A GAUSS TYPE FUNCTIONAL EQUATION

SILVIA TOADER, THEMISTOCLES M. RASSIAS, and GHEORGHE TOADER

(Received 30 March 2000)

ABSTRACT. Gauss' functional equation (used in the study of the arithmetic-geometric mean) is generalized by replacing the arithmetic mean and the geometric mean by two arbitrary means.

2000 Mathematics Subject Classification. Primary 39B22; Secondary 26E60.

1. Introduction. By mean we understand a function $M : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies the condition

$$\min(a, b) \leq M(a, b) \leq \max(a, b) \quad \forall a, b > 0. \quad (1.1)$$

The mean is called symmetric if

$$M(a, b) = M(b, a) \quad \forall a, b > 0. \quad (1.2)$$

Usual examples are the power means given by

$$P_n(a, b) = \left(\frac{a^n + b^n}{2} \right)^{1/n} \quad (1.3)$$

for $n \neq 0$, while for $n = 0$ it is the geometric mean

$$P_0(a, b) = G(a, b) = \sqrt{ab}. \quad (1.4)$$

Of course, the arithmetic mean is $A = P_1$.

If M is a mean and $p : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a strictly monotonous function, the expression

$$M(p)(a, b) = p^{-1}[M(p(a), p(b))]$$
(1.5)

defines another mean $M(p)$ which is called M -quasi mean (see [1]). For example, the power means are A -quasi means. More exactly $P_n = A(e_n)$, where

$$e_n(x) = x^n \quad \text{for } n \neq 0, \quad e_0(x) = \ln x. \quad (1.6)$$

In what follows, we refer to another famous example of mean. Given two positive numbers a and b , we define successively the terms

$$a_{n+1} = A(a_n, b_n), \quad b_{n+1} = G(a_n, b_n), \quad n \geq 0, \quad (1.7)$$

where $a_0 = a$ and $b_0 = b$. It is known (see [1]) that $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ are convergent to a common limit which is denoted by $A \otimes G(a, b)$. It defines the arithmetic-geometric mean of Gauss $A \otimes G$.

The following representation formula is also known (see [1])

$$A \otimes G(a, b) = [f(a, b)]^{-1}, \quad (1.8)$$

where

$$f(a, b) = \frac{1}{2 \cdot \pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{a^2 \cdot \cos^2 \theta + b^2 \cdot \sin^2 \theta}}. \quad (1.9)$$

The proof of this formula is based on the fact that the function f verifies the relation

$$f(A(a, b), G(a, b)) = f(a, b), \quad (1.10)$$

which is called Gauss' functional equation.

These results were generalized as follows. We denote

$$\begin{aligned} r_n(\theta) &= (a^n \cdot \cos^2 \theta + b^n \cdot \sin^2 \theta)^{1/n}, \quad n \neq 0, \\ r_0(\theta) &= \lim_{n \rightarrow 0} r_n(\theta) = a^{\cos^2 \theta} b^{\sin^2 \theta}. \end{aligned} \quad (1.11)$$

If $p: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a strictly monotonous function, then

$$M_{p,n}(a, b) = p^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} p(r_n(\theta)) d\theta \right) \quad (1.12)$$

defines a symmetric mean. The arithmetic-geometric mean of Gauss is obtained for $n = 2$ and $p(x) = x^{-1}$. For $n = -2$ and $p(x) = x^{-2}$ the mean can be found in [7]. The case $n = 1$ and $p = \log$ was studied in [2]. The essential step was done in [4] by the consideration of the definition (1.12) for $n = 2$ with an arbitrary p . The values $n = -1$ and $n = 1$ were studied in [5, 6]. The general case (of arbitrary n) was studied in [8] and continued in [9]. In [8], Gauss' functional equation was also replaced by a more general equation

$$F(P_q(a, b), P_s(a, b)) = F(a, b). \quad (1.13)$$

In this paper, we generalize the mean (1.12) as well as the functional equation (1.13).

2. An integral mean. We consider the strictly monotonous functions p and q . Using them, we define the functions

$$\begin{aligned} r_q(\theta) &= q^{-1} [q(a) \cdot \cos^2 \theta + q(b) \cdot \sin^2 \theta], \\ f(a, b; p, q) &= \frac{1}{2\pi} \int_0^{2\pi} p[r_q(\theta)] d\theta. \end{aligned} \quad (2.1)$$

It is easy to prove that

$$M_{p,q}(a, b) = p^{-1} [f(a, b; p, q)] \quad (2.2)$$

defines a mean. Choosing $q = e_n$ we obtain $M_{p,q} = M_{p,n}$. We so have generalized the means (1.12). On the other hand, if we let $p \circ q^{-1} = Q$, we have $M_{p,q} = M_{Q,1}(q)$. Thus, $M_{p,q}$ is a $M_{Q,1}$ -quasi mean.

It is thus enough to consider the function

$$f(a, b; p) = \frac{1}{2\pi} \int_0^{2\pi} p(a \cdot \cos^2 \theta + b \cdot \sin^2 \theta) d\theta \quad (2.3)$$

which defines the mean $M_p = M_{p,1}$ by

$$M_p(a, b) = p^{-1}[f(a, b; p)]. \quad (2.4)$$

In what follows, we assume that the function p is two times differentiable. From any of the papers [5, 6, 8, 9], we can deduce the following result.

LEMMA 2.1. *The function f defined by (2.3) has the following partial derivatives:*

$$\begin{aligned} f'_a(c, c; p) &= f'_b(c, c; p) = \frac{1}{2} \cdot p'(c), \\ f''_{aa}(c, c; p) &= f''_{bb}(c, c; p) = \frac{3}{8} \cdot p''(c), \\ f''_{ab}(c, c; p) &= \frac{1}{8} \cdot p''(c). \end{aligned} \quad (2.5)$$

3. The functional equation. We replace (1.13) by a more general functional equation

$$F(M(a, b), N(a, b)) = F(a, b), \quad (3.1)$$

where M and N are two given means. We prove the following result.

LEMMA 3.1. *If the function f , defined by (2.3), verifies the functional equation (3.1), then the function p is a solution of the differential equation*

$$\begin{aligned} p''(c) \cdot \{ [3 \cdot M'_b(c, c) + N'_b(c, c)] \cdot M'_a(c, c) + [M'_b(c, c) + 3 \cdot N'_b(c, c)] \cdot N'_a(c, c) - 1 \} \\ + 4 \cdot p''(c) \cdot [M''_{ab}(c, c) + N''_{ab}(c, c)] = 0. \end{aligned} \quad (3.2)$$

PROOF. Taking in (3.1) the partial derivatives with respect to a , we obtain

$$F'_a[M(a, b), N(a, b)] \cdot M'_a(a, b) + F'_b[M(a, b), N(a, b)] \cdot N'_a(a, b) = F'_a(a, b). \quad (3.3)$$

Taking again the derivatives with respect to b , it follows that

$$\begin{aligned} \{ F''_{aa}[M(a, b), N(a, b)] \cdot M'_b(a, b) + F''_{ab}[M(a, b), N(a, b)] \cdot N'_b(a, b) \} \cdot M'_a(a, b) \\ + \{ F''_{ab}[M(a, b), N(a, b)] \cdot M'_b(a, b) + F''_{bb}[M(a, b), N(a, b)] \cdot N'_b(a, b) \} \cdot N'_a(a, b) \\ + F'_a[M(a, b), N(a, b)] \cdot M''_{ab}(a, b) + F'_b[M(a, b), N(a, b)] \cdot N''_{ab}(a, b) = F''_{ab}(a, b). \end{aligned} \quad (3.4)$$

For $a = b = c$ and the function $F = f$, defined by (2.3), we apply Lemma 2.1 and obtain (3.2). $\square$

CONSEQUENCE 3.2. *If the function f , defined by (2.3), verifies the functional equation (3.1), where the means M and N are symmetric, the function p is a solution of the differential equation*

$$p''(c) + 4 \cdot p'(c) \cdot [M''_{ab}(c, c) + N''_{ab}(c, c)] = 0. \quad (3.5)$$

PROOF. As the means are symmetric, their partial derivatives of the first order are equal to $1/2$ (see [3]), thus (3.2) becomes (3.5). $\square$

CONSEQUENCE 3.3. If the function f , defined by (2.3), verifies the functional equation (1.13), then the function p is given by

$$p(c) = C \cdot c^{r+s-1} + D, \quad (3.6)$$

where C and D are arbitrary constants.

PROOF. We have in (3.5), $M = P_r$ and $N = P_s$. Thus

$$M''_{ab}(c, c) = \frac{1-r}{4 \cdot c}, \quad N''_{ab}(c, c) = \frac{1-s}{4 \cdot c}. \quad (3.7)$$

Replacing them in (3.5), we obtain the differential equation

$$p''(c) + \frac{2-r-s}{c} \cdot p'(c) = 0 \quad (3.8)$$

with the solution given above. $\square$

REMARK 3.4. This last result was obtained in [8]. As it is shown in [9], the condition is also sufficient for $r = -s = 1$.

REMARK 3.5. Equation (3.1) can be further generalized at

$$F(g(M(a, b)), g(N(a, b))) = h(F(a, b)), \quad (3.9)$$

where g and h are two given functions. We have in view the following result given in [2]. The function f , defined by (2.3), verifies the relation

$$f(A^2(a, b), G^2(a, b); \log) = 2 \cdot f(a, b; \log). \quad (3.10)$$

4. Special means. A problem which is studied for the integral means defined in [4, 5, 6, 8, 9] is that of the determination of the cases in which the mean reduces at a given one, usually a power mean. Similar results can be given also in more general circumstances. We prove the following lemma.

LEMMA 4.1. *If for a given mean N , we have $M_p = N$, then the function p verifies the equation*

$$p''(c) \cdot [8 \cdot N'_a(c, c) \cdot N'_b(c, c) - 1] + 8 \cdot p'(c) \cdot N''_{ab}(c, c) = 0. \quad (4.1)$$

PROOF. In the given hypotheses, we have

$$f(a, b; p) = p[N(a, b)]. \quad (4.2)$$

Taking the partial derivatives with respect to a , we have

$$f'_a(a, b; p) = p'[N(a, b)] \cdot N'_a(a, b). \quad (4.3)$$

Then we take the derivative with respect to b , we obtain

$$f''_{ab}(a, b; p) = p''[N(a, b)] \cdot N'_a(a, b) \cdot N'_b(a, b) + p'[N(a, b)] \cdot N''_{ab}(a, b). \quad (4.4)$$

For $a = b = c$, Lemma 2.1 gives (4.1). $\square$

CONSEQUENCE 4.2. If we have $M_p = N$, with the symmetric mean N , then the function p verifies the equation

$$p''(c) + 8 \cdot p'(c) \cdot N_{ab}''(c, c) = 0. \quad (4.5)$$

CONSEQUENCE 4.3. If we have $M_p = P_r$, then the function p is given by

$$p(c) = C \cdot c^{2r-1} + D, \quad (4.6)$$

where C and D are arbitrary constants.

REMARK 4.4. In [9], it is shown that the above condition is also sufficient for $r = 0$, $1/2$, and 1 .

REFERENCES

- [1] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and their Inequalities*, Mathematics and its Applications (East European Series), vol. 31, D. Reidel Publishing Co., Massachusetts, 1988, translated and revised from the Serbo-Croatian. [MR 89d:26003](#). [Zbl 687.26005](#).
- [2] B. C. Carlson, *Invariance of an integral average of a logarithm*, Amer. Math. Monthly **82** (1975), 379–382. [MR 50#13628](#). [Zbl 301.33001](#).
- [3] D. M. E. Foster and G. M. Phillips, *A generalization of the Archimedean double sequence*, J. Math. Anal. Appl. **101** (1984), no. 2, 575–581. [MR 85g:40001](#). [Zbl 555.40002](#).
- [4] H. Haruki, *New characterizations of the arithmetic-geometric mean of Gauss and other well-known mean values*, Publ. Math. Debrecen **38** (1991), no. 3-4, 323–332. [MR 92f:39014](#). [Zbl 746.39004](#).
- [5] H. Haruki and T. M. Rassias, *A new analogue of Gauss' functional equation*, Int. J. Math. Math. Sci. **18** (1995), no. 4, 749–756. [MR 97a:39025](#). [Zbl 855.39019](#).
- [6] ———, *New characterizations of some mean-values*, J. Math. Anal. Appl. **202** (1996), no. 1, 333–348. [MR 97f:39026](#). [Zbl 878.39005](#).
- [7] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematics Studies, no. 27, Princeton University Press, New Jersey, 1951. [MR 13,270d](#). [Zbl 044.38301](#).
- [8] G. Toader, *Some mean values related to the arithmetic-geometric mean*, J. Math. Anal. Appl. **218** (1998), no. 2, 358–368. [MR 98m:26017](#). [Zbl 892.26015](#).
- [9] G. Toader and T. M. Rassias, *New properties of some mean values*, J. Math. Anal. Appl. **232** (1999), no. 2, 376–383. [MR 2000c:26020](#). [Zbl 924.39016](#).

SILVIA TOADER: DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY, RO-3400 CLUJ, ROMANIA

THEMISTOCLES M. RASSIAS: DEPARTMENT OF MATHEMATICS, NATIONAL TECHNICAL UNIVERSITY, 15780 ATHENS, GREECE

E-mail address: trassias@math.ntua.gr

GHEORGHE TOADER: DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY, RO-3400 CLUJ, ROMANIA

E-mail address: gheorghe.toader@math.utcluj.ro