COMMON FIXED POINT THEOREMS IN 2 NON-ARCHIMEDEAN Menger PM-SPACE

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ABSTRACT. We introduce the concept of a 2 non-Archimedean Menger PM-space and prove a common fixed point theorem for weak compatible mappings of type (A).

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1. Introduction. Cho et al. [1] proved a common fixed point theorem for compatible mappings of type (A) in non-Archimedean (NA) Menger PM-space. The aim of this paper is to generalize the results of Cho et al. [1] for weak compatible mappings of type (A) in a 2 NA Menger PM-space.

We first give some definitions and notations.

DEFINITION 1.1. Let X be any nonempty set and D the set of all left-continuous distribution functions. An ordered pair (X, F) is said to be a 2 non-Archimedean probabilistic metric space (briefly 2 NA PM-space) if F is a mapping from $X \times X \times X$ into D satisfying the following conditions where the value of F at $x, y, z \in X \times X \times X$ is represented by $F_{x,y,z}$ or $F(x, y, z)$ for all $x, y, z \in X$ such that

- (i) $F_{x,y,z}(t) = 1$ for all $t > 0$ if and only if at least two of the three points are equal.
- (ii) $F_{x,y,z} = F_{x,z,y} = F_{z,y,x}$.
- (iii) $F_{x,y,z}(0) = 0$.
- (iv) If $F_{x,y,s}(t_1) = F_{x,s,z}(t_2) = F_{s,y,z}(t_3) = 1$, then $F_{x,y,z}(\max\{t_1, t_2, t_3\}) = 1$.

DEFINITION 1.2. A t -norm is a function $\Delta : [0, 1] \times [0, 1] \times [0, 1] \rightarrow [0, 1]$ which is associative, commutative, nondecreasing in each coordinate, and $\Delta(a, 1, 1) = a$ for every $a \in [0, 1]$.

DEFINITION 1.3. A 2 NA Menger PM-space is an ordered triplet (X, F, Δ) , where Δ is a t -norm and (X, F) is a 2 NA PM-space satisfying the following condition

$$F_{x,y,z}(\max\{t_1, t_2, t_3\}) \geq \Delta(F_{x,y,s}(t_1), F_{x,s,z}(t_2), F_{s,y,z}(t_3)) \quad \forall x, y, z \in X, t_1, t_2, t_3 \geq 0. \quad (1.1)$$

DEFINITION 1.4. Let (X, F, t) be a 2 NA Menger PM-space and t a continuous t -norm, then (X, F, t) is Hausdorff in the topology induced by the family of neighborhoods

$$\{U_x(\epsilon, \lambda, a_1, a_2, \dots, a_n); x, a_i \in X, \epsilon > 0, i = 1, 2, \dots, n, n \in \mathbb{Z}^+\}, \quad (1.2)$$

where $\mathbb{Z}^+$ is the set of all positive integers and

$$\begin{aligned} U_x(\epsilon, \lambda, a_1, a_2, \dots, a_n) &= \{y \in X; F_{x,y,a_i}(\epsilon) > 1 - \lambda, 1 \leq i \leq n\} \\ &= \cap_{i=1}^n \{y \in X; F_{x,y,a_i}(\epsilon) > 1 - \lambda, 1 \leq i \leq n\}. \end{aligned} \quad (1.3)$$

DEFINITION 1.5. A 2 NA Menger PM-space (X, F) is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that

$$g(F_{x,y,z}(t)) \leq g(F_{x,y,a}(t)) + g(F_{x,a,z}(t)) + g(F_{a,y,z}(t)) \quad \forall x, y, z, a \in X, t \geq 0, \quad (1.4)$$

where $\Omega = \{g \mid g : [0, 1] \rightarrow [0, \infty) \text{ is continuous, strictly decreasing, } g(1) = 0, \text{ and } g(0) < \infty\}$.

DEFINITION 1.6. A 2 NA Menger PM-space (X, F, Δ) is said to be of type $(D)_g$ if there exists a $g \in \Omega$ such that

$$g(\Delta(t_1, t_2, t_3)) \leq g(t_1) + g(t_2) + g(t_3) \quad \forall t_1, t_2, t_3 \in [0, 1]. \quad (1.5)$$

REMARK 1.7. If 2 NA Menger PM-space (X, F, Δ) is of type $(D)_g$, then (X, F, Δ) is of type $(C)_g$.

Throughout this paper, let (X, F, Δ) be a complete 2 NA Menger PM-space with a continuous strictly increasing t -norm Δ . Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a function satisfying the condition (Φ) ϕ is upper semi-continuous from right and $\phi(t) < t$ for all $t > 0$.

LEMMA 1.8. If a function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfies the condition (Φ) , then we get

- (1) For all $t \geq 0$, $\lim_{n \rightarrow \infty} \phi^n(t) = 0$, where $\phi^n(t)$ is the n th iteration of $\phi(t)$.
- (2) If $\{t_n\}$ is a nondecreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$, $n = 1, 2, \dots$. Then $\lim_{n \rightarrow \infty} t_n = 0$. In particular, if $t \leq \phi(t)$ for all $t \geq 0$, then $t = 0$.

LEMMA 1.9. Let $\{y_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} F_{y_n, y_{n+1}, a}(t) = 1$ for all $t > 0$. If the sequence $\{y_n\}$ is not a Cauchy sequence in X , then there exist $\epsilon_0 > 0$, $t_0 > 0$, and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that

- (i) $m_i > n_i + 1$ and $n_i \rightarrow \infty$ as $i \rightarrow \infty$.
- (ii) $F_{y_{m_i}, y_{n_i}, a}(t_0) < 1 - \epsilon_0$ and $F_{y_{m_i-1}, y_{n_i}, a}(t_0) \geq 1 - \epsilon_0$, $i = 1, 2, \dots$

DEFINITION 1.10. Let $A, S : X \rightarrow X$ be mappings, A and S are said to be compatible if

$$\lim_{n \rightarrow \infty} g(F_{ASx_n, SAx_n, a}(t)) = 0 \quad \forall t > 0, a \in X, \quad (1.6)$$

when $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = z = \lim_{n \rightarrow \infty} Sx_n \quad \text{for some } z \in X. \quad (1.7)$$

DEFINITION 1.11. Let $A, S : X \rightarrow X$ be mappings, A and S are said to be compatible of type (A) if

$$\lim_{n \rightarrow \infty} g(F_{ASx_n, SSx_n, a}(t)) = 0 = \lim_{n \rightarrow \infty} g(F_{SAx_n, AAx_n, a}(t)) \quad \forall t > 0, a \in X, \quad (1.8)$$

when $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = z = \lim_{n \rightarrow \infty} Sx_n \quad \text{for some } z \in X. \quad (1.9)$$

DEFINITION 1.12. Let $A, S : X \rightarrow X$ be mappings, A and S are said to be weak compatible of type (A) if

$$\begin{aligned} \lim_{n \rightarrow \infty} g(F_{ASx_n, SSx_n, a}(t)) &\geq \lim_{n \rightarrow \infty} g(F_{Sx_n, SSx_n, a}(t)), \\ \lim_{n \rightarrow \infty} g(F_{Sx_n, Ax_n, a}(t)) &\geq \lim_{n \rightarrow \infty} g(F_{ASx_n, Ax_n, a}(t)) \quad \forall t > 0, a \in X, \end{aligned} \quad (1.10)$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = z = \lim_{n \rightarrow \infty} Sx_n \quad \text{for some } z \in X. \quad (1.11)$$

PROPOSITION 1.13. Let $A, S : X \rightarrow X$ be continuous mappings. If A and S are compatible of type (A), then they are weak compatible of type (A).

PROOF. Suppose that A and S are compatible of type (A). Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = z = \lim_{n \rightarrow \infty} Sx_n \quad \text{for some } z \in X, \quad (1.12)$$

then

$$\begin{aligned} \lim_{n \rightarrow \infty} g(F_{Sx_n, SSx_n, a}(t)) &= 0 \\ &\leq \lim_{n \rightarrow \infty} g(F_{ASx_n, SSx_n, a}(t)) \\ &\Rightarrow \lim_{n \rightarrow \infty} g(F_{ASx_n, SSx_n, a}(t)) \geq \lim_{n \rightarrow \infty} g(F_{Sx_n, SSx_n, a}(t)). \end{aligned} \quad (1.13)$$

Similarly, we can show that

$$\begin{aligned} \lim_{n \rightarrow \infty} g(F_{Sx_n, Ax_n, a}(t)) &= 0 \\ &\geq \lim_{n \rightarrow \infty} g(F_{ASx_n, Ax_n, a}(t)). \end{aligned} \quad (1.14)$$

Therefore, A and S are weak compatible of type (A). $\square$

PROPOSITION 1.14. Let $A, S : X \rightarrow X$ be weak compatible mappings of type (A). If one of A and S is continuous, then A and S are compatible of type (A).

PROOF. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = z = \lim_{n \rightarrow \infty} Sx_n \quad \text{for some } z \in X. \quad (1.15)$$

Suppose S is continuous so $SSx_n, Sx_n \rightarrow Sz$ as $n \rightarrow \infty$. Since A and S are weak compatible of type (A), so we have

$$\begin{aligned} \lim_{n \rightarrow \infty} g(F_{ASx_n, SSx_n, a}(t)) &\geq \lim_{n \rightarrow \infty} g(F_{Sx_n, SSx_n, a}(t)) \\ &= \lim_{n \rightarrow \infty} g(F_{Sz, Sz, a}(t)) = 0. \end{aligned} \quad (1.16)$$

Thus

$$\lim_{n \rightarrow \infty} g(F_{ASx_n, SSx_n, a}(t)) = 0. \quad (1.17)$$

Similarly,

$$\lim_{n \rightarrow \infty} g(F_{Sx_n, Ax_n, a}(t)) = 0. \quad (1.18)$$

Hence A and S are compatible of type (A). $\square$

PROPOSITION 1.15. *Let $A, S : X \rightarrow X$ be continuous mappings. Then A and S are compatible of type (A) if and only if A and S are weak compatible of type (A).*

Note that Proposition 1.15 is a direct consequence of Propositions 1.13 and 1.14.

PROPOSITION 1.16. *Let $A, S : X \rightarrow X$ be mappings. If A and S are weak compatible of type (A) and $Az = Sz$ for some $z \in X$. Then $SAz = AAz = ASz = SSz$.*

PROOF. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = z$, $n = 1, 2, \dots$, and $Az = Sz$ for some $z \in X$. Then we have $Ax_n, Sx_n \rightarrow Sz$ as $n \rightarrow \infty$. Since A and S are weak compatible of type (A) so

$$\begin{aligned} \lim_{n \rightarrow \infty} g(F_{ASx_n, SSx_n, a}(t)) &\geq \lim_{n \rightarrow \infty} g(F_{S Ax_n, SSx_n, a}(t)), \\ \lim_{n \rightarrow \infty} g(F_{S Ax_n, AAx_n, a}(t)) &\geq \lim_{n \rightarrow \infty} g(F_{ASx_n, AAx_n, a}(t)). \end{aligned} \quad (1.19)$$

Now

$$\begin{aligned} \lim_{n \rightarrow \infty} g(F_{SAz, AAz, a}(t)) &= \lim_{n \rightarrow \infty} g(F_{S Ax_n, AAx_n, a}(t)) \geq \lim_{n \rightarrow \infty} g(F_{ASx_n, AAx_n, a}(t)) \\ &= g(F_{ASz, SSz, a}(t)). \end{aligned} \quad (1.20)$$

Since $Sz = Az$, then $SAz = AAz$. Similarly, we have $ASz = SSz$. But $Az = Sz$ for $z \in X$ implies that $AAz = ASz = SAz = SSz$. $\square$

PROPOSITION 1.17. *Let $A, S : X \rightarrow X$ be weak compatible mappings of type (A) and let $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} Ax_n = z = \lim_{n \rightarrow \infty} Sx_n$ for some $z \in X$, then*

- (1) $\lim_{n \rightarrow \infty} ASx_n = Sz$ if S is continuous at z .
- (2) $SAz = ASz$ and $Az = Sz$ if A and S are continuous at z .

PROOF. Suppose that S is continuous and $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = z = \lim_{n \rightarrow \infty} Sx_n \quad \text{for some } z \in X, \quad (1.21)$$

so

$$SSx_n \rightarrow Sz \quad \text{as } n \rightarrow \infty. \quad (1.22)$$

Since A and S are weak compatible of type (A), we have

$$\begin{aligned} g(F_{ASx_n, Sz, a}(t)) &= \lim_{n \rightarrow \infty} g(F_{ASx_n, SSx_n, a}(t)) \\ &\geq \lim_{n \rightarrow \infty} g(F_{S Ax_n, SSx_n, a}(t)) \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned} \quad (1.23)$$

for all $t > 0$ which implies that $ASx_n \rightarrow Sz$ as $n \rightarrow \infty$.

(2) Suppose that A and S are continuous at z . Since $Ax_n \rightarrow z$ as $n \rightarrow \infty$ and S is continuous at z , by Proposition 1.17(1) $ASx_n \rightarrow Sz$ as $n \rightarrow \infty$. On the other hand, since $Sx_n \rightarrow z$ as $n \rightarrow \infty$ and A is also continuous at z , $ASx_n \rightarrow Az$ as $n \rightarrow \infty$. Thus $Az = Sz$ by the uniqueness of the limit and so by Proposition 1.16, $SAz = AAz = ASz = SSz$. Therefore, we have $ASz = SAz$. $\square$

THEOREM 1.18. *Let $A, B, S, T : X \rightarrow X$ be mappings satisfying*

- (i) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

- (ii) the pairs A, S and B, T are weak compatible of type (A) ,
- (iii) S and T is continuous,
- (iv) $g(F_{Ax,By,a}(t)) \leq \phi(\max\{g(F_{Sx,Ty,a}(t)), g(F_{Sx,Ax,a}(t)), g(F_{Ty,By,a}(t)), (1/2)(g(F_{Sx,By,a}(t)) + g(F_{Ty,Ax,a}(t)))\})$,

for all $t > 0$, $a \in X$ where a function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfies the condition (Φ) . Then by (i) since $A(X) \subset T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this x_1 , we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on, inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \quad \text{for } n = 0, 1, 2, \dots \quad (1.24)$$

First we prove the following lemma.

LEMMA 1.19. Let $A, S : X \rightarrow X$ be mappings satisfying conditions (i) and (iv), then the sequence $\{y_n\}$ defined by (1.24), such that

$$\lim_{n \rightarrow \infty} g(F_{y_n, y_{n+1}}(t)) = 0 \quad \forall t > 0, a \in X, \quad (1.25)$$

is a Cauchy sequence in X .

PROOF. Since $g \in \Omega$, it follows that $\lim_{n \rightarrow \infty} (F_{y_n, y_{n+1}, a}(t)) = 0$ for all $a > 0$, $a \in X$ if and only if $\lim_{n \rightarrow \infty} g(F_{y_n, y_{n+1}, a}(t)) = 0$ for all $t > 0$. By Lemma 1.9, if $\{y_n\}$ is not a Cauchy sequence in X , there exist $\epsilon_0 > 0$, $t_0 > 0$, and two sequences $\{m_i\}, \{n_i\}$ of positive integers such that

- (A) $m_i > n_i + 1$ and $n_i \rightarrow \infty$ as $i \rightarrow \infty$,
- (B) $g(F_{y_{m_i}, y_{n_i}, a}(t_0)) > g(1 - \epsilon_0)$ and $g(F_{y_{m_i-1}, y_{n_i}, a}(t_0)) \leq g(1 - \epsilon_0)$, $i = 1, 2, \dots$, since $g(t) = 1 - t$. Thus we have

$$\begin{aligned} g(1 - \epsilon_0) &< g(F_{y_{m_i}, y_{n_i}, a}(t_0)) \\ &\leq g(F_{y_{m_i}, y_{n_i}, y_{m_i-1}}(t_0)) + g(F_{y_{m_i}, y_{m_i-1}, a}(t_0)) + g(F_{y_{m_i-1}, y_{n_i}, a}(t_0)) \\ &\leq g(F_{y_{m_i}, y_{n_i}, y_{m_i-1}}(t_0)) + g(F_{y_{m_i}, y_{m_i-1}, a}(t_0)) + g(1 - \epsilon_0). \end{aligned} \quad (1.26)$$

As $i \rightarrow \infty$ in (1.26), we have

$$\lim_{n \rightarrow \infty} g(F_{y_{m_i}, y_{n_i}, a}(t_0)) = g(1 - \epsilon_0). \quad (1.27)$$

On the other hand, we have

$$\begin{aligned} g(1 - \epsilon_0) &< g(F_{y_{m_i}, y_{n_i}, a}(t_0)) \\ &\leq g(F_{y_{m_i}, y_{n_i}, y_{n_i+1}}(t_0)) + g(F_{y_{m_i}, y_{n_i+1}, a}(t_0)) + g(F_{y_{n_i+1}, y_{n_i}, a}(t_0)). \end{aligned} \quad (1.28)$$

Now, consider $g(F_{y_{m_i}, y_{n_i+1}, a}(t_0))$ in (1.28), assume that both n_i and m_i are even. Then

by (iv), we have

$$\begin{aligned}
 g(F_{y_{m_i}, y_{n_i+1}, a}(t_0)) &= g(F_{Ax_{m_i}, Bx_{n_i+1}, a}(t_0)) \\
 &\leq \phi(\max\{g(F_{Sx_{m_i}, Tx_{n_i+1}, a}(t_0)), \\
 &\quad g(F_{Sx_{m_i}, Ax_{m_i}, a}(t_0)), g(F_{Tx_{n_i+1}, Bx_{n_i+1}, a}(t_0)), \\
 &\quad \frac{1}{2}(g(F_{Sx_{m_i}, Bx_{n_i+1}, a}(t_0)) + g(F_{Tx_{n_i+1}, Bx_{n_i+1}, a}(t_0)))\}) \\
 &= \phi(\max\{g(F_{y_{m_i-1}, y_{n_i}, a}(t_0)), \\
 &\quad g(F_{y_{m_i-1}, y_{m_i}, a}(t_0)), g(F_{y_{n_i}, y_{n_i+1}, a}(t_0)), \\
 &\quad \frac{1}{2}(g(F_{y_{m_i-1}, y_{n_i+1}, a}(t_0)) + g(F_{y_{n_i}, y_{m_i}, a}(t_0)))\}).
 \end{aligned} \tag{1.29}$$

By (1.27), (1.28), and (1.29), letting $i \rightarrow \infty$ in (1.29), we have

$$g(1 - \epsilon_0) \leq \phi(\max\{g(1 - \epsilon_0), 0, 0, g(1 - \epsilon_0)\}) = \phi(g(1 - \epsilon_0)) < g(1 - \epsilon_0), \tag{1.30}$$

which is a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in X . $\square$

Now, we prove our main theorem.

If we prove $\lim_{n \rightarrow \infty} g(F_{y_n, y_{n+1}, a}(t)) = 0$ for all $t > 0$, then by Lemma 1.19, the sequence $\{y_n\}$ defined by (1.24) is a Cauchy sequence in X .

First we prove $\lim_{n \rightarrow \infty} g(F_{y_n, y_{n+1}, a}(t)) = 0$ for all $t > 0$. In fact, by Theorem 1.18(iv) and (1.24), we have

$$\begin{aligned}
 g(F_{y_{2n}, y_{2n+1}, a}(t)) &= g(F_{Ax_{2n}, Bx_{2n+1}, a}(t)) \\
 &\leq \phi(\max\{g(F_{Sx_{2n}, Tx_{2n+1}, a}(t)), \\
 &\quad g(F_{Sx_{2n}, Ax_{2n}, a}(t)), g(F_{Tx_{2n}, Bx_{2n+1}, a}(t)), \\
 &\quad \frac{1}{2}(g(F_{Sx_{2n}, Bx_{2n+1}, a}(t)) + g(F_{Tx_{2n+1}, Ax_{2n}, a}(t)))\}) \\
 &= \phi(\max\{g(F_{y_{2n-1}, y_{2n}, a}(t)), \\
 &\quad g(F_{y_{2n-1}, y_{2n}, a}(t)), g(F_{y_{2n}, y_{2n+1}, a}(t)), \\
 &\quad \frac{1}{2}(g(F_{y_{2n-1}, y_{2n+1}, a}(t)) + g(1))\}) \\
 &\leq \phi(\max\{g(F_{y_{2n-1}, y_{2n}, a}(t)), g(F_{y_{2n}, y_{2n+1}, a}(t)), \\
 &\quad g(F_{y_{2n-1}, y_{2n}, a}(t)) + g(F_{y_{2n}, y_{2n+1}, a}(t))\})
 \end{aligned} \tag{1.31}$$

if $g(F_{y_{2n-1}, y_{2n}, a}(t)) \leq g(F_{y_{2n}, y_{2n+1}, a}(t))$ for all $t > 0$, then by Theorem 1.18(iv), $g(F_{y_{2n}, y_{2n+1}, a}(t)) \leq \phi(g(F_{y_{2n}, y_{2n+1}, a}(t)))$ and thus, by Lemma 1.8, $g(F_{y_{2n}, y_{2n+1}, a}(t)) = 0$ for all $t > 0$. Similarly, we have $g(F_{y_{2n+1}, y_{2n+2}, a}(t)) = 0$, thus we have $\lim_{n \rightarrow \infty} g(F_{y_n, y_{n+1}, a}(t)) = 0$ for all $t > 0$. On the other hand, if $g(F_{y_{2n-1}, y_{2n}, a}(t)) \geq g(F_{y_{2n}, y_{2n+1}, a}(t))$, then by Theorem 1.18(iv), we have

$$g(F_{y_{2n}, y_{2n+1}, a}(t)) \leq \phi(g(F_{y_{2n-1}, y_{2n}, a}(t))) \quad \forall t > 0. \tag{1.32}$$

Similarly,

$$g(F_{y_{2n+1}, y_{2n+2}, a}(t)) \leq \phi(g(F_{y_{2n}, y_{2n+1}, a}(t))) \quad \forall t > 0, \quad (1.33)$$

hence

$$g(F_{y_n, y_{n+1}, a}(t)) \leq \phi(g(F_{y_{n-1}, y_n, a}(t))) \quad \forall t > 0, \quad n = 1, 2, 3, \dots, \quad (1.34)$$

therefore by [Lemma 1.8](#),

$$\lim_{n \rightarrow \infty} g(F_{y_n, y_{n+1}, a}(t)) = 0 \quad \forall t > 0, \quad (1.35)$$

which implies that $\{y_n\}$ is a Cauchy sequence in X by [Lemma 1.19](#). Since (X, F, Δ) is complete, the sequence $\{y_n\}$ converges to a point $z \in X$ and so the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$, and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to the limit z .

Now, suppose that T is continuous. Since B and T are weak compatible of type (A), by [Proposition 1.17](#), BTx_{2n+1}, TTx_{2n+1} tend to Tz as n tends to ∞ . Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in [Theorem 1.18\(iv\)](#), we have

$$\begin{aligned} & g(F_{Ax_{2n}, BTx_{2n+1}, a}(t)) \\ & \leq \phi(\max\{g(F_{Sx_{2n}, TTx_{2n+1}, a}(t)), g(F_{Sx_{2n}, Ax_{2n}, a}(t)), g(F_{TTx_{2n+1}, BTx_{2n+1}, a}(t)), \\ & \quad \frac{1}{2}(g(F_{Sx_{2n}, BTx_{2n+1}, a}(t)) + g(F_{TTx_{2n+1}, Ax_{2n}, a}(t)))\}) \quad \forall t > 0. \end{aligned} \quad (1.36)$$

Letting $n \rightarrow \infty$ in (1.36), we get

$$\begin{aligned} g(F_{z, Tz, a}(t)) & \leq \phi(\max\{g(F_{z, Tz, a}(t)), g(F_{z, z, a}(t)), g(F_{Tz, Tz, a}(t)), \\ & \quad \frac{1}{2}(g(F_{z, Tz, a}(t)) + g(F_{Tz, z, a}(t)))\}) \\ & = \phi(g(F_{z, Tz, a}(t))) \quad \forall t > 0, \end{aligned} \quad (1.37)$$

which means that $g(F_{z, Tz, a}(t)) = 0$ for all $t > 0$ by [Lemma 1.8](#) and so we have $Tz = z$. Again replacing x by x_{2n} and y by z in [Theorem 1.18\(iv\)](#), we have

$$\begin{aligned} g(F_{Ax_{2n}, Bz, a}(t)) & \leq \phi(\max\{g(F_{Sx_{2n}, Tz, a}(t)), g(F_{Sx_{2n}, Ax_{2n}, a}(t)), g(F_{Tz, Bz, a}(t)), \\ & \quad \frac{1}{2}(g(F_{Sx_{2n}, Bz, a}(t)) + g(F_{Tz, Ax_{2n}, a}(t)))\}) \quad \forall t > 0. \end{aligned} \quad (1.38)$$

Letting $n \rightarrow \infty$ in (1.38), we get

$$\begin{aligned} g(F_{z, Bz, a}(t)) & \leq \phi(\max\{g(F_{z, z, a}(t)), g(F_{z, Bz, a}(t)), g(F_{Bz, Bz, a}(t)), \\ & \quad \frac{1}{2}(g(F_{z, Bz, a}(t)) + g(F_{z, z, a}(t)))\}) \quad \forall t > 0, \end{aligned} \quad (1.39)$$

which implies that $g(F_{z, Bz, a}(t)) \leq \phi(g(F_{z, Bz, a}(t)))$ for all $t > 0$ and so we have $Bz = z$. Since $B(x) \subset S(X)$, there exists a point $w \in X$ such that $Bz = Sw = z$. By using condition [Theorem 1.18\(iv\)](#) again, we have

$$\begin{aligned} g(F_{Aw, z, a}(t)) & = g(F_{Aw, Bz, a}(t)) \\ & \leq \phi(\max\{g(F_{Sw, Tz, a}(t)), g(F_{Sw, Aw, a}(t)), g(F_{Tz, Bz, a}(t)), \\ & \quad \frac{1}{2}(g(F_{Sw, Bz, a}(t)) + g(F_{Tz, Aw, a}(t)))\}) \\ & \leq \phi(g(F_{Aw, z, a}(t))) \quad \forall t > 0, \end{aligned} \quad (1.40)$$

which means that $Aw = z$. Since A and S are weak compatible mappings of type (A) and $Aw = Sw = z$, by [Proposition 1.16](#), $Az = ASw = SSw = Sz$. Again by using [Theorem 1.18\(iv\)](#), we have $Az = z$.

Therefore, $Az = Bz = Sz = Tz = z$, that is, z is a common fixed point of the given mappings A, B, S, T . The uniqueness of the common fixed point z follows easily from [Theorem 1.18\(iv\)](#).

REMARK 1.20. In [Theorem 1.18](#), if S and T are continuous, then by [Proposition 1.15](#), the theorem is true even though the pairs A, S and B, T are compatible of type (A) instead of the condition (ii).

Application

THEOREM 1.21. Let (X, F, t) be a complete 2 NA Menger PM-space and A, B, S , and T be the mappings from the product $X \times X$ to X such that

$$\begin{aligned} A(X \times \{y\}) &\subseteq T(X \times \{y\}), & B(X \times \{y\}) &\subseteq S(X \times \{y\}), \\ g(F_{A(T(x,y),y),T(A(x,y),y),a}(t)) &\leq g(F_{A(x,y),T(x,y),a}(t)), \\ g(F_{B(S(x,y),y),S(B(x,y),y),a}(t)) &\leq g(F_{B(x,y),S(x,y),a}(t)), \end{aligned} \quad (1.41)$$

for all $t > 0$. If S and T are continuous with respect to their direct argument and

$$\begin{aligned} &g(F_{A(x,y),B(x',y'),a}(t)) \\ &\leq \phi(\max\{g(F_{S(x,y),T(x',y'),a}(t)), \\ &\quad g(F_{S(x,y),A(x,y),a}(t)), g(F_{T(x',y'),B(x',y'),a}(t)), \\ &\quad \frac{1}{2}(g(F_{S(x,y),B(x',y'),a}(t)) + g(F_{T(x',y'),A(x,y),a}(t)))\}) \end{aligned} \quad (1.42)$$

for all $t > 0$ and x, y, x', y' in X , then there exists only one point b in X such that

$$A(b, y) = S(b, y) = B(b, y) = T(b, y) \quad \forall y \text{ in } X. \quad (1.43)$$

PROOF. By (1.42),

$$\begin{aligned} &g(F_{A(x,y),B(x',y'),a}(t)) \\ &\leq \phi(\max\{g(F_{S(x,y),T(x',y'),a}(t)), \\ &\quad g(F_{S(x,y),A(x,y),a}(t)), g(F_{T(x',y'),B(x',y'),a}(t)), \\ &\quad \frac{1}{2}(g(F_{S(x,y),B(x',y'),a}(t)) + g(F_{T(x',y'),A(x,y),a}(t)))\}) \end{aligned} \quad (1.44)$$

for all $t > 0$, therefore by [Theorem 1.18](#), for each y in X , there exists only one $x(y)$ in X such that

$$A(x(y), y) = S(x(y), y) = B(x(y), y) = T(x(y), y) = x(y), \quad (1.45)$$

for every y, y' in X

$$\begin{aligned}
 g(F_{x(y), x(y'), a}(t)) &= g(F_{A(x(y), y), A(x(y'), y'), a}(t)) \\
 &\leq \phi(\max\{g(F_{A(x, y), A(x, y'), a}(t)), \\
 &\quad g(F_{A(x, y), A(x, y), a}(t)), g(F_{T(x', y'), A(x', y'), a}(t)), \\
 &\quad \frac{1}{2}(g(F_{A(x, y), A(x', y'), a}(t)) + g(F_{A(x', y'), A(x, y), a}(t)))\}) \\
 &= g(F_{x(y), x(y'), a}(t)).
 \end{aligned}
 \tag{1.46}$$

This implies that $x(y) = x(y')$ and hence $x(\cdot)$ is some constant $b \in X$ so that

$$A(b, y) = b = T(b, y) = S(b, y) = B(b, y) \quad \forall y \text{ in } X. \tag{1.47}$$

□

REFERENCES

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