IJMMS 27:1 (2001) 53–63
PII. S0161171201010882
http://ijmms.hindawi.com


# FUZZY $r$-CONTINUOUS AND FUZZY $r$-SEMICONTINUOUS MAPS

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(Received 7 September 2000)

Abstract. We introduce a new notion of fuzzy $r$-interior which is an extension of Chang's fuzzy interior. Using fuzzy $r$-interior, we define fuzzy $r$-semiopen sets and fuzzy $r$-semicontinuous maps which are generalizations of fuzzy semiopen sets and fuzzy semicontinuous maps in Chang's fuzzy topology, respectively. Some basic properties of fuzzy $r$-semiopen sets and fuzzy $r$-semicontinuous maps are investigated.

2000 Mathematics Subject Classification. 54A40.

**1. Introduction.** Chang [2] introduced fuzzy topological spaces. Some authors [3, 5, 6, 7, 8] introduced other definitions of fuzzy topology as generalizations of Chang's fuzzy topology.

In this note, we introduce a new notion of fuzzy $r$-interior in a similar method by which Chattopadhyay and Samanta [4] introduced the notion of fuzzy closure. It determines a fuzzy topology and it is an extension of Chang's fuzzy interior.

Using fuzzy $r$-interior, we define fuzzy $r$-semiopen sets and fuzzy $r$-semicontinuous maps which are generalizations of fuzzy semiopen sets and fuzzy semicontinuous maps in Chang's fuzzy topology, respectively. Some basic properties of fuzzy $r$-semiopen sets and fuzzy $r$-semicontinuous maps are investigated.

**2. Preliminaries.** In this note, let $I$ denote the unit interval $[0,1]$ of the real line and $I_0 = (0,1]$. A member $\mu$ of $I^X$ is called a fuzzy subset of $X$. For any $\mu \in I^X$, $\mu^c$ denotes the complement $1-\mu$. By $\tilde{0}$ and $\tilde{1}$ we denote constant maps on $X$ with value 0 and 1, respectively. All other notation are standard notation of fuzzy set theory.

Recall that a *Chang's fuzzy topology* (see [2]) on $X$ is a family $T$ of fuzzy sets in $X$ which satisfies the following properties:

(1) $\tilde{0}, \tilde{1} \in T$;
(2) if $\mu_1, \mu_2 \in T$, then $\mu_1 \wedge \mu_2 \in T$;
(3) if $\mu_i \in T$ for each $i$, then $\bigvee \mu_i \in T$.

The pair $(X,T)$ is called a *Chang's fuzzy topological space*.

Hence a Chang's fuzzy topology on $X$ can be regarded as a map $T : I^X \to \{0,1\}$ which satisfies the following three conditions:

(1) $T(\tilde{0}) = T(\tilde{1}) = 1$;
(2) if $T(\mu_1) = T(\mu_2) = 1$, then $T(\mu_1 \wedge \mu_2) = 1$;
(3) if $T(\mu_i) = 1$ for each $i$, then $T(\bigvee \mu_i) = 1$.

But fuzziness in the concept of openness of a fuzzy subset is absent in the above Chang's definition of fuzzy topology. So for fuzzifying the openness of a fuzzy subset, some authors [3, 5, 6] gave other definitions of fuzzy topology.

**Definition 2.1** (see [3, 7, 8]). A *fuzzy topology* on $X$ is a map $\mathcal{T}: I^X \to I$ which satisfies the following properties:
(1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$,
(2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$,
(3) $\mathcal{T}(\bigvee \mu_i) \geq \bigwedge \mathcal{T}(\mu_i)$.
The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*.

**Definition 2.2** (see [3]). A *family of closed sets* in $X$ is a map $\mathcal{F}: I^X \to I$ satisfying the following properties:
(1) $\mathcal{F}(\tilde{0}) = \mathcal{F}(\tilde{1}) = 1$,
(2) $\mathcal{F}(\mu_1 \vee \mu_2) \geq \mathcal{F}(\mu_1) \wedge \mathcal{F}(\mu_2)$,
(3) $\mathcal{F}(\bigwedge \mu_i) \geq \bigwedge \mathcal{F}(\mu_i)$.

Let $\mathcal{T}$ be a fuzzy topology on $X$ and $\mathcal{F}_{\mathcal{T}}: I^X \to I$ a map defined by $\mathcal{F}_{\mathcal{T}}(\mu) = \mathcal{T}(\mu^c)$. Then $\mathcal{F}_{\mathcal{T}}$ is a family of closed sets in $X$. Also, let $\mathcal{F}$ be a family of closed sets in $X$ and $\mathcal{T}_{\mathcal{F}}: I^X \to I$ a map defined by $\mathcal{T}_{\mathcal{F}}(\mu) = \mathcal{F}(\mu^c)$. Then $\mathcal{T}_{\mathcal{F}}$ is a fuzzy topology on $X$.

The notions of fuzzy semiopen, semiclosed sets and the weaker forms of fuzzy continuity which are related to our discussion, can be found in [1, 9].

**Definition 2.3** (see [4]). Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy $r$-closure* is defined by

$$\operatorname{cl}(\mu, r) = \bigwedge \{\rho \in I^X \mid \mu \leq \rho,\ \mathcal{F}_{\mathcal{T}}(\rho) \geq r\}. \tag{2.1}$$

From now on, for $r \in I_0$ we will call $\mu$ a *fuzzy $r$-open set* of $X$ if $\mathcal{T}(\mu) \geq r$, $\mu$ a *fuzzy $r$-closed set* of $X$ if $\mathcal{F}(\mu) \geq r$. Note that $\mu$ is fuzzy $r$-closed if and only if $\mu = \operatorname{cl}(\mu, r)$.

Let $(X, \mathcal{T})$ be a fuzzy topological space. For an $r$-cut $\mathcal{T}_r = \{\mu \in I^X \mid \mathcal{T}(\mu) \geq r\}$, it is obvious that $(X, \mathcal{T}_r)$ is a Chang's fuzzy topological space for all $r \in I_0$.

## 3. Fuzzy $r$-interior.

Now, we are going to define the fuzzy interior operator in $(X, \mathcal{T})$.

**Definition 3.1.** Let $(X, \mathcal{T})$ be a fuzzy topological space. For each $\mu \in I^X$ and each $r \in I_0$, the *fuzzy $r$-interior of $\mu$* is defined as follows:

$$\operatorname{int}(\mu, r) = \bigvee \{\rho \in I^X \mid \mu \geq \rho,\ \mathcal{T}(\rho) \geq r\}. \tag{3.1}$$

The operator $\operatorname{int}: I^X \times I_0 \to I^X$ is called the *fuzzy interior operator* in $(X, \mathcal{T})$.

Obviously, $\operatorname{int}(\mu, r)$ is the greatest fuzzy $r$-open set which is contained in $\mu$ and $\operatorname{int}(\mu, r) = \mu$ for any fuzzy $r$-open set $\mu$. Moreover, we have the following results.

**Theorem 3.2.** *Let $(X, \mathcal{T})$ be a fuzzy topological space and $\operatorname{int}: I^X \times I_0 \to I^X$ the fuzzy interior operator in $(X, \mathcal{T})$. Then for $\mu, \rho \in I^X$ and $r, s \in I_0$,*
(1) $\operatorname{int}(\tilde{0}, r) = \tilde{0}$, $\operatorname{int}(\tilde{1}, r) = \tilde{1}$,
(2) $\operatorname{int}(\mu, r) \leq \mu$,
(3) $\operatorname{int}(\mu, r) \geq \operatorname{int}(\mu, s)$ *if* $r \leq s$,
(4) $\operatorname{int}(\mu \wedge \rho, r) = \operatorname{int}(\mu, r) \wedge \operatorname{int}(\rho, r)$,
(5) $\operatorname{int}(\operatorname{int}(\mu, r), r) = \operatorname{int}(\mu, r)$,
(6) *if* $r = \bigvee \{s \in I_0 \mid \operatorname{int}(\mu, s) = \mu\}$, *then* $\operatorname{int}(\mu, r) = \mu$.

**Proof.** (1), (2), and (5) are obvious. (3) Let $r \le s$. Then every fuzzy $s$-open set is also fuzzy $r$-open. Hence we have

$$\begin{aligned} \operatorname{int}(\mu,r) &= \bigvee\{\rho \in I^X \mid \mu \ge \rho,\ \mathcal{T}(\rho) \ge r\} \\ &\ge \bigvee\{\rho \in I^X \mid \mu \ge \rho,\ \mathcal{T}(\rho) \ge s\} \\ &= \operatorname{int}(\mu,s). \end{aligned} \tag{3.2}$$

(4) Since $\mu\wedge\rho \le \mu$ and $\mu\wedge\rho \le \rho$, $\operatorname{int}(\mu\wedge\rho,r) \le \operatorname{int}(\mu,r)$ and $\operatorname{int}(\mu\wedge\rho,r) \le \operatorname{int}(\rho,r)$. Thus $\operatorname{int}(\mu\wedge\rho,r) \le \operatorname{int}(\mu,r)\wedge\operatorname{int}(\rho,r)$. Conversely, it is clear that $\mu\wedge\rho \ge \operatorname{int}(\mu,r)\wedge \operatorname{int}(\rho,r)$. Also,

$$\mathcal{T}(\operatorname{int}(\mu,r)\wedge\operatorname{int}(\rho,r)) \ge \mathcal{T}(\operatorname{int}(\mu,r))\wedge\mathcal{T}(\operatorname{int}(\rho,r)) \ge r\wedge r = r. \tag{3.3}$$

So, by the definition of fuzzy $r$-interior, $\operatorname{int}(\mu\wedge\rho,r) \ge \operatorname{int}(\mu,r)\wedge\operatorname{int}(\rho,r)$. Hence $\operatorname{int}(\mu\wedge\rho,r) = \operatorname{int}(\mu,r)\wedge\operatorname{int}(\rho,r)$.

(6) Note that $\mathcal{T}(\mu) \ge r$ if and only if $\operatorname{int}(\mu,r) = \mu$. Suppose that $\operatorname{int}(\mu,r) \ne \mu$. Then $\mathcal{T}(\mu) < r$ and hence there is an $\alpha \in I$ such that $\mathcal{T}(\mu) < \alpha < r$. Since $r = \bigvee\{s \in I_0 \mid \operatorname{int}(\mu,s) = \mu\}$, there is an $s \in I$ such that $\mathcal{T}(\mu) < \alpha < s \le r$ and $\operatorname{int}(\mu,s) = \mu$. Since $\mathcal{T}(\mu) < s$, $\operatorname{int}(\mu,s) \ne \mu$. This is a contradiction. □

**Theorem 3.3.** *Let* $\operatorname{int} : I^X \times I_0 \to I^X$ *be a map satisfying (1), (2), (3), (4), (5), and (6) of Theorem 3.2. Let* $\mathcal{T} : I^X \to I$ *be a map defined by*

$$\mathcal{T}(\mu) = \bigvee\{r \in I_0 \mid \operatorname{int}(\mu,r) = \mu\}. \tag{3.4}$$

*Then* $\mathcal{T}$ *is a fuzzy topology on* $X$ *such that* $\operatorname{int} = \operatorname{int}_{\mathcal{T}}$.

**Proof.** (i) By (1), $\mathcal{T}(\tilde{0}) = 1 = \mathcal{T}(\tilde{1})$.

(ii) Suppose that $\mathcal{T}(\mu_1\wedge\mu_2) < \mathcal{T}(\mu_1)\wedge\mathcal{T}(\mu_2)$. Then there is an $\alpha \in I$ such that $\mathcal{T}(\mu_1\wedge\mu_2) < \alpha < \mathcal{T}(\mu_1)\wedge\mathcal{T}(\mu_2)$. So, there are $s_1, s_2 \in I$ such that $\alpha < s_i \le \mathcal{T}(\mu_i)$ and $\operatorname{int}(\mu_i,s_i) = \mu_i$ for each $i = 1,2$. Let $s = s_1\wedge s_2$. Then $\operatorname{int}(\mu_i,s) \ge \operatorname{int}(\mu_i,s_i) = \mu_i$ and hence $\operatorname{int}(\mu_i,s) = \mu_i$ for each $i = 1,2$. By (4), $\operatorname{int}(\mu_1\wedge\mu_2,s) = \operatorname{int}(\mu_1,s)\wedge\operatorname{int}(\mu_2,s) = \mu_1\wedge\mu_2$. Thus

$$\alpha > \mathcal{T}(\mu_1\wedge\mu_2) = \bigvee\{r \in I_0 \mid \operatorname{int}(\mu_1\wedge\mu_2,r) = \mu_1\wedge\mu_2\} \ge s = s_1\wedge s_2 > \alpha. \tag{3.5}$$

This is a contradiction. Therefore $\mathcal{T}(\mu_1\wedge\mu_2) \ge \mathcal{T}(\mu_1)\wedge\mathcal{T}(\mu_2)$.

(iii) Suppose $\mathcal{T}(\bigvee\mu_i) < \bigwedge\mathcal{T}(\mu_i)$. Then there is an $\alpha \in I$ such that $\mathcal{T}(\bigvee\mu_i) < \alpha < \bigwedge\mathcal{T}(\mu_i)$. So for each $i$, there is an $s_i \in I$ such that $\alpha < s_i \le \mathcal{T}(\mu_i)$ and $\operatorname{int}(\mu_i,s_i) = \mu_i$. Let $s = \bigwedge s_i$. Then $\operatorname{int}(\mu_i,s) \ge \operatorname{int}(\mu_i,s_i) = \mu_i$ and hence $\operatorname{int}(\bigvee\mu_i,s) \ge \operatorname{int}(\mu_i,s) = \mu_i$ for each $i$. Thus $\operatorname{int}(\bigvee\mu_i,s) \ge \bigvee\mu_i$ and hence $\operatorname{int}(\bigvee\mu_i,s) = \bigvee\mu_i$. Hence

$$\alpha > \mathcal{T}\Big(\bigvee\mu_i\Big) \ge s \ge \alpha. \tag{3.6}$$

This is a contradiction. Therefore $\mathcal{T}(\bigvee\mu_i) \ge \bigwedge\mathcal{T}(\mu_i)$.

Next we will show that $\operatorname{int} = \operatorname{int}_{\mathcal{T}}$. Note that for $s \le r$,

$$\operatorname{int}(\mu,r) = \operatorname{int}(\operatorname{int}(\mu,r),r) \le \operatorname{int}(\operatorname{int}(\mu,r),s) \le \operatorname{int}(\mu,r). \tag{3.7}$$

So $\text{int}(\mu,r) = \text{int}(\text{int}(\mu,r),s)$ for $s \le r$ and $\text{int}(\mu,r) \le \mu$. Thus

$$\begin{aligned}\text{int}_{\mathcal{T}}(\mu,r) &= \bigvee \{\rho \in I^X \mid \rho \le \mu,\ \mathcal{T}(\rho) \ge r\} \\ &= \bigvee \left\{\rho \in I^X \mid \rho \le \mu,\ \bigvee \{s \in I_0 \mid \text{int}(\rho,s) = \rho\} \ge r\right\} \\ &= \bigvee \{\rho \in I^X \mid \rho \le \mu,\ \text{int}(\rho,s) = \rho \text{ for } s \le r\} \\ &\ge \text{int}(\mu,r).\end{aligned} \tag{3.8}$$

On the other hand, let $\rho \le \mu$ and $\text{int}(\rho,s) = \rho$ for $s \le r$. Then by (6), $\rho = \text{int}(\rho,r) \le \text{int}(\mu,r)$. Thus

$$\text{int}_{\mathcal{T}}(\mu,r) = \bigvee \{\rho \in I^X \mid \rho \le \mu,\ \text{int}(\rho,s) = \rho \text{ for } s \le r\} \le \text{int}(\mu,r). \tag{3.9}$$

Therefore, $\text{int}_{\mathcal{T}}(\mu,r) = \text{int}(\mu,r)$. Hence the theorem follows. □

If $\text{int} : I^X \times I_0 \to I^X$ is a fuzzy interior operator on $X$, then for each $r \in I_0$, $\text{int}_r : I^X \to I^X$ defined by

$$\text{int}_r(\mu) = \text{int}(\mu,r) \tag{3.10}$$

is a Chang's fuzzy interior operator on $X$.

Fuzzy $r$-interior is an extension of the Chang's fuzzy interior.

**Theorem 3.4.** *An operator* $\text{int} : I^X \times I_0 \to I^X$ *is a fuzzy interior for the fuzzy topological space* $(X,\mathcal{T})$ *if and only if for any* $r \in I_0$, $\text{int}_r : I^X \to I^X$ *is a Chang's fuzzy interior for the Chang's fuzzy topological space* $(X,\mathcal{T}_r)$.

**Proof.** (⇒). This direction (⇒) is obvious.

(⇐). (1), (2), (4), and (5) are obvious.

(3) Let $r \le s$. Then $\mathcal{T}_r \supseteq \mathcal{T}_s$ and hence $\text{int}(\mu,r) = \text{int}_r(\mu) = \bigvee\{\rho \in I^X \mid \rho \le \mu,\ \rho \in \mathcal{T}_r\} \ge \bigvee\{\rho \in I^X \mid \rho \le \mu,\ \rho \in \mathcal{T}_s\} = \text{int}_s(\mu) = \text{int}(\mu,s)$.

(6) Suppose that $\text{int}(\mu,r) \ne \mu$. Then $\text{int}_r(\mu) = \text{int}(\mu,r) \ne \mu$. So $\mu \notin \mathcal{T}_r$ and hence $\mathcal{T}(\mu) < r$. Thus there is an $\alpha \in I$ such that $\mathcal{T}(\mu) < \alpha < r$. Since $r = \bigvee\{s \in I_0 \mid \text{int}(\mu,s) = \mu\}$, there is an $s \in I_0$ such that $\mathcal{T}(\mu) < \alpha < s \le r$ and $\text{int}(\mu,s) = \text{int}_s(\mu) = \mu$. Since $\mathcal{T}(\mu) < s$, $\mu \notin \mathcal{T}_s$ and hence $\text{int}_s(\mu) \ne \mu$. It is a contradiction. □

For a family $\{\mu_i\}_{i\in\Gamma}$ of fuzzy sets in a fuzzy topological space $X$ and $r \in I_0$, $\bigvee \text{cl}(\mu_i,r) \le \text{cl}(\bigvee \mu_i,r)$, and the equality holds when $\Gamma$ is a finite set. Similarly $\bigwedge \text{int}(\mu_i,r) \ge \text{int}(\bigwedge \mu_i,r)$ and $\bigwedge \text{int}(\mu_i,r) = \text{int}(\bigwedge \mu_i,r)$ for a finite set $\Gamma$.

**Theorem 3.5.** *For a fuzzy set* $\mu$ *in a fuzzy topological space* $X$ *and* $r \in I_0$,

(1) $\text{int}(\mu,r)^c = \text{cl}(\mu^c,r)$.

(2) $\text{cl}(\mu,r)^c = \text{int}(\mu^c,r)$.

**Proof.**

$$\begin{aligned}\text{int}(\mu,r)^c &= \left(\bigvee \{\rho \in I^X \mid \rho \le \mu,\ \mathcal{T}(\rho) \ge r\}\right)^c \\ &= \bigwedge \{\rho^c \in I^X \mid \rho^c \ge \mu^c,\ \mathcal{F}_{\mathcal{T}}(\rho^c) \ge r\} \\ &= \text{cl}(\mu^c,r).\end{aligned} \tag{3.11}$$

Similarly we can show (2). □

## 4. Fuzzy $r$-semiopen sets

**Definition 4.1.** Let $\mu$ be a fuzzy set in a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$. Then $\mu$ is said to be

(1) *fuzzy $r$-semiopen* if there is a fuzzy $r$-open set $\rho$ in $X$ such that $\rho \le \mu \le \mathrm{cl}(\rho,r)$,

(2) *fuzzy $r$-semiclosed* if there is a fuzzy $r$-closed set $\rho$ in $X$ such that $\mathrm{int}(\rho,r) \le \mu \le \rho$.

**Theorem 4.2.** *Let $\mu$ be a fuzzy set in a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$. Then the following statements are equivalent:*

(1) *$\mu$ is a fuzzy $r$-semiopen set.*

(2) *$\mu^c$ is a fuzzy $r$-semiclosed set.*

(3) $\mathrm{cl}(\mathrm{int}(\mu,r),r) \ge \mu$.

(4) $\mathrm{int}(\mathrm{cl}(\mu^c,r),r) \le \mu^c$.

**Proof.** (1)⇔(2). The proof follows from Theorem 3.5.

(1)⇒(3). Let $\mu$ be a fuzzy $r$-semiopen set of $X$. Then there is a fuzzy $r$-open set $\rho$ in $X$ such that $\rho \le \mu \le \mathrm{cl}(\rho,r)$. Since $\mathcal{T}(\rho) \ge r$ and $\mu \ge \rho$, $\mathrm{int}(\mu,r) \ge \rho$. Hence $\mathrm{cl}(\mathrm{int}(\mu,r),r) \ge \mathrm{cl}(\rho,r) \ge \mu$.

(3)⇒(1). Let $\mathrm{cl}(\mathrm{int}(\mu,r),r) \ge \mu$ and take $\rho = \mathrm{int}(\mu,r)$. Since $\mathcal{T}(\mathrm{int}(\mu,r)) \ge r$, $\rho$ is a fuzzy $r$-open set. Also, $\rho = \mathrm{int}(\mu,r) \le \mu \le \mathrm{cl}(\mathrm{int}(\mu,r),r) = \mathrm{cl}(\rho,r)$. Hence $\mu$ is a fuzzy $r$-semiopen set.

(2)⇔(4). The proof is similar to the proof of (1)⇔(3). □

**Theorem 4.3.** (1) *Any union of fuzzy $r$-semiopen sets is fuzzy $r$-semiopen.*

(2) *Any intersection of fuzzy $r$-semiclosed sets is fuzzy $r$-semiclosed.*

**Proof.** (1) Let $\{\mu_i\}$ be a collection of fuzzy $r$-semiopen sets. Then for each $i$, there is a fuzzy $r$-open set $\rho_i$ such that $\rho_i \le \mu_i \le \mathrm{cl}(\rho_i,r)$. Since $\mathcal{T}(\bigvee \rho_i) \ge \bigwedge \mathcal{T}(\rho_i) \ge r$, $\bigvee \rho_i$ is a fuzzy $r$-open set. Moreover,

$$\bigvee \rho_i \le \bigvee \mu_i \le \bigvee \mathrm{cl}(\rho_i,r) \le \mathrm{cl}\Big(\bigvee \rho_i, r\Big). \tag{4.1}$$

Hence $\bigvee \mu_i$ is a fuzzy $r$-semiopen set.

(2) It follows from (1) using Theorem 4.2. □

**Definition 4.4.** Let $(X,\mathcal{T})$ be a fuzzy topological space. For each $r \in I_0$ and for each $\mu \in I^X$, the *fuzzy $r$-semiclosure* is defined by

$$\mathrm{scl}(\mu,r) = \bigwedge \{\rho \in I^X \mid \mu \le \rho,\ \rho \text{ is fuzzy } r\text{-semiclosed}\} \tag{4.2}$$

and the *fuzzy $r$-semi-interior* is defined by

$$\mathrm{sint}(\mu,r) = \bigvee \{\rho \in I^X \mid \mu \ge \rho,\ \rho \text{ is fuzzy } r\text{-semiopen}\}. \tag{4.3}$$

Obviously $\mathrm{scl}(\mu,r)$ is the smallest fuzzy $r$-semiclosed set which contains $\mu$ and $\mathrm{sint}(\mu,r)$ is the greatest fuzzy $r$-semiopen set which is contained in $\mu$. Also, $\mathrm{scl}(\mu,r)=\mu$ for any fuzzy $r$-semiclosed set $\mu$ and $\mathrm{sint}(\mu,r) = \mu$ for any fuzzy $r$-semiopen set $\mu$. Moreover, we have

$$\mathrm{int}(\mu,r) \le \mathrm{sint}(\mu,r) \le \mu \le \mathrm{scl}(\mu,r) \le \mathrm{cl}(\mu,r). \tag{4.4}$$

Also, we have the following results:

(1) $\operatorname{scl}(\tilde{0},r)=\tilde{0}$, $\operatorname{scl}(\tilde{1},r)=\tilde{1}$, $\operatorname{sint}(\tilde{0},r)=\tilde{0}$, $\operatorname{sint}(\tilde{1},r)=\tilde{1}$.

(2) $\operatorname{scl}(\mu,r)\geq\mu$, $\operatorname{sint}(\mu,r)\leq\mu$.

(3) $\operatorname{scl}(\mu\vee\rho,r)\geq\operatorname{scl}(\mu,r)\vee\operatorname{scl}(\rho,r)$, $\operatorname{sint}(\mu\wedge\rho,r)\leq\operatorname{sint}(\mu,r)\wedge\operatorname{sint}(\rho,r)$.

(4) $\operatorname{scl}(\operatorname{scl}(\mu,r),r)=\operatorname{scl}(\mu,r)$, $\operatorname{sint}(\operatorname{sint}(\mu,r),r)=\operatorname{sint}(\mu,r)$.

**Remark 4.5.** It is obvious that every fuzzy $r$-open ($r$-closed) set is fuzzy $r$-semiopen ($r$-semiclosed). The converse does not hold as in Example 4.6. It also shows that the intersection (union) of any two fuzzy $r$-semiopen ($r$-semiclosed) sets need not be fuzzy $r$-semiopen ($r$-semiclosed). Even the intersection (union) of a fuzzy $r$-semiopen ($r$-semiclosed) set with a fuzzy $r$-open ($r$-closed) set may fail to be fuzzy $r$-semiopen ($r$-semiclosed).

**Example 4.6.** Let $X=I$ and $\mu_1,\mu_2$ and $\mu_3$ be fuzzy sets of $X$ defined as

$$\mu_1(x)=\begin{cases}0, & \text{if } 0\leq x\leq \frac{1}{2},\\ 2x-1, & \text{if } \frac{1}{2}\leq x\leq 1;\end{cases}$$

$$\mu_2(x)=\begin{cases}1, & \text{if } 0\leq x\leq \frac{1}{4},\\ -4x+2, & \text{if } \frac{1}{4}\leq x\leq \frac{1}{2},\\ 0, & \text{if } \frac{1}{2}\leq x\leq 1;\end{cases} \tag{4.5}$$

$$\mu_3(x)=\begin{cases}0, & \text{if } 0\leq x\leq \frac{1}{4},\\ \frac{1}{3}(4x-1), & \text{if } \frac{1}{4}\leq x\leq 1.\end{cases}$$

Define $\mathcal{T}:I^X\to I$ by

$$\mathcal{T}(\mu)=\begin{cases}1 & \text{if } \mu=\tilde{0},\tilde{1},\\ \frac{1}{2} & \text{if } \mu=\mu_1,\mu_2,\ \mu_1\vee\mu_2,\\ 0 & \text{otherwise.}\end{cases} \tag{4.6}$$

Then clearly $\mathcal{T}$ is a fuzzy topology on $X$.

(1) Note that $\operatorname{cl}(\mu_1,1/2)=\mu_2^c$. Since $\mu_1\leq\mu_3\leq\operatorname{cl}(\mu_1,1/2)$ and $\mu_1$ is a fuzzy $1/2$-open set, $\mu_3$ is a fuzzy $1/2$-semiopen set. But $\mu_3$ is not a fuzzy $1/2$-open set, because $\mathcal{T}(\mu_3)=0$.

(2) In view of Theorem 4.2, $\mu_3^c$ is a fuzzy $1/2$-semiclosed set which is not a fuzzy $1/2$-closed set.

(3) Note that $\mu_2$ is fuzzy $1/2$-open and hence fuzzy $1/2$-semiopen. Since $\tilde{0}$ is the only fuzzy $1/2$-open set contained in $\mu_2\wedge\mu_3$ and $\operatorname{cl}(\tilde{0},1/2)=\tilde{0}$, $\mu_2\wedge\mu_3$ is not a fuzzy $1/2$-semiopen set.

(4) Clearly $\mu_2^c$ and $\mu_3^c$ are fuzzy $1/2$-semiclosed sets, but $\mu_2^c\vee\mu_3^c=(\mu_2\wedge\mu_3)^c$ is not a fuzzy $1/2$-semiclosed set.

The next two theorems show the relation between $r$-semiopenness and semiopenness.

**Theorem 4.7.** *Let $\mu$ be a fuzzy set in a fuzzy topological space $(X,\mathcal{T})$ and $r \in I_0$. Then $\mu$ is fuzzy $r$-semiopen ($r$-semiclosed) in $(X,\mathcal{T})$ if and only if $\mu$ is fuzzy semiopen (semiclosed) in $(X,\mathcal{T}_r)$.*

**Proof.** The proof is straightforward. □

Let $(X,T)$ be a Chang's fuzzy topological space and $r \in I_0$. Recall [3] that a fuzzy topology $T^r : I^X \to I$ is defined by

$$T^r(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ r & \text{if } \mu \in T - \{\tilde{0}, \tilde{1}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

**Theorem 4.8.** *Let $\mu$ be a fuzzy set in a Chang's fuzzy topological space $(X,T)$ and $r \in I_0$. Then $\mu$ is fuzzy semiopen (semiclosed) in $(X,T)$ if and only if $\mu$ is fuzzy $r$-semiopen ($r$-semiclosed) in $(X,T^r)$.*

**Proof.** The proof is straightforward. □

## 5. Fuzzy $r$-continuous and fuzzy $r$-semicontinuous maps

**Definition 5.1.** Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a map from a fuzzy topological space $X$ to another fuzzy topological space $Y$ and $r \in I_0$. Then $f$ is called

(1) a *fuzzy $r$-continuous* map if $f^{-1}(\mu)$ is a fuzzy $r$-open set of $X$ for each fuzzy $r$-open set $\mu$ of $Y$, or equivalently, $f^{-1}(\mu)$ is a fuzzy $r$-closed set of $X$ for each fuzzy $r$-closed set $\mu$ of $Y$,

(2) a *fuzzy $r$-open* map if $f(\mu)$ is a fuzzy $r$-open set of $Y$ for each fuzzy $r$-open set $\mu$ of $X$,

(3) a *fuzzy $r$-closed* map if $f(\mu)$ is a fuzzy $r$-closed set of $Y$ for each fuzzy $r$-closed set $\mu$ of $X$,

(4) a *fuzzy $r$-homeomorphism* if $f$ is bijective, fuzzy $r$-continuous and fuzzy $r$-open.

**Theorem 5.2.** *Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *$f$ is a fuzzy $r$-continuous map.*

(2) *$f(\mathrm{cl}(\rho,r)) \le \mathrm{cl}(f(\rho),r)$ for each fuzzy set $\rho$ of $X$.*

(3) *$\mathrm{cl}(f^{-1}(\mu),r) \le f^{-1}(\mathrm{cl}(\mu,r))$ for each fuzzy set $\mu$ of $Y$.*

(4) *$f^{-1}(\mathrm{int}(\mu,r)) \le \mathrm{int}(f^{-1}(\mu),r)$ for each fuzzy set $\mu$ of $Y$.*

**Proof.** (1)⇒(2). Let $f$ be fuzzy $r$-continuous and $\rho$ any fuzzy set of $X$. Since $\mathrm{cl}(f(\rho),r)$ is fuzzy $r$-closed of $Y$, $f^{-1}(\mathrm{cl}(f(\rho),r))$ is fuzzy $r$-closed of $X$. Thus

$$\mathrm{cl}(\rho,r) \le \mathrm{cl}(f^{-1}f(\rho),r) \le \mathrm{cl}(f^{-1}(\mathrm{cl}(f(\rho),r)),r) = f^{-1}(\mathrm{cl}(f(\rho),r)). \tag{5.1}$$

Hence

$$f(\mathrm{cl}(\rho,r)) \le f f^{-1}(\mathrm{cl}(f(\rho),r)) \le \mathrm{cl}(f(\rho),r). \tag{5.2}$$

(2)⇒(3). Let $\mu$ be any fuzzy set of $Y$. By (2),

$$f(\mathrm{cl}(f^{-1}(\mu),r)) \le \mathrm{cl}(ff^{-1}(\mu),r) \le \mathrm{cl}(\mu,r). \tag{5.3}$$

Thus

$$\mathrm{cl}(f^{-1}(\mu),r) \le f^{-1}f(\mathrm{cl}(f^{-1}(\mu),r)) \le f^{-1}(\mathrm{cl}(\mu,r)). \tag{5.4}$$

(3)⇒(4). Let $\mu$ be any fuzzy set of $Y$. Then $\mu^c$ is a fuzzy set of $Y$. By (3),

$$\mathrm{cl}(f^{-1}(\mu)^c,r) = \mathrm{cl}(f^{-1}(\mu^c),r) \le f^{-1}(\mathrm{cl}(\mu^c,r)). \tag{5.5}$$

By Theorem 3.5,

$$f^{-1}(\mathrm{int}(\mu,r)) = f^{-1}(\mathrm{cl}(\mu^c,r))^c \le \mathrm{cl}(f^{-1}(\mu)^c,r)^c = \mathrm{int}(f^{-1}(\mu),r). \tag{5.6}$$

(4)⇒(1). Let $\mu$ be any fuzzy $r$-open set of $Y$. Then $\mathrm{int}(\mu,r) = \mu$. By (4),

$$f^{-1}(\mu) = f^{-1}(\mathrm{int}(\mu,r)) \le \mathrm{int}(f^{-1}(\mu),r) \le f^{-1}(\mu). \tag{5.7}$$

So $f^{-1}(\mu) = \mathrm{int}(f^{-1}(\mu),r)$ and hence $f^{-1}(\mu)$ is fuzzy $r$-open of $X$. Thus $f$ is fuzzy $r$-continuous. □

**Theorem 5.3.** *Let $(X,\mathcal{T})$, $(Y,\mathcal{U})$ and $(Z,\mathcal{V})$ be three fuzzy topological spaces and $r \in I_0$. If $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ and $g : (Y,\mathcal{U}) \to (Z,\mathcal{V})$ are fuzzy $r$-continuous ($r$-open, $r$-closed) maps, then so is $g \circ f : (X,\mathcal{T}) \to (Z,\mathcal{V})$.*

**Proof.** The proof is straightforward. □

**Definition 5.4.** Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a map from a fuzzy topological space $X$ to another fuzzy topological space $Y$ and $r \in I_0$. Then $f$ is called

(1) a *fuzzy $r$-semicontinuous* map if $f^{-1}(\mu)$ is a fuzzy $r$-semiopen set of $X$ for each fuzzy $r$-open set $\mu$ of $Y$, or equivalently, $f^{-1}(\mu)$ is a fuzzy $r$-semiclosed set of $X$ for each fuzzy $r$-closed set $\mu$ of $Y$,

(2) a *fuzzy $r$-semiopen* map if $f(\mu)$ is a fuzzy $r$-semiopen set of $Y$ for each fuzzy $r$-open set $\mu$ of $X$,

(3) a *fuzzy $r$-semiclosed* map if $f(\mu)$ is a fuzzy $r$-semiclosed set of $Y$ for each fuzzy $r$-closed set $\mu$ of $X$.

**Theorem 5.5.** *Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a map and $r \in I_0$. Then the following statements are equivalent:*

(1) *$f$ is a fuzzy $r$-semicontinuous map.*

(2) $f(\mathrm{scl}(\rho,r)) \le \mathrm{cl}(f(\rho),r)$ *for each fuzzy set $\rho$ of $X$.*

(3) $\mathrm{scl}(f^{-1}(\mu),r) \le f^{-1}(\mathrm{cl}(\mu,r))$ *for each fuzzy set $\mu$ of $Y$.*

(4) $f^{-1}(\mathrm{int}(\mu,r)) \le \mathrm{sint}(f^{-1}(\mu),r)$ *for each fuzzy set $\mu$ of $Y$.*

**Proof.** The proof is similar to Theorem 5.2. □

**Remark 5.6.** Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ and $g : (Y,\mathcal{U}) \to (Z,\mathcal{V})$ be maps and $r \in I_0$. Then the following statements are true.

(1) If $f$ is fuzzy $r$-semicontinuous and $g$ is fuzzy $r$-continuous then $g \circ f$ is fuzzy $r$-semicontinuous.

(2) If $f$ is fuzzy $r$-open and $g$ is fuzzy $r$-semiopen then $g \circ f$ is fuzzy $r$-semiopen.

(3) If $f$ is fuzzy $r$-closed and $g$ is fuzzy $r$-semiclosed then $g \circ f$ is fuzzy $r$-semiclosed.

**Remark 5.7.** In view of Remark 4.5, a fuzzy $r$-continuous ($r$-open, $r$-closed, resp.) map is also a fuzzy $r$-semicontinuous ($r$-semiopen, $r$-semiclosed, resp.) map for each $r \in I_0$. The converse does not hold as in the following example.

**Example 5.8.** (1) A fuzzy $r$-semicontinuous map need not be a fuzzy $r$-continuous map.

Let $(X, \mathcal{T})$ be a fuzzy topological space as described in Example 4.6 and let $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ be defined by $f(x) = x/2$. Note that $f^{-1}(\tilde{0}) = \tilde{0}$, $f^{-1}(\tilde{1}) = \tilde{1}$, $f^{-1}(\mu_1) = \tilde{0}$ and $f^{-1}(\mu_2) = \mu_1^c = f^{-1}(\mu_1 \vee \mu_2)$. Since $\text{cl}(\mu_2, 1/2) = \mu_1^c$, $\mu_1^c$ is a fuzzy $1/2$-semiopen set and hence $f$ is a fuzzy $1/2$-semicontinuous map. On the other hand, $\mathcal{T}(f^{-1}(\mu_2)) = \mathcal{T}(\mu_1^c) = 0 < 1/2$, and hence $f^{-1}(\mu_2)$ is not a fuzzy $1/2$-open set. Thus $f$ is not a fuzzy $1/2$-continuous map.

(2) A fuzzy $r$-semiopen map need not be a fuzzy $r$-open map.

Let $(X, \mathcal{T})$ be as in (1). Define $\mathcal{T}_1 : I^X \to I$ by

$$\mathcal{T}_1(\mu) = \begin{cases} 1 & \text{if } \mu = \tilde{0}, \tilde{1}, \\ \dfrac{1}{2} & \text{if } \mu = \mu_3, \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Consider the map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T})$ defined by $f(x) = x$. Then $f(\tilde{0}) = \tilde{0}$, $f(\tilde{1}) = \tilde{1}$ and $f(\mu_3) = \mu_3$ are fuzzy $1/2$-semiopen sets of $(X, \mathcal{T})$ and hence $f$ is a fuzzy $1/2$-semiopen map. On the other hand, $\mathcal{T}(f(\mu_3)) = \mathcal{T}(\mu_3) = 0 < 1/2$, and hence $f(\mu_3)$ is not a fuzzy $1/2$-open set. Thus $f$ is not a fuzzy $1/2$-open map.

(3) A fuzzy $r$-open (hence $r$-semiopen) map need not be a fuzzy $r$-semiclosed map.

Let $X = I$ and $\mu$, $\rho$, and $\lambda$ be fuzzy sets of $X$ defined as

$$\begin{aligned} &\mu(x) = 1 - x; \\ &\rho(x) = \begin{cases} -2x + 1 & \text{if } 0 \le x \le \dfrac{1}{2}, \\ 0 & \text{if } \dfrac{1}{2} \le x \le 1; \end{cases} \\ &\lambda(x) = \begin{cases} 1 & \text{if } 0 \le x \le \dfrac{1}{2}, \\ 0 & \text{if } \dfrac{1}{2} < x \le 1. \end{cases} \end{aligned} \tag{5.9}$$

Define $\mathcal{T}_1 : I^X \to I$ and $\mathcal{T}_2 : I^X \to I$ by

$$\mathcal{T}_1(\nu) = \begin{cases} 1 & \text{if } \nu = \tilde{0}, \tilde{1}, \\ \dfrac{1}{2} & \text{if } \nu = \mu, \\ 0 & \text{otherwise;} \end{cases} \qquad \mathcal{T}_2(\nu) = \begin{cases} 1 & \text{if } \nu = \tilde{0}, \tilde{1}, \lambda, \\ \dfrac{1}{2} & \text{if } \nu = \rho, \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on $X$. Consider the map $f:(X,\mathcal{T}_1)\to(X,\mathcal{T}_2)$ defined by $f(x)=x/2$. It is easy to see that $f(\tilde{0})=\tilde{0}$, $f(\mu)=\rho$ and $f(\tilde{1})=\lambda$. Thus $f$ is a fuzzy 1/2-open map and hence a fuzzy 1/2-semiopen map. On the other hand, because the only fuzzy 1/2-closed set containing $\lambda$ is $\tilde{1}$, $\lambda=f(\tilde{1})$ is not a fuzzy 1/2-semiclosed set of $(X,\mathcal{T}_2)$. Thus $f$ is not a fuzzy 1/2-semiclosed map.

(4) A fuzzy $r$-closed (hence $r$-semiclosed) map need not be a fuzzy $r$-semiopen map.

Let $X=I$ and $\mu$, $\rho$, and $\lambda$ be fuzzy sets of $X$ defined as

$$\mu(x)=1-x;\qquad \rho(x)=\begin{cases}-2x+1 & \text{if } 0\le x\le \frac{1}{2},\\ 1 & \text{if } \frac{1}{2}<x\le 1;\end{cases}\qquad \lambda(x)=\begin{cases}0 & \text{if } 0\le x\le \frac{1}{2},\\ 1 & \text{if } \frac{1}{2}<x\le 1.\end{cases}\tag{5.11}$$

Define $\mathcal{T}_1: I^X\to I$ and $\mathcal{T}_2: I^X\to I$ by

$$\mathcal{T}_1(\nu)=\begin{cases}1 & \text{if } \nu=\tilde{0},\tilde{1},\\ \frac{1}{2} & \text{if } \nu=\mu,\\ 0 & \text{otherwise;}\end{cases}\qquad \mathcal{T}_2(\nu)=\begin{cases}1 & \text{if } \nu=\tilde{0},\tilde{1},\lambda,\\ \frac{1}{2} & \text{if } \nu=\rho,\\ 0 & \text{otherwise.}\end{cases}\tag{5.12}$$

Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on $X$. Consider the map $f:(X,\mathcal{T}_1)\to(X,\mathcal{T}_2)$ defined by $f(x)=x/2$. It is easy to see that $f(\tilde{0})=\tilde{0}$, $f(\mu^c)=\rho^c$ and $f(\tilde{1})=\lambda^c$. Thus $f$ is a fuzzy 1/2-closed map and hence a fuzzy 1/2-semiclosed map. On the other hand, the only fuzzy 1/2-open set contained in $\lambda^c$ is $\tilde{0}$, hence $\lambda^c=f(\tilde{1})$ is not a fuzzy 1/2-semiopen set of $(X,\mathcal{T}_2)$. Thus $f$ is not a fuzzy 1/2-semiopen map.

The next two theorems show that a fuzzy continuous (open, closed, semicontinuous, semiopen, semiclosed, resp.) map is a special case of a fuzzy $r$-continuous ($r$-open, $r$-closed, $r$-semicontinuous, $r$-semiopen, $r$-semiclosed, resp.) map.

**Theorem 5.9.** *Let $f:(X,\mathcal{T})\to(Y,\mathcal{U})$ be a map from a fuzzy topological space $X$ to another fuzzy topological space $Y$ and $r\in I_0$. Then $f$ is fuzzy $r$-continuous ($r$-open, $r$-closed, $r$-semicontinuous, $r$-semiopen, $r$-semiclosed, resp.) if and only if $f:(X,\mathcal{T}_r)\to(Y,\mathcal{U}_r)$ is fuzzy continuous (open, closed, semicontinuous, semiopen, semiclosed, resp.).*

**Proof.** The proof is straightforward. □

**Theorem 5.10.** *Let $f:(X,T)\to(Y,U)$ be a map from a Chang's fuzzy topological space $X$ to another Chang's fuzzy topological space $Y$ and $r\in I_0$. Then $f$ is fuzzy continuous (open, closed, semicontinuous, semiopen, semiclosed, resp.) if and only if $f:(X,T^r)\to(Y,U^r)$ is fuzzy $r$-continuous ($r$-open, $r$-closed, $r$-semicontinuous, $r$-semiopen, $r$-semiclosed, resp.).*

**Proof.** The proof is straightforward. □

## References

[1] K. K. Azad, *On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl. **82** (1981), no. 1, 14–32. MR 82k:54006. Zbl 511.54006.

[2] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190. MR 38#5153. Zbl 167.51001.

[3] K. C. Chattopadhyay, R. N. Hazra, and S. K. Samanta, *Gradation of openness: fuzzy topology*, Fuzzy Sets and Systems **49** (1992), no. 2, 237–242. MR 93f:54004. Zbl 762.54004.

[4] K. C. Chattopadhyay and S. K. Samanta, *Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness*, Fuzzy Sets and Systems **54** (1993), no. 2, 207–212. MR 93k:54016. Zbl 809.54005.

[5] R. N. Hazra, S. K. Samanta, and K. C. Chattopadhyay, *Fuzzy topology redefined*, Fuzzy Sets and Systems **45** (1992), no. 1, 79–82. MR 92m:54013. Zbl 756.54002.

[6] A. A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems **48** (1992), no. 3, 371–375. MR 93e:54006. Zbl 783.54007.

[7] A. P. Šostak, *On a fuzzy topological structure*, Rend. Circ. Mat. Palermo (2) Suppl. (1985), no. 11, 89–103. MR 88h:54015. Zbl 638.54007.

[8] ______, *Two decades of fuzzy topology: the main ideas, concepts and results*, Russian Math. Surveys **44** (1989), no. 6, 125–186, [translated from Uspekhi Mat. Nauk **44** (1989) no. 6(270), 99–147 (Russian). MR 91a:54010]. Zbl 716.54004.

[9] T. H. Yalvaç, *Semi-interior and semiclosure of a fuzzy set*, J. Math. Anal. Appl. **132** (1988), no. 2, 356–364. MR 89f:54014. Zbl 645.54007.

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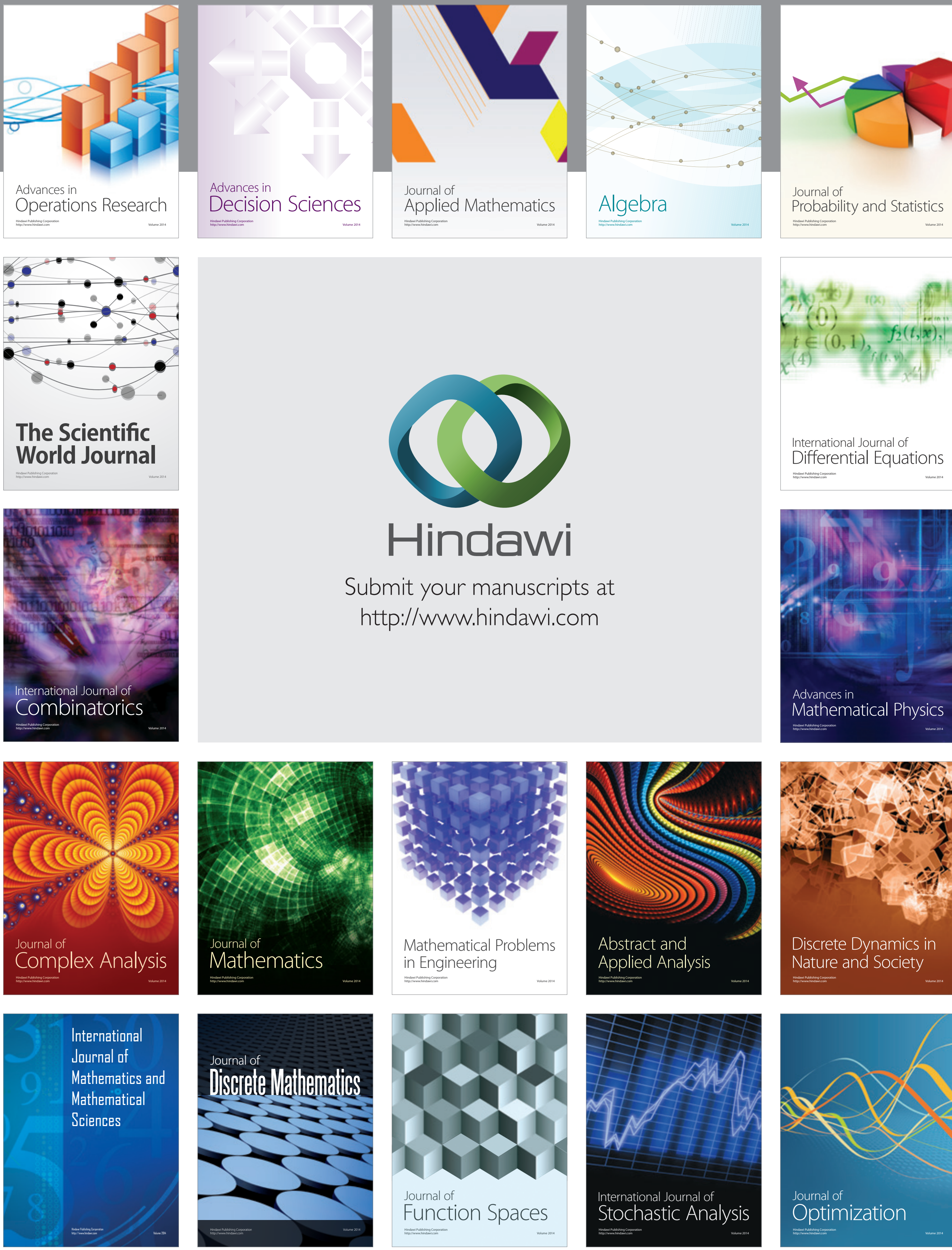
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