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PAIRS OF PATHS AND CRITICAL POINTS

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ABSTRACT. Two sufficient conditions are presented, in terms of the values taken by a holomorphic function $f(z)$ on a pair of smooth paths intersecting at a point z_0 in its domain, implying that $f'(z_0) = 0$.

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In the present paper, we present two sufficient conditions expressed in terms of the values taken by a holomorphic function f on a pair of smooth paths intersecting at a point z_0 in the domain of f, with tangent vectors at z_0 linearly independent over $\mathbb{R}$, implying that $f'(z_0) = 0$.

THEOREM 1. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function, where $D \subset \mathbb{C}$ is a domain and let $\gamma, \Gamma : (0,1) \to D$ be two smooth (C^1) paths. Assume the following:*
- (i) *for a certain $z_0 \in D$ and some $t_1, t_2 \in (0,1)$ we have $z_0 = \gamma(t_1) = \Gamma(t_2)$;*
- (ii) *$\gamma'(t_1)$ and $\Gamma'(t_2)$ linearly independent over $\mathbb{R}$ (i.e., non-collinear),*
- (iii) *$|f(z)|$ takes a constant value on the subset $\gamma((0,1)) \cup \Gamma((0,1))$ of D. Then $f'(z_0) = 0$.*

PROOF. Let $f = u + iv$, $\gamma = \gamma_1 + i\gamma_2$, and $\Gamma = \Gamma_1 + i\Gamma_2$, where u, v are real-valued functions while γ_1, γ_2, Γ_1, Γ_2 are real-valued smooth paths. The assumption (iii) can be written as

$$u^2(\gamma(t)) + v^2(\gamma(t)) = u^2(\Gamma(t)) + v^2(\Gamma(t)) = c \tag{1}$$

for any $t \in (0,1)$, where c is some constant. Note first that if $c = 0$, from (1) together with the identity theorem of the holomorphic functions it follows that $f(z) = 0$ for any $z \in D$. This being the case, we assume $c \neq 0$ from now on. We differentiate (1) with respect to t. We then have, for any $t \in (0,1)$,

$$\frac{d}{dt}\left(u^2(\gamma(t)) + v^2(\gamma(t))\right) = 0, \tag{2}$$

that is, by using the chain rule,

$$\begin{aligned} &2u(\gamma(t))u_x(\gamma(t))\gamma_1'(t) + 2u(\gamma(t))u_y(\gamma(t))\gamma_2'(t) \\ &\quad + 2v(\gamma(t))v_x(\gamma(t))\gamma_1'(t) + 2v(\gamma(t))v_y(\gamma(t))\gamma_2'(t) = 0 \end{aligned} \tag{3}$$

together with the similar relation for Γ:

$$\begin{aligned} &2u(\Gamma(t))u_x(\Gamma(t))\Gamma_1'(t) + 2u(\Gamma(t))u_y(\Gamma(t))\Gamma_2'(t) \\ &\quad + 2v(\Gamma(t))v_x(\Gamma(t))\Gamma_1'(t) + 2v(\Gamma(t))v_y(\Gamma(t))\Gamma_2'(t) = 0 \end{aligned} \tag{4}$$

holding also for any $t \in (0,1)$. By using the Cauchy-Riemann equations in (3) and (4), respectively, we get, after a convenient grouping of terms,

$$u(\gamma(t))[u_x(\gamma(t))\gamma_1'(t)-v_x(\gamma(t))\gamma_2'(t)]+v(\gamma(t))[u_x(\gamma(t))\gamma_2'(t)+v_x(\gamma(t))\gamma_1'(t)]=0, \tag{5}$$

$$u(\Gamma(t))[u_x(\Gamma(t))\Gamma_1'(t)-v_x(\Gamma(t))\Gamma_2'(t)]+v(\Gamma(t))[u_x(\Gamma(t))\Gamma_2'(t)+v_x(\Gamma(t))\Gamma_1'(t)]=0, \tag{6}$$

for any $t \in (0,1)$. By specializing $t = t_1$ in (5) and $t = t_2$ in (6), we obtain

$$\begin{aligned} u(z_0)[u_x(z_0)\gamma_1'(t_1)-v_x(z_0)\gamma_2'(t_1)]+v(z_0)[u_x(z_0)\gamma_2'(t_1)+v_x(z_0)\gamma_1'(t_1)]=0,\\ u(z_0)[u_x(z_0)\Gamma_1'(t_2)-v_x(z_0)\Gamma_2'(t_2)]+v(z_0)[u_x(z_0)\Gamma_2'(t_2)+v_x(z_0)\gamma_1'(t_2)]=0. \end{aligned} \tag{7}$$

Since $u^2(z_0)+v^2(z_0) = c \neq 0$, it follows from (7) that

$$(u(z_0), v(z_0)) \neq (0,0) \tag{8}$$

is a nontrivial solution of the linear homogeneous system

$$\begin{aligned} X[u_x(z_0)\gamma_1'(t_1)-v_x(z_0)\gamma_2'(t_1)]+Y[u_x(z_0)\gamma_2'(t_1)+v_x(z_0)\gamma_1'(t_1)] = 0,\\ X[u_x(z_0)\Gamma_1'(t_2)-v_x(z_0)\Gamma_2'(t_2)]+Y[u_x(z_0)\Gamma_2'(t_2)+v_x(z_0)\gamma_1'(t_2)] = 0, \end{aligned} \tag{9}$$

and so

$$\begin{vmatrix} u_x(z_0)\gamma_1'(t_1)-v_x(z_0)\gamma_2'(t_1) & u_x(z_0)\gamma_2'(t_1)+v_x(z_0)\gamma_1'(t_1)\\ u_x(z_0)\Gamma_1'(t_2)-v_x(z_0)\Gamma_2'(t_2) & u_x(z_0)\Gamma_2'(t_2)+v_x(z_0)\gamma_1'(t_2) \end{vmatrix} = 0. \tag{10}$$

By expanding the determinant, equation (10) can be rewritten as

$$(u_x^2(z_0)+v_x^2(z_0))(\gamma_1'(t_1)\Gamma_2'(t_2)-\Gamma_1'(t_2)\gamma_2'(t_1)) = 0. \tag{11}$$

On the other hand, the assumption (iii) can be rewritten as

$$\begin{vmatrix} \gamma_1'(t_1) & \gamma_2'(t_1)\\ \Gamma_1'(t_2) & \Gamma_2'(t_2) \end{vmatrix} \neq 0. \tag{12}$$

Finally, from (11) and (12) it follows that

$$u_x^2(z_0)+v_x^2(z_0) = 0, \tag{13}$$

that is, $u_x(z_0) = v_x(z_0) = 0$. This, together with the Cauchy-Riemann relations [1] implies $u_y(z_0) = v_x(z_0) = 0$ and so $f'(z_0) = 0$. This concludes the proof of Theorem 1. □

The following exercise represents an interesting corollary of Theorem 1.

Corollary 2. *Let $D \subset \mathbb{C}$ be a domain which contains the square $[-1,1]\times[-1,1]$. Assume that $f : D \to \mathbb{C}$ is a holomorphic function with the property that there exists $c \in \mathbb{R}_+^*$ such that*

$$|f(x+i0)| = c = \left|f\left(x+i\sin\left(\frac{1}{x}\right)\right)\right| \tag{14}$$

for any $x \in (0,1)$. Then f is a constant function.

Proof. Let $\gamma, \Gamma : (0,1) \to \mathbb{C}$ defined by

$$\gamma(t) = (t,0), \qquad \Gamma(t) = \left(t, \sin\left(\frac{1}{t}\right)\right), \tag{15}$$

respectively. We have

$$\gamma'(t) = (1,0), \qquad \Gamma'(t) = \left(1, -\frac{1}{t^2}\cos\left(\frac{1}{t}\right)\right), \tag{16}$$

for any $t \in (0,1)$. Consider the sequence

$$t_k = \frac{1}{k\pi} \in (0,1) \tag{17}$$

convergent to 0. This choice of the sequence makes sure that

$$\gamma(t_k) = \Gamma(t_k) = (t_k, 0) \tag{18}$$

for any $k \geq 1$. We also have $\gamma'(t_k) = (1,0)$ and $\Gamma'(t_k) = (1, -k^2(-1)^k\pi^2)$ which implies immediately that $\gamma(t_k)$ and $\Gamma(t_k)$ are linearly independent over $\mathbb{R}$ for any $k \geq 1$. By Theorem 1,

$$f'(t_k + i0) = 0 \tag{19}$$

holds true for any $k \geq 1$. Since f' is holomorphic and $t_k \to 0 \in D$ ($z = 0 \in D$ is an accumulation point for the zeros of f'), it follows that $f'(z) = 0$ for any $z \in D$, that is, f is a constant on D. □

Another result of similar flavour is the following theorem.

Theorem 3. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic on an open neighborhood V of z_0, and let $\gamma_1, \gamma_2 : (0,1) \to V$ be a pair of C^1 paths such that for some $t_1, t_2 \in (0,1)$, we have $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1)$, $\gamma_2'(t_2)$ are linearly independent over $\mathbb{R}$. We also assume that $f(\gamma_k(t)) \in \mathbb{R}$, $k = 1,2$ for any $t \in (0,1)$. Then, under the above assumptions, $f'(z_0) = 0$. If, in addition, $\arg(\gamma_1')$, $\arg(\gamma_2')$ are constant functions, then there exists a nonnegative integer n and a holomorphic function h defined on some open neighborhood of 0 such that $f(z) = h((z - z_0)^n)$ for $z \in V$.*

Proof. Let ϕ be the angle between $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$. Consider two sequences $\{x_n\}, \{y_n\}$ of numbers from $(0,1)$ such that $\lim_{n\to\infty} x_n = t_1$ while $\lim_{n\to\infty} y_n = t_2$. Then

$$\begin{aligned} f'(z_0) &= \lim_{n\to\infty} \frac{f(\gamma_1(x_n)) - f(\gamma_1(t_1))}{\gamma_1(x_n) - \gamma_1(t_1)} \\ &= \lim_{n\to\infty} \frac{(f(\gamma_1(x_n)) - f(\gamma_1(t_1)))/(x_n - t_1)}{(\gamma_1(x_n) - \gamma_1(t_1))/(x_n - t_1)} \in \mathbb{R}e^{-i\arg(\gamma_1'(t_1))}. \end{aligned} \tag{20}$$

In a similar way, it is shown that

$$f'(z_0) \in \mathbb{R}e^{-i\arg(\gamma_2'(t_2))}. \tag{21}$$

From (20) and (21), together with the assumption that $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ are linearly independent over $\mathbb{R}$, it follows that $f'(z_0)$ has to be zero. This concludes the proof of the first part of the theorem. We assume now that $\arg(\gamma_1')$, $\arg(\gamma_2')$ are constant functions, say $\arg(\gamma_k') = c_k$, $k = 1,2$, where $c_1 \neq c_2$. Then, keeping in mind that $f(\gamma_k(t)) \in \mathbb{R}$, $k = 1,2$ for any $t \in (0,1)$, we see that

$$f'(\gamma_k(t)) \in \mathbb{R}e^{-ic_k} \tag{22}$$

for $k = 1,2$ and $t \in (0,1)$. By induction on r, we can show that

$$f^{(r)}(\gamma_k(t)) \in \mathbb{R}e^{-irc_k} \tag{23}$$

holds true for any nonnegative integer r where $k = 1,2$ and $t \in (0,1)$. Indeed, for $r = 0$ and $r = 1$, equation (23) is already shown. Assuming that (23) is true, by differentiation we get

$$f^{(r+1)}(\gamma_k(t))\gamma_k'(t) \in \mathbb{R}e^{-irc_k}. \tag{24}$$

From (24) and the fact that $\arg(\gamma_k'(t)) = c_k$, it follows that

$$f^{(r+1)}(\gamma_k(t)) \in \mathbb{R}e^{-i(r+1)c_k} \tag{25}$$

which concludes the inductive proof of (23). By specializing $t = t_1$ and then $t = t_2$ in (23), it follows that

$$f^{(r)}(z_0) \in \mathbb{R}e^{-irc_1} \cap \mathbb{R}e^{-irc_2} \tag{26}$$

for any $r = 0,1,2,\dots$. From (26) it follows that, for any given r, either $f^{(r)}(z_0) = 0$ or $e^{ir\phi} \in \mathbb{R}$ (i.e., $r\phi \in 2\pi\mathbb{Z}$). At this moment we distinguish two cases. First, if $\phi/\pi \in \mathbb{R} \setminus \mathbb{Q}$, it follows that $f^{(r)}(z_0) = 0$ for any $r = 0,1,2,\dots$ which implies that $f(z)$ is constant on a neighborhood of z_0 and this being the case the choice $h = \text{constant} = c$ would work. We consider now the second case, when $\phi = m\pi/n$, where $0 < m < n$, $m,n \in \mathbb{Z}_{>0}$, $(m,n) = 1$. From (26) it follows that $f^{(r)}(z_0) = 0$ for any r which is not divisible by n, since in this case $e^{ir\phi} = e^{irm\pi/n} \notin \mathbb{R}$. Therefore, on some neighborhood of z_0 the power series expansion of f has the form

$$f(z) = \sum_{l\le 0} a_{\ln}(z-z_0)^{\ln} = \sum_{l\ge 0} a_{\ln}[(z-z_0)^n]^l. \tag{27}$$

If we denote

$$h(z) := \sum_{l\ge 0} a_{ln} z^l, \tag{28}$$

it follows that h is holomorphic on some neighborhood of 0 and satisfies $f(z) = h((z-z_0)^n)$. This concludes the proof of Theorem 3. □

References

[1] L. V. Ahlfors, *Complex Analysis. An Introduction to the Theory of Analytic Functions of one Complex Variable*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1953. MR 14,857a. Zbl 052.07002.

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