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GEOMETRIC PROPERTIES OF COMPOSITION OPERATORS BELONGING TO SCHATTEN CLASSES

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Abstract. We investigate the connection between the geometry of the image domain of an analytic function mapping the unit disk into itself and the membership of the composition operator induced by this function in the Schatten classes. The purpose is to provide solutions to Lotto's conjectures and show a new compact composition operator which is not in any of the Schatten classes.

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1. Introduction. Let D denote the unit disk in the complex plane $\mathbb{C}$ and let H^p denote the Hardy space of functions

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \tag{1.1}$$

analytic in D such that

$$\|f\|_p^p=\sup_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta<\infty, \tag{1.2}$$

where $0<p<\infty$. Let ϕ be an analytic function mapping D into itself. The composition operator C_ϕ (induced by ϕ) on H^p is defined by

$$C_\phi(f)(z)=f(\phi(z)),\quad z\in D. \tag{1.3}$$

It is well known that C_ϕ is a bounded linear operator on H^p. The compactness of this operator is characterized in Shapiro [4] by the following criterion.

Shapiro's compactness criterion. The operator C_ϕ is compact if and only if

$$\lim_{|z|\to 1}\frac{N_\phi(z)}{\log(1/|z|)}=0, \tag{1.4}$$

where

$$N_\phi(w)=\sum_{z\in\phi^{-1}(w)}\log\frac{1}{|z|},\quad w\in D-\{\phi(0)\}. \tag{1.5}$$

Let $d\lambda=(1-|z|^2)^{-2}dA$ be the Mobius invariant measure on D. Let $S_p(H^2)$ be the Schatten ideal of operators on the Hilbert space H^2 for $p>0$. D. H. Luecking and K. Zhu [3] proved the following theorem.

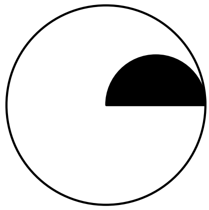

FIGURE 1.1

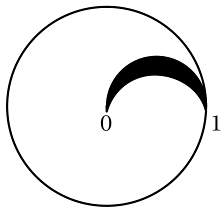

FIGURE 1.2

LUECKING-ZHU THEOREM. *The composition operator $C_\phi \in S_p(H^2)$ if and only if*

$$\frac{N_\phi(z)}{\log(1/|z|)} \in L^{p/2}(d\lambda). \tag{1.6}$$

B. A. Lotto [2] began the investigation of the connection between the geometry of $\phi(D)$ and the membership of composition operators in the Schatten classes.

Let

$$\phi(z) = \frac{1}{1 - ig(z)}, \quad \text{where } g(z) = \sqrt{i\frac{1-z}{1+z}}. \tag{1.7}$$

Then ϕ is a Riemann map from D onto the semi-disk

$$\left\{ z : \operatorname{Im}(z) > 0 \text{ and } \left| z - \frac{1}{2} \right| < \frac{1}{2} \right\} \tag{1.8}$$

with $\phi(1) = 1$ (see Figure 1.1).

Lotto proved that C_ϕ is a compact operator but not a Hilbert-Schmidt (i.e., $S_2(H^2)$) operator. His investigation led to the following conjecture.

LOTTO'S CONJECTURE 1. *The composition operator C_ϕ belongs to the Schatten ideal S_p, if $p > 2$.*

Suppose ψ is a univalent map from D onto a crescent shaped region bounded by the semi-circle

$$\left\{ z : \operatorname{Im}(z) \geq 0 \text{ and } \left| z - \frac{1}{2} \right| = \frac{1}{2} \right\}, \tag{1.9}$$

and a circular arc in the upper half of D that joins 0 to 1 (see Figure 1.2).

Lotto proved that C_ψ is a Hilbert-Schmidt operator and issued the following challenge.

Lotto's conjecture 2. *Given p, $0 < p < \infty$, there exists a simple example of a domain G_p with $G_p \subset D$, or there are easily verifiable geometric conditions on G_p, such that the Riemann map from D onto G_p induces a compact operator that is not in S_p.*

He described a way to produce a compact composition operator which is not in any Schatten ideal if his conjecture 2 is true. Here we want to point out that Tom Carrol and Carl C. Cowen gave such an example in [1]. But the function ϕ that induces the desired compact non-Schatten class operator is described in terms of its "model for iteration." In general, it is hard to visualize the domain $\phi(D)$ and how the geometry of this domain prevents C_ϕ to belong to S_p for all p, $0 < p < \infty$.

The goal of this paper is to prove both Lotto's conjectures. We establish Lotto's conjecture 1 in Section 3 and Lotto's conjecture 2 in Section 4. In Section 5, we follow Lotto's method to construct a Riemann map that induces a compact composition operator which is not in any Schatten ideals.

2. Background and terminology.

For infinite-dimensional Hilbert space H and compact operator T on H, we define singular numbers for T by

$$s_n = s_n(T) = \inf\{\|T-K\| : K \text{ has rank} \le n\}. \tag{2.1}$$

We know that the compact operators are exactly those T for which $s_n(T) \to 0$. By definition the finite rank operators are those for which s_n is eventually zero. In between are the Schatten classes. Specifically, the Schatten p-class $S_p(H)$, $0 < p < \infty$, consists of those T for which

$$\sum_{n=0}^{\infty} s_n^p < \infty. \tag{2.2}$$

The class $S_1(H)$ is the trace class and $S_2(H)$ is the famous Hilbert-Schmidt class. Clearly, $S_p(H) \subset S_q(H)$, if $0 < p < q < \infty$.

We denote the set of bounded operators on H by $B(H)$ and the set of compact operators on H by $K(H)$. We have the following lemma.

Lemma 2.1 (see [3]). *The Schatten class $S_p(H)$, $0 < p < \infty$, is a two-sided ideal in $B(H)$.*

Let Ω_n $(n = 0, 1, 2)$ be simply connected domains such that $\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset D$ and ρ_n be univalent maps from D onto Ω_n, respectively. The following useful corollaries are easy consequences of this lemma (see [2]).

Corollary 2.2. *If $C_{\rho_1} \in S_p(H)$ for some p, $0 < p < \infty$, then $C_{\rho_0} \in S_p(H)$.*

Corollary 2.3. *If $C_{\rho_1} \notin S_p(H)$ for some p, $0 < p < \infty$, then $C_{\rho_2} \notin S_p(H)$.*

Corollary 2.4. *Suppose that Ω is the image of Ω_1 under an automorphism of the unit disk D and ρ is a univalent analytic function which maps D onto Ω. Then $C_{\rho_1} \in S_p(H)$ if and only if $C_\rho \in S_p(H)$.*

If ϕ is univalent, we have

$$N_\phi(w) = \log \frac{1}{|\phi^{-1}(w)|}, \tag{2.3}$$

that is equivalent to $1-|\phi^{-1}(w)|$, as $|\phi^{-1}(w)| \to 1$. Thus, Shapiro's compactness criterion becomes the following corollary.

Corollary 2.5. *Suppose that ϕ is a univalent selfmap of D. The composition operator C_ϕ is compact on H^2 if and only if*

$$\lim_{|z|\to 1} \frac{1-|\phi(z)|}{1-|z|} = \infty. \tag{2.4}$$

Luecking-Zhu theorem implies the following corollary.

Corollary 2.6. *Suppose that ϕ is a univalent selfmap of D into itself. The composition operator $C_\phi \in S_p(H^2)$ if and only if*

$$\chi_{\phi(D)} \cdot \frac{1-|\phi^{-1}(z)|}{1-|z|} \in L^{p/2}(d\lambda). \tag{2.5}$$

We use Corollaries 2.5 and 2.6 to prove our theorems.

3. Proof of Lotto's conjecture 1

Theorem 3.1. *Let ϕ be a Riemann map from D onto the semi-disk*

$$G = \left\{z : \mathrm{Im}(z) > 0 \text{ and } \left|z-\frac{1}{2}\right| < \frac{1}{2}\right\}, \tag{3.1}$$

such that $\phi(1) = 1$. Then the composition operator C_ϕ induced by ϕ belongs to Schatten ideals S_p for all $p > 2$.

Proof. Since ∂G contacts ∂D only at $z = 1$, we only need to consider what happens when $z \in G$ closes to 1. For small $\varepsilon > 0$, let $\Delta(\varepsilon) = \{z : |z-1| < \varepsilon\}$. By Corollary 2.6, we need to prove

$$\int_{G\cap\Delta(\varepsilon)} \left(\frac{1-|\phi^{-1}(z)|}{1-|z|}\right)^{p/2} \frac{dA(z)}{(1-|z|^2)^2} < \infty, \quad \text{if } p > 2. \tag{3.2}$$

It is not difficult to find such ϕ so that $\phi^{-1}(z) = (z^2-(z-1)^2 i)/(z^2+(z-1)^2 i)$ and

$$1-|\phi^{-1}(z)|^2 = -\frac{4\,\mathrm{Im}\,(\bar{z}(z-1))^2}{|z^2+(z-1)^2 i|}. \tag{3.3}$$

Let $z = a+bi \in G$, then $\mathrm{Im}(\bar{z}(z-1))^2 \approx 2ab(a-1) \approx \mathrm{Im}(z-1)^2$, as $z \to 1$. Where $\approx$ means comparable. $A(t)$ and $B(t)$ are comparable if there are positive constants C_1 and C_2 such that $C_1 A(t) \le B(t) \le C_2 A(t)$. So we have

$$1-|\phi^{-1}(z)|^2 \approx -\mathrm{Im}(z-1)^2, \quad \text{as } z \longrightarrow 1,\ z \in G. \tag{3.4}$$

Let $z = 1-re^{-i\theta} \in G$. Then $1-z = re^{-i\theta} \in G$ which implies that $r^2 \le r\cos\theta$. Thus

$$1-|z| \approx 1-|z|^2 = r^2+2r\cos\theta \approx r\cos\theta. \tag{3.5}$$

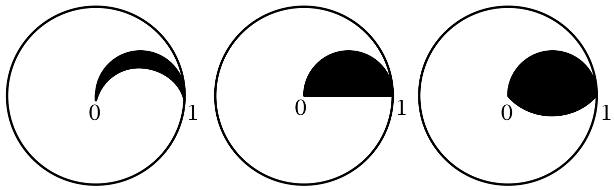

FIGURE 4.1

Now we use (3.4) and (3.5) to show that (3.2) is true. In fact,

$$\begin{aligned}
&\int_{G\cap\Delta(\varepsilon)}\left(\frac{1-|\phi^{-1}(z)|}{1-|z|}\right)^{p/2}\frac{dA(z)}{(1-|z|^2)^2}\\
&\qquad\approx\int_{G\cap\Delta(\varepsilon)}\left(\frac{1-|\phi^{-1}(z)|^2}{1-|z|}\right)^{p/2}\frac{dA(z)}{(1-|z|)^2}\\
&\qquad\approx\int_{G\cap\Delta(\varepsilon)}\left(\frac{-\operatorname{Im}(z-1)^2}{1-|z|}\right)^{p/2}\frac{dA(z)}{(1-|z|)^2}\\
&\qquad\approx\int_{G\cap\Delta(\varepsilon)}\left(\frac{r^2\sin 2\theta}{r\cos\theta}\right)^{p/2}\frac{r\,dr\,d\theta}{(r\cos\theta)^2},\quad z=1-re^{-i\theta}.
\end{aligned}\tag{3.6}$$

The last integral is finite if and only if the following integral:

$$\int_G\left(\frac{r^2\sin 2\theta}{r\cos\theta}\right)^{p/2}\frac{r\,dr\,d\theta}{(r\cos\theta)^2}\tag{3.7}$$

is finite. But

$$\begin{aligned}
\int_G\left(\frac{r^2\sin 2\theta}{r\cos\theta}\right)^{p/2}\frac{r\,dr\,d\theta}{(r\cos\theta)^2}&=2^{p/2}\int_0^{\pi/2}\int_0^{\cos\theta}(r\sin\theta)^{p/2}\frac{dr\,d\theta}{r\cos^2\theta}\\
&\approx\int_0^{\pi/2}\frac{d\theta}{(\pi/2-\theta)^{2-p/2}}<\infty,\quad\text{if and only if } p>2.
\end{aligned}\tag{3.8}$$

This completes the proof of Theorem 3.1. □

4. Geometric characterization of Schatten ideals. For $0<\alpha<1$, let G_α represents one of the shaded regions in Figure 4.1.

Each of the regions is bounded by the semi-circle

$$\left\{z:\operatorname{Im}(z)\ge 0\text{ and }\left|z-\frac{1}{2}\right|=\frac{1}{2}\right\},\tag{4.1}$$

and a circular arc that is inside of D joining 0 to 1 (see Figure 4.1). These two arcs form angles of $\alpha\pi$ at 0 and 1. Define ϕ_α to be one of the Riemann maps from D onto G_α such that $\phi_\alpha(1)=1$.

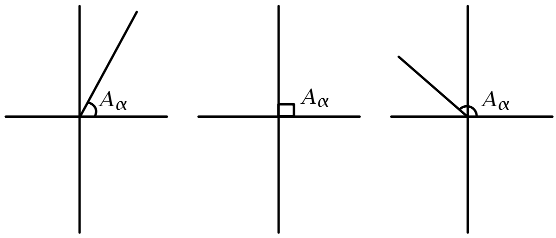

FIGURE 4.2

THEOREM 4.1. *Suppose that C_{ϕ_α} is the composition operator induced by ϕ_α. Then*
(a) *C_{ϕ_α} does not belong to Schatten ideal $S_{2\alpha/(1-\alpha)}(H^2)$.*
(b) *$C_{\phi_\alpha} \in S_p(H^2)$ for any $p > 2\alpha/(1-\alpha)$.*

Theorem 4.1 gives Lotto's conjecture 2 a positive answer: given any $0 < p < \infty$, we can pick $\alpha = p/(p+2) \in (0,1)$, then $G_\alpha \subset D$ is the domain for which the Riemann map from D onto G_α induces a composition operator that is not in $S_p(H^2)$.

LEMMA 4.2. *Let $\tau(z) = i(1/z-1)$. Then τ maps G_α onto the sector A_α (see Figure 4.2), where the two sides of the sector form an angle of $\alpha\pi$. Moreover,*

$$1-|z|^2 \approx \rho^2 + 2\rho\sin\theta, \quad \textit{if } \tau(z) = \rho e^{i\theta},\ \rho \longrightarrow 0. \tag{4.2}$$

PROOF. It is easy to verify that $\tau(G_\alpha) = A_\alpha$ and

$$\begin{aligned} 1-|z|^2 &= 1-\left|\frac{1}{1-i\tau}\right|^2 \\ &\approx \rho^2 + 2\rho\sin\theta, \quad \text{if } \tau(z) = \rho e^{i\theta},\ \rho \longrightarrow 0. \end{aligned} \tag{4.3}$$

□

LEMMA 4.3. *Suppose that $w(\sigma) = (1+i\sigma)/(1-i\sigma)$, then w maps the upper-half plane H^+ onto D and*

$$1-|w| \approx \operatorname{Im}\sigma, \quad \textit{as } \sigma \longrightarrow 0\ (\textit{or } w \longrightarrow 1),\ \sigma \in H^+. \tag{4.4}$$

PROOF. Let $w = re^{i\theta}$. Then

$$\sigma = i\frac{1-re^{i\theta}}{1+re^{i\theta}} \approx \frac{1}{2}\theta + \frac{1}{2}(1-r)i, \quad \text{as } w \longrightarrow 1. \tag{4.5}$$

Therefore,

$$\operatorname{Im}\sigma \approx 1-r. \tag{4.6}$$

On the other hand, we have

$$1-|w|^2 \approx -i(\theta+(1-r)i) + i(\theta-(1-r)i) = 2(1-r) \approx \operatorname{Im}\sigma. \tag{4.7}$$

□

Now we can begin to prove Theorem 4.1.

PROOF OF THEOREM 4.1. Let $\psi_\alpha = \phi_\alpha^{-1}$. We can decompose ψ_α

$$w = \psi_\alpha(z) = w(\tau^{1/\alpha}(z)). \tag{4.8}$$

By Corollary 2.6, we need to show that

$$\int_{G_\alpha}\left(\frac{1-|\psi_\alpha(z)|}{1-|z|}\right)^{p/2}\frac{dA(z)}{(1-|z|^2)^2} \tag{4.9}$$

is finite for $p > 2\alpha/(1-\alpha)$. Actually we only need to consider what happens to (4.9) when z closes to 1. So we consider

$$\begin{aligned}
&\int_{G_\alpha\cap\Delta(\varepsilon)}\left(\frac{1-|\psi_\alpha(z)|}{1-|z|}\right)^{p/2}\frac{dA(z)}{(1-|z|^2)^2}\\
&\quad\approx\int_{G_\alpha\cap\Delta(\varepsilon)}\left(\frac{1-|w(z)|}{1-|z|}\right)^{p/2}\frac{dA(z)}{(1-|z|)^2}\\
&\quad\approx\iint_{\tau(G_\alpha\cap\Delta(\varepsilon))}\left(\frac{\operatorname{Im}\sigma}{\rho^2+2\rho\sin\theta}\right)^{p/2}\frac{\rho\,d\rho\,d\theta}{(\rho^2+2\rho\sin\theta)^2},
\end{aligned} \tag{4.10}$$

where σ is in the upper-half plane, $(\rho,\theta)\in\tau(G_\alpha\cap\Delta(\varepsilon))\subset A_\alpha$ and $\sigma=\tau^{1/\alpha}$. Without loss of generality, we may assume that $\tau(G_\alpha\cap\Delta(\varepsilon))=\{(\rho,\theta)\in A_\alpha:\rho<1\}$. Now we consider three cases: $0<\alpha<1/2$; $\alpha=1/2$; and $1/2<\alpha<1$.

Case 1 ($0<\alpha<1/2$). Define

$$\Omega_1=\{(\rho,\theta)\in\tau(G_\alpha\cap\Delta(\varepsilon)):0<\rho\le 2\sin\theta\},$$

$$\Omega_2=\{(\rho,\theta)\in\tau(G_\alpha\cap\Delta(\varepsilon)):\rho>2\sin\theta\},$$

$$\begin{aligned}
&\int_{G_\alpha}\left(\frac{1-|\psi_\alpha(z)|}{1-|z|}\right)^{p/2}\frac{dA(z)}{(1-|z|^2)^2}\\
&\quad\approx\int_{\Omega_1}\left(\frac{\operatorname{Im}\sigma}{\rho^2+2\rho\sin\theta}\right)^{p/2}\frac{\rho\,d\rho\,d\theta}{(\rho^2+2\rho\sin\theta)^2}\\
&\qquad+\iint_{\Omega_2}\left(\frac{\operatorname{Im}\sigma}{\rho^2+2\rho\sin\theta}\right)^{p/2}\frac{\rho\,d\rho\,d\theta}{(\rho^2+2\rho\sin\theta)^2}\\
&\quad\approx\iint_{\Omega_1}\left(\frac{\rho^{1/\alpha}\sin(\theta/\alpha)}{\rho\sin\theta}\right)^{p/2}\frac{\rho\,d\rho\,d\theta}{(\rho\sin\theta)^2}\\
&\qquad+\iint_{\Omega_2}\left(\frac{\rho^{1/\alpha}\sin(\theta/\alpha)}{\rho^2}\right)^{p/2}\frac{\rho\,d\rho\,d\theta}{(\rho^2)^2}\\
&\quad\approx\int_0^{\alpha\pi}d\theta\int_0^{2\sin\theta}\left(\frac{\rho^{1/\alpha}\sin(\theta/\alpha)}{\rho\sin\theta}\right)^{p/2}\frac{\rho\,d\rho}{(\rho\sin\theta)^2}\\
&\qquad+\int_0^{\alpha\pi}d\theta\int_{2\sin\theta}^{1}\left(\frac{\rho^{1/\alpha}\sin(\theta/\alpha)}{\rho^2}\right)^{p/2}\frac{\rho\,d\rho\,d\theta}{(\rho^2)^2}\\
&\quad\approx\int_0^{\alpha\pi}\frac{1}{\theta^{2-(1/\alpha-1)(p/2)}}\,d\theta.
\end{aligned} \tag{4.11}$$

The last integral converges if and only if $2-(1/\alpha-1)(p/2)<1$, that is, $p>2\alpha/(1-\alpha)$. This simultaneously proves both parts (a) and (b) of Theorem 4.1 for $0<\alpha<1/2$.

Case 2 ($\alpha = 1/2$)**.** That is what exactly Theorem 3.1 is about.
Case 3 ($1/2 < \alpha < 1$)**.** Define

$$\begin{aligned}
\Omega_1 &= \left\{(\rho,\theta) \in \tau(G_\alpha \cap \Delta(\varepsilon)) : 0 < \rho \le 2\sin\theta \text{ and } \theta < \frac{\pi}{4}\right\},\\
\Omega_2 &= \{(\rho,\theta) \in \tau(G_\alpha \cap \Delta(\varepsilon)) : \rho > 2\sin\theta\},\\
\Omega_3 &= \left\{(\rho,\theta) \in \tau(G_\alpha \cap \Delta(\varepsilon)) : 0 < \rho \le 2\sin\theta \text{ and } \theta \ge \frac{\pi}{4}\right\},\\
\int_{\Omega_3} \left(\frac{1-|\psi_\alpha(z)|}{1-|z|}\right)^{p/2} \frac{dA(z)}{(1-|z|^2)^2} &\approx \iint_{\Omega_3} \left(\frac{\operatorname{Im}\sigma}{\rho^2+2\rho\sin\theta}\right)^{p/2} \frac{\rho\, d\rho\, d\theta}{(\rho^2+2\rho\sin\theta)^2}\\
&\approx \int_{\pi/4}^{\alpha\pi} \frac{1}{\theta^2}\, d\theta \int_0^1 \rho^{(1/\alpha-1)(p/2-2)}\, d\rho.
\end{aligned} \tag{4.12}$$

The last integral is finite if and only if $p > 2\alpha/(1-\alpha)$. Using the same proof as in Case 1, we have

$$\begin{aligned}
&\int_{\Omega_1} \left(\frac{1-|\psi_\alpha(z)|}{1-|z|}\right)^{p/2} \frac{dA(z)}{(1-|z|^2)^2} < \infty \quad \text{if and only if } p > \frac{2\alpha}{1-\alpha},\\
&\int_{\Omega_2} \left(\frac{1-|\psi_\alpha(z)|}{1-|z|}\right)^{p/2} \frac{dA(z)}{(1-|z|^2)^2} < \infty \quad \text{if and only if } p > \frac{2\alpha}{1-\alpha}.
\end{aligned} \tag{4.13}$$

This completes the proof of Theorem 4.1. □

Let

$$T_{a,\theta}(z) = \frac{z+a}{1+az} e^{i\theta}, \quad -1 < a < 1,\ 0 \le \theta < 2\pi. \tag{4.14}$$

$T_{a,\theta}$ is an automorphism of D and maps G_α onto $\Omega_{\alpha,a,\theta}$, where $\Omega_{\alpha,a,\theta}$ can have different shapes and positions, depending on the parameters. By choosing appropriate parameters, we can apply Theorems 3.1 and 4.1 and Corollary 2.4 to different regions. We can also apply these theorems with Corollaries 2.2 and 2.3. In Section 5, we use Theorem 4.1 and Corollary 2.4 to construct a non-Schatten class compact composition operator.

5. A non-Schatten class composition operator. Based on Lotto's suggestion, we successfully constructed a compact composition operator that is not in any Schatten ideals. Here is the process of the construction.

Let $\theta_n = \pi/(n+1)$, $z_n = e^{i\theta_n}$, $r_n = (1/2)\sin\theta_n$, and $c_n = (1-r_n)z_n$, where $n = 1,2,3,\ldots$. Define Ω_n to be the region bounded by the semi-circle

$$\{z : \operatorname{Im}(z) \ge 0 \text{ and } |z - |c_n|| = r_n\}, \tag{5.1}$$

and a circular arc that is inside D joining $1-2r_n$ to 1 (it is actually a line segment when $n = 1$). These two arcs form an angle of $(n+1)/(n+3)\pi$ at 1. Let

$$\Omega'_n = \{z e^{i\theta_n} : z \in \Omega_n\}, \tag{5.2}$$

$$\Omega = \cup_{n=1}^{\infty} \Omega'_n. \tag{5.3}$$

We have the following theorem.

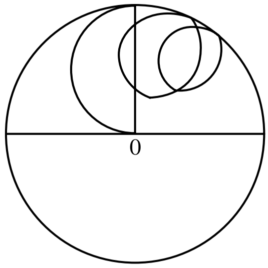

FIGURE 5.1

THEOREM 5.1. *Suppose that Ω is defined by (5.3). Then we have*

(1) *Ω is a simply connected domain contained in the upper half of D.*

(2) *Any Riemann map ϕ that maps D onto Ω induces a compact composition operator C_ϕ. But C_ϕ does not belong to any Schatten ideal $S_p(H^2)$, $p > 0$.*

PROOF. We estimate the distance between the centers c_{n-1} and c_n of Ω'_{n-1} and $\Omega'_n (n \geq 2)$:

$$\begin{aligned} |c_{n-1} - c_n| &= |(1-r_n)z_n - ((1-r_{n-1})z_{n-1})| \\ &= \left| \left(1 - \frac{1}{2}\sin\frac{\pi}{n+1}\right)e^{i(\pi/(n+1))} - \left(\left(1-\frac{1}{2}\sin\frac{\pi}{n}\right)e^{i(\pi/n)}\right)\right| \\ &= O\left(\frac{1}{n^2}\right). \end{aligned} \tag{5.4}$$

But the radius r_n of Ω'_n is $(1/2)\sin(\pi/(n+1)) \geq 1/(1+n)$. Thus Ω'_{n-1} and Ω'_n overlap and Ω is simply connected.

Since

$$\operatorname{Im} c_n = (1-r_n)\operatorname{Im} z_n = (1-r_n)\sin\frac{\pi}{n+1} > \frac{1}{2}\sin\frac{\pi}{n+1} = r_n. \tag{5.5}$$

Ω'_n lies in the upper half of D. Therefore $\Omega \subset D$ is in the upper plane. By the construction of Ω, we know that Ω touches the boundary of D only at z_n, $n = 1,2,3,\ldots$, and 1. One can see that at z_n, ϕ is not conformal (see [4, 5]). Thus ϕ has no finite angular derivative at z_n. Note that $\Omega \subset D$ is in the upper plane and $z_n \to 1$ as $n \to \infty$, ϕ is not conformal at 1 either. By the angular derivative criterion for compactness (see [5]), we know C_ϕ is compact. Let ϕ_n be a Riemann map that maps D onto Ω'_n and C_n be the induced composition operator. Let G_α be the region defined in Section 4 and ψ_α be a Riemann map from D onto G_α. By Theorem 4.1, we know that the composition operator induced by $\psi_{(n+1)/(n+3)}$ does not belong to Schatten ideal S_{n+1}. Let

$$\eta_n(z) = \frac{z + (1-2r_n)}{1 + (1-2r_n)z}e^{i\theta}, \quad z \in D. \tag{5.6}$$

Then η_n is an automorphism of D. Moreover, $\eta_n(G_{(n+1)/(n+3)}) = \Omega'_n$ (see Figure 5.2). Thus, according to Corollary 2.4, $C_n \notin S_{n+1}(H^2)$. But $\Omega'_n \subset \Omega$ for any positive integer n, $C_\phi \notin S_{n+1}(H^2)$ by Corollary 2.3 for any n. We know that $S_p(H^2) \subset S_q(H^2)$ if $p < q$. Therefore C_ϕ does not belong to any Schatten classes. This completes the proof of Theorem 5.1. □

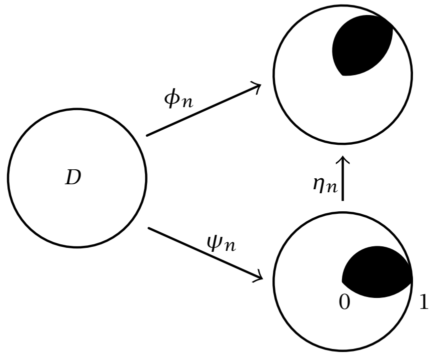

FIGURE 5.2

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