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BOUNDARY CONTROL PROBLEM WITH AN INFINITE NUMBER OF VARIABLES

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ABSTRACT. Using a previous result by Gali and El-Saify (1983) and the theory of Kotarski (1989), and Lions (1971), we formulate the boundary control problem for a system governed by Neumann problem involving selfadjoint elliptic operator of 2ℓth order with an infinite number of variables. The inequalities which characterize the optimal control in terms of the adjoint system are obtained, it is studied in order to construct algorithms attainable to numerical computations for the approximation of the control.

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1. Functions spaces. This section covers the basic notations, definitions, and properties which are necessary to present this work.

Let $(p_k(t))_{k=1}^{\infty}$ be a sequence of weights, fixed in all that follows, such that

$$0 < p_k(t) \in C^{\infty}(\mathbb{R}^1), \quad \int_{\mathbb{R}^1} p_k(t)\,dt = 1, \tag{1.1}$$

with respect to it we introduce on the region $\mathbb{R}^{\infty} = \mathbb{R}^1 \times \mathbb{R}^1 \times \cdots$, the measure $d\rho(x)$ by setting [1, 2, 3],

$$d\rho(x) = p_1(x_1)dx_1 \otimes p_2(x_2)dx_2 \otimes \cdots, \quad \left(\mathbb{R}^{\infty} \ni x = (x_k)_{k=1}^{\infty},\ x_k \in \mathbb{R}^1\right). \tag{1.2}$$

On $\mathbb{R}^{\infty}$ we construct the space $L^2(\mathbb{R}^{\infty}, d\rho(x))$ with respect to this measure [3], that is, $L^2(\mathbb{R}^{\infty}, d\rho(x))$ is the space of quadratic integrable functions on $\mathbb{R}^{\infty}$. We will often set $L^2(\mathbb{R}^{\infty}, d\rho(x)) = L^2(\mathbb{R}^{\infty})$.

We next consider a Sobolev space in the case of an unbounded region. For functions which are $\ell = 1, 2, \ldots$ times continuously differentiable up to the boundary Γ of $\mathbb{R}^{\infty}$ and which vanish in a neighborhood of ∞, we introduce the scalar product

$$(\phi, \psi)_{W^{\ell}(\mathbb{R}^{\infty})} = \sum_{|\alpha| \le \ell} (D^{\alpha}\phi, D^{\alpha}\psi)_{L^2(\mathbb{R}^{\infty})}, \tag{1.3}$$

where D^{α} is defined by

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}\cdots}, \quad |\alpha| = \sum_{i=1}^{\infty} \alpha_i \tag{1.4}$$

and the differentiation is in the sense of generalized function, and after the completion, we obtain the Sobolev space $W^{\ell}(\mathbb{R}^{\infty})$.

As in the case of a bounded region, the spaces $W^{\ell}(\mathbb{R}^\infty)$ form a sequence of positive spaces. We can construct the negative spaces $W^{-\ell}(\mathbb{R}^\infty)$ with respect to the zero space $W^0(\mathbb{R}^\infty) = L^2(\mathbb{R}^\infty)$ and then we have the following equipped [1, 3],

$$\begin{gathered} W^{\ell}(\mathbb{R}^\infty) \subseteq L^2(\mathbb{R}^\infty) = W^0(\mathbb{R}^\infty) \subseteq W^{-\ell}(\mathbb{R}^\infty), \\ \|\phi\|_{W^{\ell}(\mathbb{R}^\infty)} \geq \|\phi\|_{L^2(\mathbb{R}^\infty)} \geq \|\phi\|_{W^{-\ell}(\mathbb{R}^\infty)}. \end{gathered} \tag{1.5}$$

As in [7] by using the Laplace-Beltrami operator $(-\Delta_\Gamma)$ on Γ, we choose the scalar product on the space $H^{\ell-\beta-1/2}(\Gamma)$, fractional order Sobolev spaces, as

$$(\phi,\psi)_{H^{\ell-\beta-1/2}(\Gamma)} = \int_\Gamma (-\Delta_\Gamma)^{\ell-\beta-1/2}\phi\cdot\psi\, d\Gamma. \tag{1.6}$$

2. Facts and results. Let $\pi(\phi,\psi)$ be a continuous bilinear form which has the representation

$$\pi(\phi,\psi) = (A\phi,\psi)_{L^2(\mathbb{R}^\infty)}, \quad A \in L(W^{\ell}(\mathbb{R}^\infty), W^{-\ell}(\mathbb{R}^\infty)), \quad \phi,\psi \in W^{\ell}(\mathbb{R}^\infty), \tag{2.1}$$

where A is a bounded selfadjoint elliptic operator of 2ℓth order with an infinite number of variables which takes the form, [2, 4, 8, 9],

$$\begin{aligned} (A\phi)(x) &= \sum_{|\alpha|\leq\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}\frac{1}{\sqrt{p_k(x_k)}}\frac{\partial^{2\alpha}}{\partial x_k^{2\alpha}}\left(\sqrt{p_k(x_k)}\,\phi(x)\right)+q(x)\phi(x) \\ &= \sum_{|\alpha|\leq\ell}\sum_{k=1}^{\infty}(-1)^{|\alpha|}(D_k^{2\alpha}\phi)(x)+q(x)\phi(x), \end{aligned} \tag{2.2}$$

where

$$(D_k^{\alpha}\phi)(x) = \frac{1}{\sqrt{p_k(x_k)}}\frac{\partial^{\alpha}}{\partial x_k^{\alpha}}\left(\sqrt{p_k(x_k)}\,\phi(x)\right), \tag{2.3}$$

and the potential $q(x)$ is a real-valued function from $L^2(\mathbb{R}^\infty)$ such that $q(x) \geq c_0 > 0$, c_0 is a constant [5].

The above bilinear form is coercive in $W^{\ell}(\mathbb{R}^\infty)$ that means

$$\pi(\phi,\phi) \geq c\|\phi\|^2_{W^{\ell}(\mathbb{R}^\infty)}, \quad \phi \in W^{\ell}(\mathbb{R}^\infty),\ c \text{ constant}. \tag{2.4}$$

Since

$$\begin{aligned} \pi(\phi,\phi) &= \sum_{|\alpha|\leq\ell}\sum_{k=1}^{\infty}(D_k^{\alpha}\phi, D_k^{\alpha}\phi)_{L^2((\mathbb{R}^\infty))}+(q\phi,\phi)_{L^2((\mathbb{R}^\infty))} \\ &\geq \|\phi\|_{W^{\ell}(\mathbb{R}^\infty)}+c_0\|\phi\|_{W^{\ell}(\mathbb{R}^\infty)} \\ &\geq c\|\phi\|_{W^{\ell}(\mathbb{R}^\infty)}, \quad 1 \geq c > 0. \end{aligned} \tag{2.5}$$

From the coerciveness condition (2.4) and using the Lax-Milgram lemma [6, 7] we have the following lemma which define the Neumann problem for the operator A and enables us to obtain the state of our control problem.

Lemma 2.1. *If (2.4) is satisfied then there exists a unique element $y \in W^{\ell}(\mathbb{R}^{\infty})$ satisfying Neumann problem*

$$\begin{aligned} Ay &= f, \quad \text{in } \mathbb{R}^{\infty}, \\ \frac{\partial^{\beta} y}{\partial \nu_A^{\beta}} &= h \quad \text{on } \Gamma, \end{aligned} \tag{2.6}$$

where $\partial^{\beta} y / \partial \nu_A^{\beta} = \sum_{|\beta| \le \ell - 1} \sum_{k=1}^{\infty} (D_k^{\beta} y) \cos(n, x_k)$ on Γ, $\cos(n, x_k) = k$th direction cosine of n, n being the normal at Γ.

Proof. From the coerciveness condition and using the Lax-Milgram lemma [6] there exists a unique $y \in W^{\ell}(\mathbb{R}^{\infty})$ such that

$$\pi(y, \psi) = L(\psi) \quad \forall \psi \in W^{\ell}(\mathbb{R}^{\infty}) \tag{2.7}$$

which is known as the variational Neumann problem where $L(\psi)$ is a continuous linear form on $W^{\ell}(\mathbb{R}^{\infty})$ and take the form

$$L(\psi) = \int_{\mathbb{R}^{\infty}} f \psi \, d\rho(x) + \int_{\Gamma} h \psi \, d\Gamma, \quad f \in L^2(\mathbb{R}^{\infty}), \ h \in H^{\ell - \beta - 1/2}(\Gamma). \tag{2.8}$$

Equation (2.7) is equivalent to

$$Ay = f \quad \text{on } \mathbb{R}^{\infty}, \tag{2.9}$$

multiplying both sides by ψ and applying Green's formula, we have

$$\begin{gathered} \int_{\mathbb{R}^{\infty}} (Ay) \, d\rho(x) = \int_{\mathbb{R}^{\infty}} f \psi \, d\rho(x), \\ \int_{\mathbb{R}^{\infty}} \sum_{|\alpha| \le \ell} \sum_{k=1}^{\infty} (-1)^{|\alpha|} (D_k^{2\alpha} y) \psi \, d\rho(x) + \int_{\mathbb{R}^{\infty}} q \cdot y \psi \, d\rho(x) = \int_{\mathbb{R}^{\infty}} f \psi \, d\rho(x), \\ \int_{\mathbb{R}^{\infty}} \sum_{|\alpha| \le \ell} \sum_{k=1}^{\infty} (D_k^{\alpha} y)(D_k^{\alpha} \psi) \, d\rho(x) - \int_{\Gamma} \sum_{|\beta| \le \ell - 1} \sum_{k=1}^{\infty} \psi (D_k^{\alpha} y) \cos(n, x_k) \, d\Gamma \\ + \int_{\mathbb{R}^{\infty}} q \cdot y \psi \, d\rho(x) = \int_{\mathbb{R}^{\infty}} \phi \psi \, d\rho(x). \end{gathered} \tag{2.10}$$

Then

$$\pi(y, \psi) - \int_{\Gamma} \sum_{|\beta| \le \ell - 1} \sum_{k=1}^{\infty} \psi (D_k^{\beta} y) \cos(n, x_k) \, d\Gamma = \int_{\mathbb{R}^{\infty}} f \psi \, d\rho(x), \tag{2.11}$$

since $\pi(y, \psi) = L(\psi)$ we have

$$\int_{\Gamma} \left(h - \frac{\partial^{\beta} y}{\partial \nu_A^{\beta}} \right) \psi \, d\Gamma = 0, \tag{2.12}$$

so

$$h = \frac{\partial^{\beta} y}{\partial \nu_A^{\beta}} \quad \text{on } \Gamma. \tag{2.13}$$

□

3. Boundary control problem. The space $U = H^{\ell-\beta-1/2}(\Gamma)$ is the space of controls. For every control $u \in U$, $Bu \in W^{-\ell}(\mathbb{R}^\infty)$ is given by

$$(Bu,\psi) = \int_\Gamma u\psi\, d\Gamma, \quad \psi \in W^\ell(\mathbb{R}^\infty). \tag{3.1}$$

The state $y(u) \in W^\ell(\mathbb{R}^\infty)$ of the system is given by the solution of

$$\pi(y(u),\psi) = L(\psi) + (Bu,\psi) \quad \forall \psi \in W^\ell(\mathbb{R}^\infty) \tag{3.2}$$

which may be interpreted as Lemma 2.1 to

$$\begin{aligned} Ay(u) &= f \quad \text{in } \mathbb{R}^\infty, \\ \frac{\partial^\beta y(u)}{\partial \nu_A^\beta} &= h + u \quad \text{on } \Gamma. \end{aligned} \tag{3.3}$$

The observation $z(u) = y(u)$ and the cost function $J(v)$ is given by

$$\begin{aligned} J(v) &= \|y(v) - z_d\|^2_{L^2(\mathbb{R}^\infty)} + (Nv,v)_U \\ &= \int_{\mathbb{R}^\infty} (y(v) - z_d)^2 d\rho(x) + (Nv,v)_U, \end{aligned} \tag{3.4}$$

where $z_d \in L^2(\mathbb{R}^\infty)$ and $N \in L(U,U)$, N is Hermitian positive definite operator.

We wish to find

$$\inf_{v \in U_{\text{ad}}} J(v), \tag{3.5}$$

where U_{ad} (set of admissible controls) is a closed convex subset of U.

Under the given consideration we have the following theorem.

Theorem 3.1. *Assume that (2.4) holds and the cost function being given by (3.4). The optimal control u is characterized by (3.3) and*

$$\left((-\Delta_\Gamma)^{-\ell+\beta+1/2} P(u)|_\Gamma + Nu, v - u\right)_U \geq 0 \quad \forall v \in U_{\text{ad}}, \tag{3.6}$$

where $P(u)$ is the adjoint state of $y(u)$.

Outline of the proof. As the proof of theorems in [4, 6], the control $u \in U_{\text{ad}}$ is optimal if and only if

$$J'(u)(v - u) \geq 0 \quad \forall\, v \in U_{\text{ad}} \tag{3.7}$$

which may be written as

$$(y(u) - z_d, y(v) - y(u))_{L^2(\mathbb{R}^\infty)} + (Nu, v - u)_U \geq 0. \tag{3.8}$$

In order to transform (3.8), we define the adjoint state $P(u)$ as the solution of the adjoint Neumann problem

$$\begin{aligned} Ap(u) &= y(u) - z_d \quad \text{in } \mathbb{R}^\infty, \\ \frac{\partial^\beta P(u)}{\partial \nu_A^\beta} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{3.9}$$

Now, multiplying the first equation in (3.9) by $(y(v)-y(u))$ and applying Green's formula, finally taking into account the conditions in (3.3) and (3.9), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^\infty}((y(u)-z_d)(y(v)-y(u)))d\rho(x) \\
&\quad=\int_{\mathbb{R}^\infty} Ap(u)(y(v)-y(u))d\rho(x) \\
&\quad=\int_\Gamma P(u)\left(\frac{\partial^\beta y(v)}{\partial \nu_A^\beta}-\frac{\partial^\beta y(u)}{\partial \nu_A^\beta}\right)d\Gamma \\
&\qquad-\int_\Gamma \frac{\partial^\beta}{\partial \nu_A^\beta}P(u)(y(v)-y(u))d\Gamma+\int_{\mathbb{R}^\infty} P(u)A(y(v)-y(u))\,d\rho(x) \\
&\quad=\int_\Gamma P(u)(v-u)d\Gamma.
\end{aligned}\tag{3.10}$$

Hence (3.8) becomes

$$\int_\Gamma P(u)(v-u)d\Gamma+(Nu,v-u)_U\ge 0.\tag{3.11}$$

By using the definition of the scalar product in $H^{\ell-\beta-1/2}(\Gamma)$, then (3.9) is equivalent to

$$\Big((-\Delta_\Gamma)^{-\ell+\beta+1/2}P(u)|_\Gamma+Nu,v-u\Big)_U\ge 0\quad \forall v\in U_{\text{ad}}\tag{3.12}$$

which is equivalent to

$$\int_\Gamma\Big(P(u)+(-\Delta_\Gamma)^{\ell-\beta-1/2}Nu\Big)(v-u)d\Gamma\ge 0\quad \forall v\in U_{\text{ad}}\tag{3.13}$$

which completes the proof. □

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