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ON ALMOST INCREASING SEQUENCES AND ITS APPLICATIONS

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ABSTRACT. We prove a general theorem on $|\bar{N}, p_n; \delta|_k$ summability factors, which generalizes a theorem of Bor (1994) on $|\bar{N}, p_n|_k$ summability factors, under weaker conditions by using an almost increasing sequence.

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1. Introduction. Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{1.1}$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (T_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta T_{n-1}|^k < \infty \tag{1.3}$$

and it is said to be summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |\Delta T_{n-1}|^k < \infty, \tag{1.4}$$

where

$$\Delta T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{1.5}$$

In the special case when $\delta = 0$, (respectively, $p_n = 1$ for all values of n) $|\bar{N}, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n|_k$ (respectively, $|C, 1; \delta|_k$) summability.

Mishra and Srivastava [6] proved the following theorem for $|C,1|_k$ summability.

Theorem 1.1. *Let (X_n) be a positive nondecreasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \le \beta_n, \tag{1.6}$$

$$\beta_n \longrightarrow 0 \quad as\ n \longrightarrow \infty, \tag{1.7}$$

$$|\lambda_n| X_n = O(1) \quad as\ n \longrightarrow \infty, \tag{1.8}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n| X_n < \infty. \tag{1.9}$$

If

$$\sum_{n=1}^{m} \frac{1}{n}|s_n|^k = O(X_m) \quad as\ m \longrightarrow \infty, \tag{1.10}$$

then the series $\sum a_n\lambda_n$ is summable $|C,1|_k$, $k \ge 1$.

Bor [3] has generalized Theorem 1.1 for $|\bar{N},p_n|_k$ summability in the form of the following theorem.

Theorem 1.2. *Let (X_n) be a positive nondecreasing sequence and the sequences (β_n) and (λ_n) such that conditions (1.6), (1.7), (1.8), and (1.9) of Theorem 1.1 are satisfied. Furthermore, if (p_n) is a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as\ n \longrightarrow \infty, \tag{1.11}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|s_n|^k = O(X_m) \quad as\ m \longrightarrow \infty, \tag{1.12}$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N},p_n|_k$, $k \ge 1$.

It should be noted that, if we take $p_n = 1$ for all values of n, then condition (1.12) will be reduced to condition (1.10). Also, it can be noticed that in this case condition (1.11) is obvious.

2. The main result. The aim of this paper is to generalize Theorem 1.2 for $|\bar{N},p_n;\delta|_k$ summability under weaker conditions. For this we need the concept of almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \le b_n \le Bc_n$. Obviously every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. So we are weakening the hypotheses of Theorem 1.2 replacing the increasing sequence by an almost increasing sequence.

Now, we will prove the following theorem.

Theorem 2.1. *Let (X_n) be an almost increasing sequence and the sequences (β_n) and (λ_n) such that conditions (1.6), (1.7), (1.8), and (1.9) of Theorem 1.1 are satisfied.*

If (p_n) is a sequence such that condition (1.11) of Theorem 1.2 is satisfied and

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|s_n|^k = O(X_m) \quad as\ m \longrightarrow \infty, \tag{2.1}$$

$$\sum_{n=v+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right), \tag{2.2}$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N},p_n;\delta|_k$ for $k \geq 1$ and $0 \leq \delta < 1/k$.

Remark 2.2. It may be noted that if we take (X_n) as a positive nondecreasing sequence and $\delta = 0$ in this theorem, then we get Theorem 1.2. In this case, condition (2.1) reduces to condition (1.12) and condition (2.2) reduces to

$$\sum_{n=v+1}^{\infty}\frac{p_n}{P_nP_{n-1}} = O\left(\frac{1}{P_v}\right), \tag{2.3}$$

which always holds.

We need the following lemma for the proof of our theorem.

Lemma 2.3 (see [5]). *Under the conditions on (X_n), (β_n), and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (1.9) is satisfied:*

$$n\beta_nX_n = O(1) \quad as\ n \longrightarrow \infty, \tag{2.4}$$

$$\sum_{n=1}^{\infty}\beta_nX_n < \infty. \tag{2.5}$$

Proof of Theorem 2.1. Let (T_n) denotes the $(\bar{N},p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition and changing the order of summation, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{i=0}^{v}a_i\lambda_i = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v. \tag{2.6}$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v\lambda_v. \tag{2.7}$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta(P_{v-1}\lambda_v)s_v + \frac{p_n}{P_n}s_n\lambda_n \\ &= -\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_vs_v\lambda_v + \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vs_v\Delta\lambda_v + \frac{p_n}{P_n}s_n\lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3}, \end{aligned} \tag{2.8}$$

where $T_{n,i}$, $i = 1,2,3$, denotes the ith term in the sum. Since

$$|T_{n,1}+T_{n,2}+T_{n,3}|^k \le 3^k\left(|T_{n,1}|^k+|T_{n,2}|^k+|T_{n,3}|^k\right), \tag{2.9}$$

to complete the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,r}|^k<\infty \quad \text{for } r=1,2,3. \tag{2.10}$$

Now, when $k>1$ applying Hölder's inequality with indices k and k', where $1/k+1/k'=1$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,1}|^k &\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}(P_{n-1})^{-k}\left(\sum_{v=1}^{n-1}p_v|s_v||\lambda_v|\right)^k\\
&\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v|s_v|^k|\lambda_v|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\
&=O(1)\sum_{v=1}^{m}p_v|s_v|^k|\lambda_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\\
&=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|s_v|^k|\lambda_v||\lambda_v|^{k-1}\\
&=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|s_v|^k|\lambda_v|\\
&=O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k-1}|s_r|^k\\
&\quad+O(1)|\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|s_v|^k\\
&=O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&=O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)|\lambda_m|X_m\\
&=O(1)\quad \text{as } m\longrightarrow\infty,
\end{aligned} \tag{2.11}$$

by virtue of the hypotheses of Theorem 1.2 and Lemma 2.3.

Since $v\beta_v=O(1/X_v)$ by (2.4), using the fact that $P_v=O(vp_v)$ by (1.11), and $|\Delta\lambda_n|\le\beta_n$ by (1.6), and after applying Hölder's inequality again, we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|T_{n,2}\right|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}(P_{n-1})^{-k}\left(\sum_{v=1}^{n-1}P_v\left|\Delta\lambda_v\right|\left|s_v\right|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}(P_{n-1})^{-k}\left(\sum_{v=1}^{n-1}v p_v\beta_v\left|s_v\right|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}(v\beta_v)^k p_v\left|s_v\right|^k\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}(v\beta_v)^k p_v\left|s_v\right|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}} \\
&= O(1)\sum_{v=1}^{m}(v\beta_v)^k\left(\frac{P_v}{p_v}\right)^{\delta k-1}\left|s_v\right|^k \\
&= O(1)\sum_{v=1}^{m}(v\beta_v)^{k-1}(v\beta_v)\left(\frac{P_v}{p_v}\right)^{\delta k-1}\left|s_v\right|^k \\
&= O(1)\sum_{v=1}^{m}(v\beta_v)\left(\frac{P_v}{p_v}\right)^{\delta k-1}\left|s_v\right|^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k-1}|s_r|^k \\
&\quad + O(1)m\beta_m\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}\left|s_v\right|^k \\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta(v\beta_v)\right|X_v + O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1}v\left|\Delta\beta_v\right|X_v + O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1)m\beta_m X_m \\
&= O(1) \quad \text{as } m \longrightarrow \infty,
\end{aligned}
\tag{2.12}
$$

by virtue of the hypotheses of Theorem 1.2 and Lemma 2.3.

Finally, using the fact that $P_v = O(v p_v)$ by (1.11), as in $T_{n,1}$, we have

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|T_{n,3}\right|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\left|s_n\right|^k\left|\lambda_n\right| \\
&= O(1) \quad \text{as } m \longrightarrow \infty.
\end{aligned}
\tag{2.13}
$$

Therefore, we get (2.10) and this completes the proof of the theorem. □

If we take $p_n = 1$ for all values of n in this theorem, then we get a result concerning the $|C,1;\delta|_k$ summability factors.

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