IJMMS 26:10 (2001) 625–634
PII. S0161171201004628
http://ijmms.hindawi.com

ON A NONRESONANCE CONDITION BETWEEN THE FIRST AND THE SECOND EIGENVALUES FOR THE p-LAPLACIAN

A. ANANE and N. TSOULI

(Received 24 February 2000)

ABSTRACT. We are concerned with the existence of solution for the Dirichlet problem $-\triangle_p u = f(x,u)+h(x)$ in Ω, $u=0$ on $\partial\Omega$, when $f(x,u)$ lies in some sense between the first and the second eigenvalues of the p-Laplacian $\triangle_p$. Extensions to more general operators which are $(p-1)$-homogeneous at infinity are also considered.

2000 Mathematics Subject Classification. 35J65.

1. Introduction. In this paper, we are concerned with the existence of solution to the following quasilinear elliptic problem:

$$\begin{aligned} -\triangle_p u = f(x,u)+h(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 1$, Δ_p denotes the p-Laplacian $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1<p<\infty$, h belongs to $W^{-1,p'}(\Omega)$ with p' the Hölder conjugate of p and f is a Caratheodory function from $\Omega\times\mathbb{R}$ to $\mathbb{R}$ such that

$$\lambda_1 \underset{\neq}{\leq} \liminf_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \limsup_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} < \lambda_2 \quad \text{a.e. in } \Omega, \tag{1.2}$$

where λ_1 (resp., λ_2) is the first (resp., the second) eigenvalue of the problem

$$\begin{aligned} -\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

Problems of this sort have been extensively studied in the 70s and 80s in the semilinear case $p=2$. In the quasilinear case $p\neq 2$, (1.1) was investigated for $N=1$ in [6] and for $N\geq 1$ in [3]. In this latter work nonresonance is studied at the left of λ_1.

One of the difficulties to deal with the partial differential equation case $N\geq 1$ is the lack of knowledge of the spectrum of the p-Laplacian in that case. The basic properties of λ_1 were established in [2], while a variational characterization of λ_2 was derived recently in [4]. This variational characterization of λ_2 allows the study of its (strict) monotonicity dependence with respect to a weight. This is the property which is used in our approach to (1.1). The asymmetry in our assumption (1.2) between λ_1 and λ_2 also comes from that property. In fact it remains an open question whether the last strict inequality in (1.2) can be replaced by $\underset{\neq}{\leq}$.

In Section 3 we extend our existence result to more general operators. We consider

$$A(u) = f(x,u) + h(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $A = -\sum_{i=1}^{N}(\partial/\partial x_i)A_i(x,u,\nabla u)$ verifies a $(p-1)$-homogeneity condition at infinity. Such operators were studied by Anane [1] in the variational case. Here we use degree theory for mappings of type $(S)_+$ as developed by Browder [7] and Berkowits and Mustonen [5]. No variational structure is consequently needed.

2. A result for the p-Laplacian.

We seek a weak solution of (1.1), that is,

$$\begin{gathered} \text{find } u \in W_0^{1,p}(\Omega) \quad \text{such that} \quad \forall v \in W_0^{1,p}(\Omega): \\ \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v\,dx = \int_\Omega f(x,u)v\,dx + \langle h,v\rangle, \end{gathered} \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. We assume that f satisfies

$$\max_{|s|\le R}|f(x,s)| \in L^{p'}(\Omega), \quad \forall R > 0, \tag{2.2}$$

$$\lambda_1 \underset{\not\equiv}{\le} l(x) \le k(x) < \lambda_2 \quad \text{a.e. in } \Omega, \tag{2.3}$$

where

$$l(x) = \liminf_{s\to\pm\infty}\frac{f(x,s)}{|s|^{p-2}s}, \qquad k(x) = \limsup_{s\to\pm\infty}\frac{f(x,s)}{|s|^{p-2}s}. \tag{2.4}$$

The first inequality in (2.3) must be understood as “less or equal almost everywhere together with strict inequality on a set of positive measure.” We also assume that some uniformity holds in the inequalities in (2.3):

$$\begin{gathered} \forall\varepsilon > 0, \quad \exists\eta(\varepsilon) > 0 : \lambda_1 - \varepsilon \le \frac{f(x,s)}{|s|^{p-2}s}, \quad \forall|s| \ge \eta(\varepsilon), \quad \text{a.e. in } \Omega, \\ \forall\varepsilon > 0, \quad \exists\eta(\varepsilon) > 0 : \frac{f(x,s)}{|s|^{p-2}s} \le \lambda_2 + \varepsilon, \quad \forall|s| \ge \eta(\varepsilon), \quad \text{a.e. in } \Omega. \end{gathered} \tag{2.5}$$

Remark 2.1. It is clear that (2.2) and (2.5) imply the growth condition

$$|f(x,s)| \le a|s|^{p-1} + b(x) \quad \forall s \in \mathbb{R}, \text{ a.e. in } \Omega, \tag{2.6}$$

where $a > 0$ and $b(\cdot) \in L^{p'}(\Omega)$.

Remark 2.2. Equations (2.2) and (2.5) also imply

$$\begin{gathered} \forall\varepsilon > 0, \quad \exists b_\varepsilon \in L^{p'}(\Omega) \text{ such that} \\ |s|^p(\lambda_1 - \varepsilon) - b_\varepsilon(x) \le sf(x,s) \le |s|^p(\lambda_2 + \varepsilon) + b_\varepsilon(x), \\ \forall s \in \mathbb{R}, \quad \text{a.e. in } \Omega. \end{gathered} \tag{2.7}$$

Theorem 2.3. *Suppose that f satisfies (2.2), (2.3), and (2.5). Then for any $h \in W^{-1,p'}(\Omega)$, problem (2.1) admits a solution u in $W_0^{1,p}(\Omega)$.*

Proof. We denote by $(T_t)_{t\in[0,1]}$ the family of operators from $W_0^{1,p}(\Omega)$ to $W_0^{1,p}(\Omega)$ defined by

$$T_t(u) = (-\Delta_p)^{-1}\left[(1-t)\alpha|u|^{p-2}u + tf(\cdot,u) + th(\cdot)\right], \tag{2.8}$$

where α is some fixed number with $\lambda_1 < \alpha < \lambda_2$.

To prove Theorem 2.3, we first establish the following estimate:

$$\exists R > 0 \text{ such that } \forall t \in [0,1],\ \forall u \in \partial B(O,R) \text{ such that } [I - T_t](u) \neq 0, \tag{2.9}$$

where $B(O,R)$ denotes the ball of center O and radius R in $W_0^{1,p}(\Omega)$.

To prove (2.9) we assume by contradiction that

$$\forall n > 0,\quad \exists t_n \in [0,1], \exists u_n \in W_0^{1,p}(\Omega) \text{ with } ||u_n||_{1,p} = n \text{ such that } T_{t_n}(u_n) = u_n, \tag{2.10}$$

where $\|\cdot\|_{1,p}$ denotes the norm in $W_0^{1,p}(\Omega)$.

Let $w_n = u_n/n$. We can extract from (w_n) a subsequence, still denoted by (w_n), which converges weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω to $w \in W_0^{1,p}(\Omega)$. We can also suppose that t_n converges to $t \in [0,1]$. To reach a contradiction, we use the following lemmas which give various information on w_n and w.

Lemma 2.4. *The sequence g_n defined by*

$$g_n = \frac{f(x, nw_n)}{n^{p-1}} \tag{2.11}$$

is bounded in $L^{p'}(\Omega)$, and consequently, for a subsequence, g_n converges weakly to some g in $L^{p'}(\Omega)$.

Proof. This is an immediate consequence of (2.6). □

Lemma 2.5. *$w \not\equiv 0$.*

Proof. Since w_n verifies,

$$\begin{aligned}\int_\Omega |\nabla w_n|^p\,dx &= (1-t_n)\alpha\int_\Omega |w_n|^p\,dx \\ &\quad + t_n\left[\int_\Omega g_n(x)w_n(x)\,dx + \frac{1}{n^{p-1}}\langle h, w_n\rangle\right],\end{aligned} \tag{2.12}$$

we deduce from Lemma 2.4 that

$$1 = (1-t)\alpha\int_\Omega |w|^p\,dx + t\int_\Omega g(x)w(x)\,dx, \tag{2.13}$$

which clearly implies the conclusion of Lemma 2.5. □

Lemma 2.6. *$g = 0$ a.e. in $\Omega\setminus A$, where $A = \{x \in \Omega : w(x) \neq 0\}$.*

Proof. By (2.6), we have

$$|g_n(x)| \le a|w_n|^{p-1} + \frac{b(x)}{n^{p-1}} \quad \text{a.e. in } \Omega, \tag{2.14}$$

and so

$$\|g_n\|_{L^{p'}(\Omega\setminus A)} \le a\|w_n\|^{p/p'}_{L^p(\Omega\setminus A)} + \frac{1}{n^{p-1}}\|b\|_{L^{p'}(\Omega\setminus A)}, \tag{2.15}$$

which implies

$$\lim_{n\to+\infty}\|g_n\|_{L^{p'}(\Omega\setminus A)} = 0. \tag{2.16}$$

Set $D = \{x \in \Omega\setminus A : g(x) \ne 0\}$. By Lemma 2.4 we have, for $\phi(x) = \operatorname{sign}[g(x)]\chi_D(x) \in L^p(D)$

$$\lim_{n\to+\infty}\int_D g_n(x)\phi(x)\,dx = \int_D |g(x)|\,dx, \tag{2.17}$$

and consequently by (2.16),

$$\int_D |g(x)|\,dx = 0, \tag{2.18}$$

which implies $\operatorname{meas}(D) = 0$, that is, the conclusion of Lemma 2.6. □

Lemma 2.7. *Set*

$$\tilde{g}(x) = \begin{cases} \dfrac{g(x)}{|w(x)|^{p-2}w(x)} & \text{on } A, \\ \beta & \text{on } \Omega\setminus A, \end{cases} \tag{2.19}$$

where β is a fixed number with $\lambda_1 < \beta < \lambda_2$. We have

$$\lambda_1 \lneqq \tilde{g}(x) < \lambda_2 \quad \text{a.e. in } \Omega. \tag{2.20}$$

Proof. Set

$$\begin{aligned} B_l &= \left\{x \in A : w(x)g(x) < l(x)|w(x)|^p\right\}, \\ B_k &= \left\{x \in A : w(x)g(x) > k(x)|w(x)|^p\right\}. \end{aligned} \tag{2.21}$$

We first prove that $\operatorname{meas}(B_l) = 0$ and $\operatorname{meas}(B_k) = 0$.

By (2.7), we have that $\forall \varepsilon \ge 0, \exists b_\varepsilon \in L^{p'}(\Omega)$ such that

$$\begin{aligned} &-\frac{b_\varepsilon(x)}{n^p} + |w_n(x)|^p[l(x)-\varepsilon] \\ &\qquad \le w_n(x)g_n(x) \le \frac{b_\varepsilon(x)}{n^p} + |w_n(x)|^p[k(x)+\varepsilon] \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.22}$$

The first inequality gives

$$-\frac{1}{n^p}\int_{B_l} b_\varepsilon(x)\,dx + \int_{B_l}|w_n(x)|^p[l(x)-\varepsilon]\,dx \le \int_{B_l} w_n(x)g_n(x)\,dx. \tag{2.23}$$

Letting first $x \to \infty$, then $\varepsilon \to 0$, we deduce

$$\int_{B_l}\left[w(x)g(x) - |w(x)|^p l(x)\right]dx \ge 0, \tag{2.24}$$

which implies $\text{meas}(B_l) = 0$. Similarly one gets $\text{meas}(B_k) = 0$. We thus have

$$l(x) \le \tilde{g}(x) \le k(x) \quad \text{a.e. in } A. \tag{2.25}$$

Since

$$\lambda_1 < \tilde{g}(x) = \beta < \lambda_2 \quad \text{a.e. in } \Omega \setminus A, \tag{2.26}$$

we obtain the conclusion of the lemma. □

Lemma 2.8. *w is a solution of*

$$\begin{gathered} -\Delta_p w = m|w|^{p-2}w \quad \textit{in } \Omega, \\ w = 0 \quad \textit{on } \partial\Omega, \end{gathered} \tag{2.27}$$

where $m(x) = (1-t)\alpha + t\tilde{g}(x)$.

Proof. We first prove that w is a solution of

$$\begin{gathered} -\Delta_p w = (1-t)\alpha|w|^{p-2}w + tg \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.28}$$

We recall that w_n satisfies

$$\begin{gathered} -\Delta_p w_n = (1-t_n)\alpha|w_n|^{p-2}w_n + t_n\left[g_n + \frac{1}{n^{p-1}}h\right] \quad \text{in } \Omega, \\ w_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.29}$$

Since $(-\Delta_p)(w_n)$ is bounded in $W^{-1,p'}(\Omega)$, there exists a subsequence, still denoted by (w_n), and a distribution $T \in W^{-1,p'}(\Omega)$, such that $(-\Delta_p)(w_n)$ converges weakly to T in $W^{-1,p'}(\Omega)$; in particular

$$\lim_{n\to+\infty} \langle -\Delta_p w_n, w\rangle = \langle T, w\rangle. \tag{2.30}$$

We also have

$$\begin{aligned} \langle -\Delta_p w_n, w_n - w\rangle &= (1-t_n)\alpha\int_\Omega |w_n|^{p-2}w_n(w_n - w)\,dx \\ &\quad + t_n\left[\int_\Omega g_n(x)(w_n - w)\,dx + \frac{1}{n^{p-1}}\langle h, w_n - w\rangle\right], \end{aligned} \tag{2.31}$$

which implies

$$\lim_{n\to+\infty} \langle -\Delta_p w_n, w_n - w\rangle = 0, \tag{2.32}$$

and therefore

$$\lim_{n\to+\infty} \langle -\Delta_p w_n, w_n\rangle = \langle T, w\rangle. \tag{2.33}$$

Since $(-\Delta_p)$ is an operator of type (M), we deduce

$$T = -\Delta_p w. \tag{2.34}$$

Going to the limit in (2.29) then yields (2.28). But by Lemma 2.6, we have

$$(1-t)\alpha|w|^{p-2}w + tg = m|w|^{p-2}w \quad \text{a.e. in } \Omega. \tag{2.35}$$

So w is a solution of (2.27). □

We denote by $\lambda_1(\Omega, r(x))$ (resp., $\lambda_2(\Omega, r(x))$) the first (resp., the second) eigenvalue in the problem with weight

$$\begin{gathered} -\Delta_p u = \lambda r(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.36}$$

By Lemma 2.7 and the fact that $\lambda_1 < \alpha < \lambda_2$, we have

$$\lambda_1 \underset{\neq}{\leq} m(x) < \lambda_2 \quad \text{a.e. in } \Omega. \tag{2.37}$$

It follows, by the strict monotonicity property of the second eigenvalue with respect to the weight (cf. [4]), that

$$1 = \lambda_2(\Omega, \lambda_2) < \lambda_2(\Omega, m). \tag{2.38}$$

It also follows by the strict monotonicity of the first eigenvalue with respect to the weight (cf. [8]), that

$$\lambda_1(\Omega, m) < \lambda_1(\Omega, \lambda_1) = 1. \tag{2.39}$$

Consequently,

$$\lambda_1(\Omega, m) < 1 < \lambda_2(\Omega, m). \tag{2.40}$$

But by Lemmas 2.5 and 2.8, 1 is an eigenvalue of $(-\Delta_p)$ for the weight m. This contradicts the definition of the second eigenvalue $\lambda_2(\Omega, m)$. We have thus proved that the estimate (2.9) holds.

We can now conclude by a standard degree argument. Indeed T_t is clearly completely continuous, since $(\Delta_p)^{-1}$ is continuous from $W^{-1,p'}(\Omega)$ to $W_0^{1,p}(\Omega)$. Therefore,

$$\deg\left(I - T_0, B(O,R), O\right) = \deg\left(I - T_1, B(O,R), O\right). \tag{2.41}$$

Since T_0 is odd, we have, by Borsuk theorem, that $\deg(I - T_0, B(O,R), O)$ is an odd integer and so nonzero. It then follows that there exists $u \in B(O,R)$ such that $T_1(u) = u$, which proves Theorem 2.3. □

3. Generalization. Theorem 2.3 will now be extended to the case of nonhomogeneous operators. We consider the problem

$$\begin{gathered} A(u) = f(x,u) + h(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where

$$A(u) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x, u(x), \nabla u(x)). \tag{3.2}$$

The method used in Section 2 for $(-\Delta_p)$ can be adapted under suitable assumptions on A. We basically assume that A is a Leray-Lions operator which is $(p-1)$-homogeneous at infinity. Our precise assumptions are the following:

$$\text{Each } A_i(x,s,\xi) \text{ is a Carathéodory function,} \tag{3.3}$$

$$\sum_{i=1}^{N}\left[A_i(x,s,\xi)-A_i\left(x,s,\xi'\right)\right](\xi_i-\xi_i')>0, \quad \text{for a.e. } x\in\Omega,\ \text{all } s\in\mathbb{R},\ \text{all } \xi\neq\xi'\in\mathbb{R}^N, \tag{3.4}$$

$$\begin{gathered}\exists K\in L^{p'}(\Omega), \exists c(t) \text{ a function defined on } \mathbb{R}^+ \text{ with } \lim_{t\to+\infty} c(t)=0 \text{ such that}\\ \left|A_i(x,ts,t\xi)-t^{p-1}|\xi|^{p-2}\xi_i\right|\leq t^{p-1}c(t)\left[|\xi|^{p-1}+|s|^{p-1}+K(x)\right],\\ \text{for a.e. } x\in\Omega,\ \text{all } s\in\mathbb{R},\ \text{all } \xi\in\mathbb{R}^N,\ \text{all } t\in\mathbb{R}^+.\end{gathered} \tag{3.5}$$

We will be able to solve (3.1) when $f(x,s)$ lies at infinity between the first and the second eigenvalues of the p-Laplacian $(-\Delta_p)$, in the sense of (1.2).

Remark 3.1. Equation (3.5) is a hypothesis which means that A is asymptotically homogeneous to $(-\Delta_p)$. An example of an operator which verifies (3.3), (3.4), and (3.5) is the following regularized version of the p-Laplacian:

$$A=-\Delta_{p,\epsilon}=-\operatorname{div}\left[(\epsilon+|\nabla u|^2)^{(p-2)/2}\nabla u\right] \tag{3.6}$$

with $\epsilon>0$.

Remark 3.2. Equations (3.3), (3.4), and (3.5) imply the following usual growth and coercivity conditions:

$$\begin{gathered}\exists c_4>0,\ \exists K_4\in L^{p'}(\Omega) \text{ such that } \left|A_i(x,s,\xi)\right|\leq c_4\left(|\xi|^{p-1}+|s|^{p-1}+K_4(x)\right),\\ \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R},\ \xi\in\mathbb{R}^N,\ \text{for } i=1,\dots,N,\end{gathered} \tag{3.7}$$

$$\begin{gathered}\exists c_5>0,\ c_5'>0,\ K_5\in L^1(\Omega) \text{ such that } \sum_{i=1}^{N}A_i(x,s,\xi)\xi_i\geq c_5|\xi|^p-c_5'|s|^p-K_5(x),\\ \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R},\ \xi\in\mathbb{R}^N.\end{gathered} \tag{3.8}$$

Indeed (3.7) follows immediately from (3.5). To verify (3.8), one observes that by (3.5) one has, for each $t>0$,

$$A_i(x,ts,t\xi)\xi_i-t^{p-1}|\xi|^{p-2}\xi_i^2\geq -t^{p-1}c(t)|\xi_i|\left[|\xi|^{p-1}+|s|^{p-1}+K(x)\right], \tag{3.9}$$

and so

$$\sum_{i=1}^{N}A_i(x,ts,t\xi)\xi_i\geq t^{p-1}|\xi|^p\left[1-Nc(t)\left(1+\frac{2}{p}\right)\right]-\frac{1}{p'}t^{p-1}|c(t)|N\left(|s|^p+|K(x)|^{p'}\right). \tag{3.10}$$

Choosing t sufficiently large yields (3.8).

Remark 3.3. Equations (3.3) and (3.5) imply that A is well defined, continuous, and bounded from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$. Equations (3.3), (3.4), and (3.5) also imply that A is of type $(S)_+$. This latter fact can be proved along similar lines as in the argument given by Berkovits and Mustonen in [5].

We are now ready to state the following theorem.

Theorem 3.4. *Assume (2.2), (2.3), (2.5), (3.3), (3.4), and (3.5). Then for any $h \in W^{-1,p'}(\Omega)$, there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of (3.1), that is,*

$$\int_\Omega \sum_{i=1}^N A_i(x,u(x),\nabla u(x)) \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f(x,u)v\, dx + \langle h,v\rangle, \quad \forall v \in W_0^{1,p}(\Omega). \tag{3.11}$$

Proof. The proof is rather similar to that of Theorem 2.3, and we will only detail below those points which really involve the operator A.

Let $(S_t)_{t\in[0,1]}$ be the family of operators from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ defined by

$$S_t(u) = tA(u) - (1-t)(\Delta_p u) - t[f(x,u)+h(x)] - (1-t)\alpha|u|^{p-2}u, \tag{3.12}$$

for some fixed number α with $\lambda_1 < \alpha < \lambda_2$. Since the operator A is of type $(S)_+$, S_t is also of type $(S)_+$. By the degree theory for mappings of type $(S)_+$, as developed in Browder [7] and Berkowits and Mustonen [5], to solve (3.1) it suffices to prove the following estimate:

$$\exists R > 0 \text{ such that } \forall t \in [0,1], \quad \forall u \in \partial B(OR) \text{ such that } S_t(u) \neq 0. \tag{3.13}$$

To prove (3.13), we assume by contradiction that

$$\forall n \in \mathbb{N},\ \exists t_n \in [0,1], \exists u_n \in W_0^{1,p}(\Omega) \text{ with } \|u_n\|_{1,p} = n, \text{ such that } S_{t_n}(u_n) = 0. \tag{3.14}$$

Let $w_n = u_n/n$. We can extract from (w_n) a subsequence, still denoted by (w_n), which converges weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω to $w \in W_0^{1,p}(\Omega)$. We can also suppose that t_n converges to $t \in [0,1]$.

In the same manner as in the proof of Theorem 2.3, to obtain a contradiction, we use Lemmas 2.4, 2.6, and 2.7 (which do not involve the operator A) together with the following two lemmas. □

Lemma 3.5. $w \not\equiv 0$.

Proof. By (3.14) we have

$$\begin{aligned}\left\langle \frac{t_n A(u_n)}{n^{p-1}} - (1-t_n)\Delta_p w_n, w_n \right\rangle &= (1-t_n)\alpha \int_\Omega |w_n|^p\, dx \\ &\quad + t_n\left[\int_\Omega g_n(x) w_n(x)\, dx + \frac{1}{n^{p-1}}\langle h, w_n\rangle\right].\end{aligned} \tag{3.15}$$

Since

$$\begin{aligned}&\left|\left\langle\frac{t_nA(u_n)}{n^{p-1}}-t_n(-\Delta_p w_n),w_n\right\rangle\right|\\&\qquad\le n^{1-p}\int_\Omega\sum_{i=1}^N\left|A_i(x,u_n,n\nabla w_n)-n^{p-1}|\nabla w_n|^{p-2}\frac{\partial w_n}{\partial x_i}\right|\cdot\left|\frac{\partial w_n}{\partial x_i}\right|dx,\end{aligned}\tag{3.16}$$

using (3.5) and the fact that $\|w_n\|_{1,p}=1$, we obtain

$$\begin{aligned}&\left|\left\langle\frac{t_nA(u_n)}{n^{p-1}}-t_n(-\Delta_p w_n),w_n\right\rangle\right|\\&\qquad\le c(n)\left[||\nabla w_n||_{L^p(\Omega)}^{p/p'}+||w_n||_{L^p(\Omega)}^{p/p'}+\|K\|_{L^{p'}(\Omega)}\right]||w_n||_{1,p}\xrightarrow{n\to+\infty}0.\end{aligned}\tag{3.17}$$

Therefore

$$1=(1-t)\alpha\int_\Omega|w|^p\,dx+t\int_\Omega g(x)w(x)\,dx,\tag{3.18}$$

which clearly implies $w\not\equiv 0$. □

Lemma 3.6. *w is a solution of*

$$\begin{aligned}-\Delta_p w&=m|w|^{p-2}w\quad in\ \Omega,\\ w&=0\quad on\ \partial\Omega,\end{aligned}\tag{3.19}$$

where $m(x)=((1-t)\alpha+t\tilde g(x))$ and $\tilde g$ is defined in Lemma 2.7.

Proof. We first show that w is a solution of

$$\begin{aligned}-\Delta_p w&=(1-t)\alpha|w|^{p-2}w+tg\quad\text{in }\Omega,\\ w&=0\quad\text{on }\partial\Omega.\end{aligned}\tag{3.20}$$

Since $(-\Delta_p)(w_n)$ is bounded in $W^{-1,p'}(\Omega)$, there exists a subsequence, still denoted by (w_n), and a distribution $T\in W^{-1,p'}(\Omega)$, such that $(-\Delta_p)(w_n)$ converges weakly to T in $W^{-1,p'}(\Omega)$. In particular

$$\lim_{n\to+\infty}\langle-\Delta_p w_n,w\rangle=\langle T,w\rangle.\tag{3.21}$$

We also have

$$\begin{aligned}\langle-\Delta_p w_n,w_n-w\rangle&=(1-t_n)\alpha\int_\Omega|w_n|^{p-2}w_n(w_n-w)\,dx\\&\quad+t_n\left[\int_\Omega g_n(x)(w_n-w)\,dx+\frac{1}{n^{p-1}}\langle h,w_n-w\rangle\right]\\&\quad-\left\langle t_n\left[\frac{A(u_n)}{n^{p-1}}+\Delta_p w_n\right],w_n-w\right\rangle,\end{aligned}\tag{3.22}$$

and since, by (3.5),

$$\begin{aligned}&\left|\left\langle t_n\left[\frac{A(u_n)}{n^{p-1}}+\Delta_p w_n\right], w_n-w\right\rangle\right|\\&\quad\le c(n)\left[\|\nabla w_n\|_{L^p(\Omega)}^{p/p'}+\|w_n\|_{L^p(\Omega)}^{p/p'}+\|K\|_{L^{p'}(\Omega)}\right]\|w_n-w\|_{1,p}\xrightarrow{n\to+\infty}0,\end{aligned}\tag{3.23}$$

we deduce

$$\lim_{n\to+\infty}\langle-\Delta_p w_n, w_n-w\rangle=0.\tag{3.24}$$

The rest of the proof of Lemma 3.6 uses the fact that $(-\Delta_p)$ is of type (M) and is similar to the proof of Lemma 2.8. □

Acknowledgement. We wish to thank J.-P. Gossez for various interesting discussions on the subject of this paper.

References

[1] A. Anane, *Etude des valeurs propres et de la résonance pour l'opérateur p-Laplacien*, Ph.D. thesis, Université Libre de Bruxelles, 1987.

[2] ———, *Simplicité et isolation de la première valeur propre du p-Laplacien avec poids* [*Simplicity and isolation of the first eigenvalue of the p-Laplacian with weight*], C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 16, 725–728 (French). MR 89e:35124. Zbl 633.35061.

[3] A. Anane and J.-P. Gossez, *Strongly nonlinear elliptic problems near resonance: a variational approach*, Comm. Partial Differential Equations **15** (1990), no. 8, 1141–1159. MR 91h:35121. Zbl 715.35029.

[4] A. Anane and N. Tsouli, *On the second eigenvalue of the p-Laplacian*, Nonlinear Partial Differential Equations (Fès, 1994), Pitman Research Notes in Mathematics Series, vol. 343, Longman, Harlow, 1996, pp. 1–9. MR 97k:35190. Zbl 854.35081.

[5] J. Berkovits and V. Mustonen, *Nonlinear Mappings of Monotone Type Classification and Degree Theory*, University of Oulu, Finland, 1988.

[6] L. Boccardo, P. Drábek, D. Giachetti, and M. Kučera, *Generalization of Fredholm alternative for nonlinear differential operators*, Nonlinear Anal. **10** (1986), no. 10, 1083–1103. MR 88f:34024. Zbl 623.34031.

[7] F. Browder, *Fixed point theory and nonlinear problems*, Proceeding of Symposia in Pure Mathematics, vol. 39, 1983.

[8] N. Tsouli, *Etude de l'ensemble nodal des fonctions propres et de la non-résonance pour l'opérateurs p-Laplacien.*, Ph.D. thesis, Université Mohammed I. Faculté des Sciences. Département de Maths. Oujda. Maroc., 1995.

A. Anane: Department of Mathematics, Faculty of Sciences, University Mohammed I, Oujda, Morocco

E-mail address: anane@sciences.univ-oujda.ac.ma

N. Tsouli: Department of Mathematics, Faculty of Sciences, University Mohammed I, Oujda, Morocco

E-mail address: tsouli@sciences.univ-oujda.ac.ma


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization