psi = B$ is the Jacobson map of Ψ. $T(\tilde{M})$ and $T(M)$ are tangent bundles of $\tilde{M}$ and M, respectively. We call $T(\tilde{M}, M)$ as the set of all vectors tangent to the submanifold $\Psi(\tilde{M})$. It is known that $B : T(\tilde{M}) \to T(\tilde{M}, M)$ is an isomorphism (see [5]).

Let $\tilde{X}$ and $\tilde{Y}$ be two C^∞ vector fields defined along $\Psi(\tilde{M})$ and tangent to $\Psi(\tilde{M})$. Let X and Y be the local extensions of $\tilde{X}$ and $\tilde{Y}$. The restriction of $[X, Y]_{\tilde{M}}$ is determined independently of the choice of these local extensions X and Y. Therefore, we can define

$$[\tilde{X}, \tilde{Y}] = [X, Y]_{\tilde{M}}. \tag{2.1}$$

Since B is an isomorphism, it is easy to see that $[B\tilde{X}, B\tilde{Y}] = B[\tilde{X}, \tilde{Y}]$ for all $\tilde{X}, \tilde{Y} \in T(\tilde{M})$. We denote by G the Riemannian metric tensor of M and put

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(B\tilde{X}, B\tilde{Y}) \quad \forall \tilde{X}, \tilde{Y} \text{ in } T(\tilde{M}), \tag{2.2}$$

where g is the Riemannian metric in M and $\tilde{g}$ is the induced metric of $\tilde{M}$.

Definition 2.1. We say that $\tilde{M}$ is an invariant submanifold of M if

(i) the tangent space $T_p(\Psi(\tilde{M}))$ of the submanifold $\Psi(\tilde{M})$ is invariant by the linear mapping F at each point p of $\Psi(\tilde{M})$,

(ii) for each $\tilde{X} \in T(\tilde{M})$, we have

$$F^{(K-1)/2}(B\tilde{X}) = B\tilde{X}'. \tag{2.3}$$

Definition 2.2. Let $\tilde{F}$ be a $(1,1)$-tensor field defined in $\tilde{M}$ such that $\tilde{F}(\tilde{X}) = \tilde{X}'$ and M is an invariant submanifold, then we have

$$F(B\tilde{X}) = B(\tilde{F}\tilde{X}), \tag{2.4a}$$

$$F^{(K-1)/2}(B\tilde{X}) = B(\tilde{F}^{(K-1)/2}\tilde{X}). \tag{2.4b}$$

We see that there are two cases for any invariant submanifold $\tilde{M}$. We assume the following cases.

Case 1. The distribution D_m is never tangential to $\Psi(\tilde{M})$.

Case 2. The distribution D_m is always tangential to $\Psi(\tilde{M})$.

We will consider Case 1 and assume that no vector field of the type mX, where $X \in T(\Psi(\tilde{M}))$ is tangential to $\Psi(\tilde{M})$.

Theorem 2.3. *An invariant submanifold $\tilde{M}$ is an almost complex manifold if the following two conditions are satisfied:*

(i) *the distribution D_m is never tangential to $\Psi(\tilde{M})$, and*

(ii) *$\tilde{F}$ in $\tilde{M}$ defines an induced almost complex structure satisfying $\tilde{F}^{K-1} = (-)^K I$.*

Proof. Applying $F^{(K-1)/2}$ in (2.4), we obtain

$$F^{(K-1)/2}(F^{(K-1)/2}(B\tilde{X})) = F^{(K-1)/2}(B(\tilde{F}^{(K-1)/2}, \tilde{X})). \tag{2.5}$$

Making use of (2.4a) in (2.5), we get

$$F^{K-1}(B\tilde{X}) = B(\tilde{F}^{K-1}\tilde{X}). \tag{2.6}$$

In order to show that vector fields of the type $B\tilde{X}$ belong to the distribution D_ℓ, we suppose that $m(B\tilde{X}) \neq 0$, then using (1.2) we have

$$m(B\tilde{X}) = (I+(-)^{K+1}F^{K-1})B\tilde{X} = B\tilde{X}+(-)^{K+1}F^{K-1}(B\tilde{X}) \tag{2.7}$$

which in view of (2.6) becomes

$$m(B\tilde{X}) = B\tilde{X}+(-)^{K+1}B(\tilde{F}^{K-1}\tilde{X}) = B[\tilde{X}+(-)^{K+1}\tilde{F}^{K-1}\tilde{X}] \tag{2.8}$$

which, contrary to our assumption, shows that $m(B\tilde{X})$ is tangential to $\Psi(\tilde{M})$. Thus $m(B\tilde{X}) = 0$.

Also, in view of (1.4b), (1.3), and (2.6) we obtain

$$\begin{aligned} B(\tilde{F}^{K-1}\tilde{X}) &= F^{K-1}(B\tilde{X}) = (-)^K\ell(B\tilde{X}) = (-)^K(I-m)B\tilde{X} \\ &= (-)^K B\tilde{X}-(-)^K mB\tilde{X}, \\ B(\tilde{F}^{K-1}\tilde{X}) &= (-)^K B\tilde{X}. \end{aligned} \tag{2.9}$$

Since B is an isomorphism, we get

$$\tilde{F}^{K-1} = (-)^K I. \tag{2.10}$$

Let $\mathcal{F}(M)$ be the ring of real-valued differentiable functions on M, and let $\mathcal{X}(M)$ be the module of derivatives of $\mathcal{F}(M)$. Then $\mathcal{X}(M)$ is Lie algebra over the real numbers and the elements of $\mathcal{X}(M)$ are called vector fields. Then M is equipped with $(1,1)$-tensor field F which is a linear map such that

$$F : \mathcal{X}(M) \longrightarrow \mathcal{X}(M). \tag{2.11}$$

Let M be of degree K and let K be a positive odd integer greater than 2. Then we consider a positive definite Riemannian metric with respect to which D_ℓ and D_m are orthogonal so that

$$g(X,Y) = g(HX,HY)+g(mX,Y), \tag{2.12}$$

where $H = F^{(K-1)/2}$ for all $X,Y \in \mathcal{X}(M)$. □

Definition 2.4. The induced metric $\tilde{g}$ defined by (2.2) is Hermitian if the following is satisfied:

$$\tilde{g}(H\tilde{X},H\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}), \quad \text{where } H = F^{(K-1)/2}. \tag{2.13}$$

Theorem 2.5. *If F-structure manifold has the following two properties, that is,*

(a) *$\tilde{M}$ is an invariant submanifold of F-structure manifold M such that distribution D_m is never tangential to $\Psi(\tilde{M})$,*

(b) *the Riemannian metric g on M is defined by (2.12).*

Then the induced metric $\tilde{g}$ of $\tilde{M}$ defined by (2.2) is Hermitian.

Proof. In view of (2.2) and (2.13) we obtain

$$\tilde{g}(\tilde{F}^{(K-1)/2}\tilde{X},\tilde{F}^{(K-1)/2}\tilde{Y}) = g(B\tilde{F}^{(K-1)/2}\tilde{X},B\tilde{F}^{(K-1)/2}\tilde{Y}). \tag{2.14}$$

Applying (2.4) and (2.12) in (2.14), we get

$$\begin{aligned}\tilde{g}(\tilde{F}^{(K-1)/2}\tilde{X},\tilde{F}^{(K-1)/2}\tilde{Y}) &= g(F^{(K-1)/2}B\tilde{X},F^{(K-1)/2}B\tilde{Y})\\ &= g(B\tilde{X},B\tilde{Y}) - g(mB\tilde{X},B\tilde{Y}).\end{aligned} \tag{2.15}$$

Since the distribution D_m is never tangential to $\Psi(\tilde{M})$, on using (2.2) we get

$$\tilde{g}(\tilde{F}^{(K-1)/2}\tilde{X},\tilde{F}^{(K-1)/2}\tilde{Y}) = g(B\tilde{X},B\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}). \tag{2.16}$$

Now, we consider the second case and assume that the distribution D_m is always tangential to $\Psi(\tilde{M})$. In view of Case 2, we have $m(B\tilde{X}) = B\tilde{X}^*$, where $\tilde{X}^* \in T(\tilde{M})$ for some $\tilde{X}^* \in T(\tilde{M})$.

We define (1,1)-tensor fields $\tilde{m}$ and $\tilde{\ell}$ in $\tilde{M}$ as follows:

$$\tilde{\ell} = (-)^K\tilde{F}^{K-1}, \qquad \tilde{m} = \tilde{I} + (-)^{K+1}\tilde{F}^{K-1}, \tag{2.17a}$$

$$\tilde{m}\tilde{X} = \tilde{X}^*, \qquad m(B\tilde{X}) = B(\tilde{m}\tilde{X}). \tag{2.17b}$$

□

Theorem 2.6. *We have*

$$B(\tilde{\ell}\tilde{X}) = \ell(B\tilde{X}). \tag{2.18}$$

Proof. In view of (2.17a), equation (2.18) assumes the following form:

$$B(\tilde{\ell}\tilde{X}) = B((-)^K\tilde{F}^{K-1}\tilde{X}) = (-)^K B(\tilde{F}^{K-1}\tilde{X}). \tag{2.19}$$

Making use of (2.6) and (2.15) in (2.19), we get

$$B(\tilde{\ell}\tilde{X}) = (-)^K\tilde{F}^{K-1}(B\tilde{X}) = \tilde{\ell}(B\tilde{X}). \tag{2.20}$$

□

Theorem 2.7. *For $\tilde{\ell}$ and $\tilde{m}$ satisfying (2.17a), we have*

$$\tilde{\ell} + \tilde{m} = \tilde{I}, \quad \tilde{\ell}^2 = \tilde{\ell}, \quad \tilde{m}^2 = \tilde{m}. \tag{2.21}$$

Proof. From (1.3) we have $\ell + m = I$, which can be written as $(\ell + m)B\tilde{X} = B\tilde{X}$, thus we have

$$\ell B\tilde{X} + mB\tilde{X} = B\tilde{X} \tag{2.22}$$

which in view of (2.17b) and (2.18) becomes

$$B(\tilde{\ell}\tilde{X}) + B(\tilde{m}\tilde{X}) = B(\tilde{\ell} + \tilde{m})\tilde{X} = B\tilde{X}. \tag{2.23}$$

Therefore $\tilde{\ell} + \tilde{m} = \tilde{I}$ since B is an isomorphism. Proof of the other relations follows in a similar manner. □

Theorem 2.7 shows that $\tilde{\ell}$ and $\tilde{m}$ defined by (2.17a) are complementary projectionoperators on $\tilde{M}$.

Theorem 2.8. *If F-structure manifold has the following property, that is, $\tilde{M}$ is an invariant submanifold of F-structure manifold M such that distribution D_m is always tangential to $\Psi(\tilde{M})$. Then there exists an induced $\tilde{F}$-structure manifold which admits a similar Riemannian metric $\tilde{g}$ satisfying*

$$\tilde{g}(\tilde{X},\tilde{Y}) = \tilde{g}(\tilde{H}\tilde{X},\tilde{H}\tilde{Y}) + \tilde{g}(\tilde{m}\tilde{X}\tilde{Y}). \tag{2.24}$$

Proof. From (2.4b) we get

$$B(\tilde{F}^{(K-1)/2}\tilde{X}) = F^{(K-1)/2}(B\tilde{X}). \tag{2.25}$$

Furthermore,

$$B(\tilde{F}^{K}\tilde{X}) = F^{K}(B\tilde{X}) \tag{2.26}$$

which in view of (1.1) and (2.4a) yields

$$B(\tilde{F}^{K}\tilde{X}) = B(-(-)^{K+1}\tilde{F}\tilde{X}) \tag{2.27}$$

which shows that $\tilde{F}$ defines an $\tilde{F}$-structure manifold which satisfies

$$\tilde{F}^{K} + (-)^{K+1}\tilde{F} = 0. \tag{2.28}$$

In consequence of (2.2), (2.4b), and (2.12) we obtain

$$\begin{aligned} \tilde{g}(\tilde{H},\tilde{X},\tilde{H}\tilde{Y}) + \tilde{g}(\tilde{m}\tilde{X},\tilde{Y}) &= g(B\tilde{H}\tilde{X},B\tilde{H}\tilde{Y}) + g(B\tilde{m}\tilde{X},B\tilde{Y}) \\ &= g(HB\tilde{X},HB\tilde{Y}) + g(mB\tilde{X},B\tilde{Y}) \\ &= g(B\tilde{X},B\tilde{Y}), \quad \text{where } \tilde{H} = \tilde{F}^{(K-1)/2} \end{aligned} \tag{2.29}$$

which in view of the fact that B is an isomorphism gives

$$\tilde{g}(\tilde{H},\tilde{X},\tilde{H}\tilde{Y}) + \tilde{g}(\tilde{m}\tilde{X},\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}). \tag{2.30}$$

□

3. Integrability conditions. The Nijenhuis tensor N of the type (1.2) of F satisfying (1.1) in M is given by (see [2])

$$N(X,Y) = [FX,FY] - F[FX,Y] - F[X,F,Y] + F^2[X,Y], \tag{3.1}$$

and the Nijenhuis tensor $\tilde{N}$ of $\tilde{F}$ satisfying (2.28) in $\tilde{M}$ is given by

$$N(\tilde{X},\tilde{Y}) = [\tilde{F}\tilde{X},\tilde{F}\tilde{Y}] - \tilde{F}[\tilde{F}\tilde{X},\tilde{Y}] - \tilde{F}[\tilde{X}\tilde{F}\tilde{Y}] + \tilde{F}^2[\tilde{X},\tilde{Y}]. \tag{3.2}$$

Theorem 3.1. *The Nijenhuis tensors N and $\tilde{N}$ of M and $\tilde{M}$ given by (3.1) and (3.2) satisfy the following relation:*

$$N(B\tilde{X},B\tilde{Y}) = B\tilde{N}(\tilde{X},\tilde{Y}). \tag{3.3}$$

Proof. We have

$$N(B\tilde{X},B\tilde{Y}) = [F(B\tilde{X}),F(B\tilde{Y})] - F[F(B\tilde{X}),B\tilde{Y}] - F[B\tilde{X},F(B\tilde{Y})] + F^2[B\tilde{X},B\tilde{Y}] \tag{3.4}$$

which in view of (2.4a) becomes

$$\begin{aligned} N(B\tilde{X},B\tilde{Y}) &= B[\tilde{F}\tilde{X},\tilde{F}\tilde{Y}]-F[B(\tilde{F}\tilde{X}),B\tilde{Y}]-F[(B\tilde{X},B\tilde{F}\tilde{Y})]+F^2[B\tilde{X},B\tilde{Y}] \\ &= B[\tilde{F}\tilde{X},\tilde{F}\tilde{Y}]-FB[\tilde{F}\tilde{X},\tilde{Y}]-FB[\tilde{X},\tilde{F}\tilde{Y}]+BF^2[\tilde{X},\tilde{Y}] \qquad (3.5) \\ &= B[\tilde{F}\tilde{X},\tilde{F}\tilde{Y}]-B\tilde{F}[\tilde{F},\tilde{X},\tilde{Y}]-B\tilde{F}[\tilde{X},\tilde{F}\tilde{Y}]+B\tilde{F}^2[\tilde{X},\tilde{Y}]=B\tilde{N}(\tilde{X},\tilde{Y}). \end{aligned}$$ □

Theorem 3.2. *The following identities hold:*

$$\begin{gathered} B\tilde{N}(\tilde{\ell}\tilde{X},\tilde{\ell}\tilde{Y})=N(\tilde{\ell}B\tilde{X},\tilde{\ell}B\tilde{Y}), \qquad B\tilde{N}(\tilde{m}\tilde{X},\tilde{m}\tilde{Y})=N(\tilde{m}B\tilde{X},\tilde{m}B\tilde{Y}), \\ B\{\tilde{m}\tilde{n}(\tilde{X},\tilde{Y})\}=mN(B\tilde{X},B\tilde{Y}). \end{gathered} \qquad (3.6)$$

Proof. The proof of (3.6) follows by virtue of Theorem 3.1, equations (1.4a), (2.4a), (2.17a), (2.17b), and (3.3). □

For $\tilde{F}$ satisfying (2.28), there exists complementary distribution $D_{\tilde{\ell}}$ and $D_{\tilde{m}}$ corresponding to the projection operators $\tilde{\ell}$ and $\tilde{m}$ in $\tilde{M}$ given by (2.17a). Then in view of the integrability conditions of $\tilde{F}$ structure we state the following theorems.

Theorem 3.3. *If D_ℓ is integrable in M, then $D_{\tilde{\ell}}$ is also integrable in $\tilde{M}$. If D_m is integrable in M, then $D_{\tilde{m}}$ is also integrable in $\tilde{M}$.*

Theorem 3.4. *If D_ℓ and D_m are both integrable in M, then $D_{\tilde{\ell}}$ and $D_{\tilde{m}}$ are also integrable in $\tilde{M}$.*

Theorem 3.5. *If F-structure is integrable in M, then the induced structure $\tilde{F}$ is also integrable in $\tilde{M}$.*

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