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ON THE STRONGLY STARLIKENESS OF MULTIVALENTLY CONVEX FUNCTIONS OF ORDER α

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Abstract. The object of the present paper is to derive some sufficient conditions for strongly starlikeness of multivalently convex functions of order α in the open unit disc.

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1. Introduction. Let $\mathcal{A}(p)$ denote the class of the functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ which are analytic in the open unit disc $\mathcal{E} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}(p)$ is called p-valently starlike if and only if the inequality

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{1.1}$$

holds for $z \in \mathcal{E}$. A function $f(z) \in \mathcal{A}(p)$ is called p-valently convex of order α $(0 \le \alpha < p)$ if and only if the inequality

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.2}$$

holds for $z \in \mathcal{E}$. We denote by $\mathcal{C}(p,\alpha)$ the family of such functions. A function $f(z) \in \mathcal{A}(p)$ is said to be strongly starlike of order α $(0 < \alpha \le 1)$ if and only if the inequality

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \tag{1.3}$$

holds for $z \in \mathcal{E}$. We also denote by $\mathrm{STS}(p,\alpha)$ the family of functions which satisfy the above inequality for the argument. From the definition, it follows that if $f(z) \in \mathrm{STS}(p,\alpha)$, then we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad \text{in } \mathcal{E} \tag{1.4}$$

or $f(z)$ is p-valently starlike in $\mathcal{E}$ and therefore $f(z)$ is p-valent in $\mathcal{E}$ (see [1, Lemma 7]).

Nunokawa [2, 3] proved the following theorems.

Theorem 1.1 (see [2]). *If $f(z) \in \mathcal{A}(p)$ satisfies*

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} < p + \frac{\alpha}{2}, \tag{1.5}$$

where $0 < \alpha \le 1$, then $f(z) \in \mathrm{STS}(p,\alpha)$.

Theorem 1.2 (see [3]). *If $f(z) \in \mathcal{A}(1)$ satisfies*

$$\left|\arg\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi}{2}\alpha(\beta) \quad \text{in } \mathcal{E}, \tag{1.6}$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\beta \quad \text{in } \mathcal{E}, \tag{1.7}$$

where

$$\begin{aligned} \alpha(\beta) &= \beta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\beta q(\beta)\sin(\pi/2)(1-\beta)}{p(\beta)+\beta q(\beta)\cos(\pi/2)(1-\beta)}\right\}, \\ p(\beta) &= (1+\beta)^{(1+\beta)/2}, \qquad q(\beta) = (1-\beta)^{(\beta-1)/2}. \end{aligned} \tag{1.8}$$

It is the purpose of the present paper to prove that if $f(z) \in \mathcal{C}(1, 1-(\alpha/2))$, then $f(z) \in \mathrm{STS}(1,\alpha)$.

In this paper, we need the following lemma.

Lemma 1.3. *Let $f(z) \in \mathcal{A}(1)$ be starlike with respect to the origin in $\mathcal{E}$. Let $C(r,\theta) = \{f(te^{i\theta}) : 0 \le t \le r < 1\}$ and $T(r,\theta)$ be the total variation of $\arg f(te^{i\theta})$ on $C(r,\theta)$, so that*

$$T(r,\theta) = \int_0^r \left|\frac{\partial}{\partial t}\arg\{f(te^{i\theta})\}\right| dt. \tag{1.9}$$

Then

$$T(r,\theta) < \pi. \tag{1.10}$$

We owe this lemma to Sheil-Small [6, Theorem 1].

2. Main theorem.

Our main theorem for the starlikeness of multivalently convex functions of order α is the following.

Theorem 2.1. *Let $f(z) \in \mathcal{A}(1)$ and*

$$1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 1-\frac{\alpha}{2} \quad \text{in } \mathcal{E}, \tag{2.1}$$

where $0 < \alpha \le 1$. Then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad \text{in } \mathcal{E}, \tag{2.2}$$

or $f(z)$ is strongly starlike of order α in $\mathcal{E}$.

Proof. We put

$$\frac{2}{\alpha}\left\{1+\frac{zf''(z)}{f'(z)}-1+\frac{\alpha}{2}\right\} = \frac{zg'(z)}{g(z)}, \tag{2.3}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. From assumption (2.1), we have

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 0 \quad \text{in } \mathscr{E}. \tag{2.4}$$

This shows that $g(z)$ is starlike and univalent in $\mathscr{E}$. With an easy calculation (cf. [4]), (2.3) gives us that

$$f'(z) = \left\{\frac{g(z)}{z}\right\}^{\alpha/2}. \tag{2.5}$$

Since

$$f'(z) \neq 0, \quad 0 < |z| < 1, \tag{2.6}$$

we easily have

$$\frac{f(z)}{zf'(z)} = \int_0^1 \frac{f'(tz)}{f'(z)}\,dt = \int_0^1 t^{-\alpha/2}\left\{\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right\}^{\alpha/2} dt, \tag{2.7}$$

where $z = re^{i\theta}$ and $0 < r < 1$. Since $g(z)$ is starlike in $\mathscr{E}$, from Lemma 1.3, we have

$$-\pi < \arg\left\{g(tre^{i\theta})\right\} - \arg\left\{g(re^{i\theta})\right\} < \pi \tag{2.8}$$

for $0 < t \leq 1$. Putting

$$\xi = \left\{\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right\}^{\alpha/2}, \tag{2.9}$$

we have

$$\arg s = \frac{\alpha}{2}\arg\left\{\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right\}. \tag{2.10}$$

From (2.8) and (2.10), s lies in the convex sector

$$\left\{s : |\arg s| \leq \frac{\pi}{2}\alpha\right\} \tag{2.11}$$

and the same is true of its integral mean of (2.7), (cf. [5, Lemma 1]). Therefore, we have

$$\left|\arg\left\{\frac{f(z)}{zf'(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad \text{in } \mathscr{E} \tag{2.12}$$

or

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \quad \text{in } \mathscr{E}. \tag{2.13}$$

This shows that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad \text{in } \mathscr{E}, \tag{2.14}$$

which completes the proof of our main theorem. $\square$

Remark 2.2. This result is sharp for the case $\alpha \to 0$ and $\alpha = 1$.

(a) For the case $\alpha \to 0$, put $f(z) = z$, then $f(z)$ is a convex function of order $1 - (\alpha/2) \to 1$ and $f(z)$ then $f(z)$ is a strongly starlike function of order $\alpha \to 0$.

(b) For the case $\alpha = 1$, put

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{1-z}. \tag{2.15}$$

Then we have

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2} \quad \text{in } \mathscr{E}, \tag{2.16}$$

and therefore $f(z)$ is a convex function of order $1/2$. From (2.10), we easily have

$$f'(z) = \frac{1}{1-z}, \qquad f(z) = \log\left\{\frac{1}{1-z}\right\}. \tag{2.17}$$

Putting $|z| = 1$, $z = e^{i\theta}$, $0 \le \theta < 2\pi$, then it follows that

$$\frac{z}{1-z} = -\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)},$$

$$\log\left\{\frac{1}{1-z}\right\} = \log\left|\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right| + i\arg\left\{\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right\}.$$

$$\begin{aligned}
\lim_{\theta\to+0}\arg\left\{\frac{zf'(z)}{f(z)}\right\} &= \lim_{\theta\to+0}\arg\left\{\frac{z/(1-z)}{\log(1/(1-z))}\right\} \\
&= \lim_{\theta\to+0}\arg\left\{-\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right\} \\
&\quad - \lim_{\theta\to+0}\arg\left\{\log\left|\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right| + i\arg\left(\frac{1}{2} + i\frac{\cos(\theta/2)}{2\sin(\theta/2)}\right)\right\} \\
&= \frac{\pi}{2}.
\end{aligned} \tag{2.18}$$

The above shows that the main theorem is sharp for the case $\alpha \to 0$ and $\alpha = 1$.

Applying the same method as above and [2], we can obtain the following result.

Theorem 2.3. *If $f(z) \in A(p)$ and satisfies*

$$p - \frac{\alpha}{2} < 1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \quad \text{in } \mathscr{E}, \tag{2.19}$$

where $0 < \alpha \le 1$, then $f(z) \in \mathrm{STS}(p,\alpha)$.

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