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ON THE RATE OF CONVERGENCE OF BOOTSTRAPPED MEANS IN A BANACH SPACE

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ABSTRACT. We establish the complete convergence for arrays of Banach space valued random elements. This result is applied to bootstrapped means of random elements to obtain their strong consistency and is derived in the spirit of Baum-Katz/Hsu-Robbins/Spitzer type convergence.

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1. Introduction. The main focus of the present investigation is the determination of the rates of convergence for strong laws of large numbers for arrays of random elements. Interestingly, this result will be applied to establish the strong consistency for bootstrapped means taking values in Banach spaces. More precisely, we present Chung type strong law of large numbers for arrays of rowwise independent random elements under conditions similar to those given by Bozorgnia et al. [1]; Hu et al. [3]; and Sung [6]. This result is of interest since it holds for an arbitrary real separable Banach space without imposing any geometric conditions. Thus, the results of this paper are more general than those presented in the papers cited above.

Strong laws of large numbers are of practical use in establishing the strong asymptotic validity of the bootstrapped mean for random elements. Furthermore, they are of considerable theoretical and practical interest in investigating the consistency of *bootstrap estimators.*

Some results on the consistency of the bootstrapped mean of random elements in Banach spaces are given in [1]. However, these consistency results impose a geometric condition (Rademacher type p) on the Banach space. Moreover, the random elements from the original sample are assumed to be independently and identically distributed (i.i.d.). However, in the present investigation we do not make any assumptions regarding the marginal or joint distributions of the random elements forming the sample and the random elements assuming values in an arbitrary real separable Banach space. A similar situation was considered for real-valued random variables by Li et al. [4, Theorem 2.1].

In order to obtain strong consistency for the bootstrapped mean, we assume the corresponding weak consistency. In this case, the main result of Bozorgnia et al. [1] can be seen as a special case of the results given in Theorem 3.2 of this paper.

2. Chung's type strong law of large numbers. First, we state the following recent theorem which forms the basis of our results.

Theorem 2.1 (see [2, Theorem 3.2]). *Let $\{k_n,\ n \geq 1\}$ be a sequence of positive integers, let $\{Y_{nk},\ 1 \leq k \leq k_n,\ n \geq 1\}$ be an array of rowwise independent random elements taking values in a real separable Banach space, and let $\{c_n,\ n \geq 1\}$ be a sequence of positive constants. Suppose that $E\|Y_{nk}\|^q < \infty$, $1 \leq k \leq k_n$, $n \geq 1$ for some $0 < q \leq 2$.*

Moreover, assume that

(1) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\{\|Y_{nk}\| > \epsilon\} < \infty$ *for all* $\epsilon > 0$,

(2) $\sum_{k=1}^{k_n} Y_{nk} \xrightarrow{P} 0$,

(3) *there exists* $J \geq 1$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\|Y_{nk}\|^q \right)^J < \infty, \tag{2.1}$$

(4) $\sum_{k=1}^{k_n} P\{\|Y_{nk}\| > \delta\} = o(1)$ *for some* $\delta > 0$ *as* $n \to \infty$ *if* $\liminf_{n\to\infty} c_n = 0$. *Then*

$$\sum_{n=1}^{\infty} c_n P\left\{ \left\| \sum_{k=1}^{k_n} Y_{nk} \right\| > \epsilon \right\} < \infty \quad \forall \epsilon > 0. \tag{2.2}$$

Recently, Bozorgnia et al. [1], Hu et al. [3], and Sung [6] proved Chung's type strong laws of large numbers for arrays of rowwise independent random variables or random elements. We now apply Theorem 2.1 to obtain a similar result in a general real separable Banach space under the assumption that the corresponding weak law of large numbers holds. Theorem 2.2 is an adaptation of Theorem 2.1.

Theorem 2.2. *Let $\{Z_{nk},\ 1 \leq k \leq k_n,\ n \geq 1\}$ be an array of rowwise independent random elements taking values in a real separable Banach space and let $\{a_n,\ n \geq 1\}$ and $\{c_n,\ n \geq 1\}$ be sequences of positive constants. Suppose that $E\psi(\|Z_{nk}\|) < \infty$ and $E\|Z_{nk}\|^q < \infty$ for some $0 < q \leq 2$ and some continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $\epsilon > 0$,*

$$\sup_{n\geq 1} \frac{\psi(a_n)}{\psi(\epsilon a_n)} < \infty. \tag{2.3}$$

Then the conditions

(i) $\sum_{n=1}^{\infty} (c_n/\psi(a_n)) \sum_{k=1}^{k_n} E\psi(\|Z_{nk}\|) < \infty$,

(ii) $(1/a_n) \sum_{k=1}^{k_n} Z_{nk} \xrightarrow{P} 0$,

(iii) *there exists* $J \geq 1$ *such that*

$$\sum_{n=1}^{\infty} \frac{c_n}{a_n^{qJ}} \left(\sum_{k=1}^{k_n} E\|Z_{nk}\|^q \right)^J < \infty, \tag{2.4}$$

(iv) $(c_n/\psi(a_n)) \sum_{k=1}^{k_n} E\psi(\|Z_{nk}\|) = o(1)$ *as* $n \to \infty$ *if* $\liminf_{n\to\infty} c_n = 0$, *imply that*

$$\sum_{n=1}^{\infty} c_n P\left\{ \left\| \sum_{k=1}^{k_n} Z_{nk} \right\| > \epsilon a_n \right\} < \infty \quad \forall \epsilon > 0. \tag{2.5}$$

Proof. The assumptions of Theorem 2.1 for the array $\{Y_{nk} = Z_{nk}/a_n,\ 1 \le k \le k_n,\ n \ge 1\}$ can be rewritten as follows.

(1) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\{\|Z_{nk}\| > \epsilon a_n\} < \infty$ for all $\epsilon > 0$,

(2) $(1/a_n)\sum_{k=1}^{k_n} Z_{nk} \xrightarrow{P} 0$,

(3) There exists $J \ge 1$ such that

$$\sum_{n=1}^{\infty} \frac{c_n}{a_n^{qJ}} \left(\sum_{k=1}^{k_n} E\|Z_{nk}\|^q \right)^J < \infty, \tag{2.6}$$

(4) $\sum_{k=1}^{k_n} P\{\|Y_{nk}\| > \delta a_n\} = o(1)$ as $n \to \infty$ if $\liminf_{n\to\infty} c_n = 0$.

Note that conditions (ii) and (2) are the same and that (iii) is equivalent to (3). For (i) and (iv), using Markov's inequality and the assumptions on the function ψ, we have the following:

$$P\{\|Z_{nk}\| > \epsilon a_n\} \le \frac{1}{\psi(\epsilon a_n)} E\psi(\|Z_{nk}\|) \le \frac{C(\epsilon)}{\psi(a_n)} E\psi(\|Z_{nk}\|). \tag{2.7}$$

□

It should be pointed out that the assumptions on the function $\psi(\cdot)$ seem to be the most natural and general for the current work.

3. The consistency of the bootstrapped mean.

We now outline the bootstrap procedure. Let $\{X_n;\ n \ge 1\}$ be a sequence of (not necessarily independent or identically distributed) random elements defined on some complete probability space $(\Omega, \mathcal{F}, P)$ which take values in a real separable Banach space. For $\omega \in \Omega$ and $n \ge 1$, let $P_n(\omega) = n^{-1}\sum_{i=1}^{n} \delta_{X_i(\omega)}$ denote the empirical measure. For $n \ge 1$, let $\{\hat{X}_{n,j}^{\omega};\ 1 \le j \le k_n\}$ be i.i.d. random elements with law $P_n(\omega)$, where k_n is a positive integer. Let $\bar{X}_n(\omega)$ denote the sample mean of $\{X_i(\omega);\ 1 \le i \le n\}$, $n \ge 1$, that is, $\bar{X}_n(\omega) = (1/n)\sum_{i=1}^{n} X_i(\omega)$.

To prove the consistency of bootstrapped mean, we use the following lemma. We formulated this simple observation as a lemma since it is frequently applied in the proof.

Lemma 3.1. *If $s > 0$, then for almost every $\omega \in \Omega$,*

$$E\|\hat{X}_{n,1}^{\omega} - \bar{X}_n(\omega)\|^s \le A_s \left[\frac{1}{n} \sum_{i=1}^{n} \|X_i(\omega)\|^s + \|\bar{X}_n(\omega)\|^s \right], \tag{3.1}$$

where $A_s = 2^{s-1}$ for $s \ge 1$ and $A_s = 1$ for $0 < s < 1$.

Proof. For almost every $\omega \in \Omega$,

$$E\|\hat{X}_{n,j}^{\omega} - \bar{X}_n(\omega)\|^s = \frac{1}{n}\sum_{i=1}^{n} \|X_i(\omega) - \bar{X}_n(\omega)\|^s \le A_s \left[\frac{1}{n} \sum_{i=1}^{n} \left(\|X_i(\omega)\|^s + \|\bar{X}_n(\omega)\|^s \right) \right], \tag{3.2}$$

by the c_r-inequalities (Loève [5, page 157]). □

We can now prove the main application presented in this paper.

Theorem 3.2. *Let $\{X_n,\ n \ge 1\}$ be a sequence of random elements taking values in a real separable Banach space and let $\{a_n,\ n \ge 1\}$, $\{b_n,\ n \ge 1\}$, $\{c_n,\ n \ge 1\}$, and $\{d_n,\ n \ge 1\}$ be sequences of positive constants. Suppose that there exists $0 < q \le 2$ and $r \ge q$*

such that

(1) $\sup_{n\geq 1}(1/d_n)\|\bar{X}_n\| < \infty$ *a.s. and* $\sup_{n\geq 1}(1/b_n)\sum_{i=1}^{n}\|X_i\|^q < \infty$ *a.s.,*

(2) $\sum_{n=1}^{\infty}(c_n/a_n^r)k_n^{r/q}d_n^r < \infty$ *and* $\sum_{n=1}^{\infty}(c_n/a_n^r)b_n^{r/q}\max\{k_n/n, (k_n/n)^{r/q}\} < \infty$,

(3) *The bootstrapped mean is weakly consistent, that is, for almost every* $\omega \in \Omega$,

$$\frac{1}{a_n}\left\|\sum_{k=1}^{k_n}(\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega))\right\| \xrightarrow{P} 0, \tag{3.3}$$

(4) $c_n(b_n^{r/q}k_n/a_n^r n) \to 0$ *and* $c_n(k_n d_n^r)/a_n^r \to 0$ *if* $\liminf_{n\to\infty} c_n = 0$.

Then the bootstrapped mean is strongly consistent, that is, for almost every $\omega \in \Omega$ *and for all* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left\{\left\|\sum_{k=1}^{k_n}(\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega))\right\| > \epsilon a_n\right\} < \infty. \tag{3.4}$$

Proof. We need only check conditions (i)–(v) of Theorem 2.2 for the array $\{Z_{nk} = \hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega),\ 1 \leq k \leq k_n,\ n \geq 1\}$ with $\psi(t) = t^r$, $t \geq 0$.

For (i), an application of Lemma 3.1 with $s = r$ yields, for almost every $\omega \in \Omega$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{c_n}{\psi(a_n)}\sum_{k=1}^{k_n}E\psi(\|Z_{nk}\|) \\
&\quad = \sum_{n=1}^{\infty}\frac{c_n}{a_n^r}\sum_{k=1}^{k_n}E\|\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega)\|^r \\
&\quad \leq A_r\left(\sum_{n=1}^{\infty}\frac{c_n k_n}{a_n^r n}\sum_{i=1}^{n}\|X_i(\omega)\|^r + \sum_{n=1}^{\infty}\frac{c_n k_n}{a_n^r}\|\bar{X}_n(\omega)\|^r\right) \\
&\quad \leq A_r\left(\sum_{n=1}^{\infty}\frac{c_n k_n b_n^{r/q}}{a_n^r n}\left[\frac{1}{b_n}\sum_{i=1}^{n}\|X_i(\omega)\|^q\right]^{r/q} + \sum_{n=1}^{\infty}\frac{c_n k_n d_n^r}{a_n^r}\left[\frac{1}{d_n}\|\bar{X}_n(\omega)\|\right]^r\right) < \infty
\end{aligned} \tag{3.5}$$

by (1) and (2) and since $k_n \leq k_n^{r/q}$.

For (ii), let $J = r/q \geq 1$. Another application of Lemma 3.1 with $s = q$ yields, for almost every $\omega \in \Omega$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{c_n}{a_n^{qJ}}\left(\sum_{k=1}^{k_n}E\|\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega)\|^q\right)^J \\
&\quad \leq A_q^J 2^{J-1}\left(\sum_{n=1}^{\infty}\frac{c_n}{a_n^r}\left(\frac{k_n}{n}\right)^{r/q}\left[\sum_{i=1}^{n}\|X_i(\omega)\|^q\right]^{r/q} + \sum_{n=1}^{\infty}\frac{c_n k_n^{r/q}}{a_n^r}\|\bar{X}_n(\omega)\|^r\right) \\
&\quad \leq A_q^J\left(\sum_{n=1}^{\infty}\frac{c_n b_n^{r/q}}{a_n^r}\left(\frac{k_n}{n}\right)^{r/q}\left[\frac{1}{b_n}\sum_{i=1}^{n}\|X_i(\omega)\|^q\right]^{r/q} + \sum_{n=1}^{\infty}c_n k_n^{r/q}\left(\frac{d_n}{a_n}\right)^r\left(\frac{1}{d_n}\|\bar{X}_n(\omega)\|\right)^r\right) < \infty,
\end{aligned} \tag{3.6}$$

by (1) and (2).

Since (iii) is the same as (3), it is automatically satisfied. Now, considering (iv), by another application of the lemma with $s = r$ and by the same argument as in the proof of (i), we have, for almost every $\omega \in \Omega$,

$$\begin{aligned}
&\frac{c_n}{\psi(a_n)} \sum_{k=1}^{k_n} E\psi(||Z_{nk}||) \\
&\quad = \frac{c_n}{a_n^r} \sum_{k=1}^{k_n} E||\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega)||^r \\
&\quad \le A_r \left(\frac{c_n k_n}{a_n^r n} \sum_{i=1}^{n} ||X_i(\omega)||^r + \frac{c_n k_n}{a_n^r} ||\bar{X}_n(\omega)||^r \right) \\
&\quad \le A_s \left(c_n \frac{k_n b_n^{r/q}}{a_n^r n} \left(\frac{1}{b_n} \sum_{i=1}^{n} ||X_i(\omega)||^q \right)^{r/q} + c_n \frac{k_n d_n^r}{a_n^r} \left(\frac{1}{d_n} ||\bar{X}_n(\omega)|| \right)^r \right) \longrightarrow 0
\end{aligned} \tag{3.7}$$

in case $\liminf_{n\to\infty} c_n = 0$, by (4) and (1). □

It is natural to question whether the conditions of Theorem 3.2 can be easily verified. We give an example verifying the conditions of Theorem 3.2 by providing an alternative proof of the main result of Bozorgnia et al. [1, Theorem 3.1]. This proof is substantially simpler than the original one.

We recall that a Banach space is said to be of *type q*, $1 \le q \le 2$, if there exists a constant $C > 0$ such that, for any sequence $X_1, \dots, X_n$ of independent, mean zero random elements taking values in the Banach space, the following inequality holds:

$$E\left\|\sum_{i=1}^{n} X_i\right\|^q \le C \sum_{i=1}^{n} E||X_i||^q. \tag{3.8}$$

Corollary 3.3. *Let $\{X, X_n,\ n \ge 1\}$ be i.i.d. random elements taking values in a real separable Banach space of type q, $1 < q \le 2$ with $E\|X\|^q < \infty$ and $EX = \mu$. Then for all $\epsilon > 0$ and for almost every $\omega \in \Omega$,*

$$\sum_{n=1}^{\infty} P\left\{\left\|\sum_{k=1}^{n} (\hat{X}_{n,k}^{\omega} - \mu)\right\| > \epsilon n\right\} < \infty. \tag{3.9}$$

Proof. The first step is to prove that, for almost every $\omega \in \Omega$,

$$\sum_{n=1}^{\infty} P\left\{\left\|\sum_{k=1}^{n} (\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega))\right\| > \epsilon n\right\} < \infty. \tag{3.10}$$

In order to establish this fact, we need to prove that the assumptions of Theorem 3.2 are satisfied. To this end, let $a_n = b_n = k_n = n$, $c_n = d_n = 1$ and $r > (q/q-1)$. It is easy then to see that (2) is satisfied. Furthermore, there is no need to check (4). However, for (1) we note that $\bar{X}_n(\omega) \to \mu$ a.s. by Mourier's strong law of large numbers in Banach spaces and $(1/n)\sum_{i=1}^{n} \|X_i\|^q \to E\|X\|^q$ a.s. by the ordinary (or Kolmogorov) strong law of large numbers.

In order to obtain weak consistency (3), we apply the Markov inequality and use the fact that the Banach space is of type q. So, for almost every $\omega \in \Omega$,

$$\begin{aligned} P\left\{\left\|\sum_{k=1}^{n}\left(\hat{X}_{n,k}^{\omega}-\bar{X}_n(\omega)\right)\right\|>\epsilon n\right\} &\leq \frac{1}{\epsilon^q n^q} E\left\|\sum_{k=1}^{n}\left(\hat{X}_{n,k}^{\omega}-\bar{X}_n(\omega)\right)\right\|^q \\ &\leq \frac{C}{\epsilon^q n^q} nE\left\|\hat{X}_{n,1}^{\omega}-\bar{X}_n(\omega)\right\|^q \\ &\leq \frac{CA_q}{\epsilon^q n^{q-1}}\left[\frac{1}{n}\sum_{i=1}^{n}\left\|X_i(\omega)\right\|^q+\left\|\bar{X}_n(\omega)\right\|^q\right] \end{aligned} \tag{3.11}$$

again by an application of Lemma 3.1 with $s = q$. So, $P\{\|\sum_{k=1}^{n}(\hat{X}_{n,j}^{\omega}-\bar{X}_n(\omega))\| > \epsilon n\} \to 0$ almost every $\omega \in \Omega$, by the same argument as in the proof of (1).

The next step is to prove that $\sum_{n=1}^{\infty} P\{\|\sum_{k=1}^{n}(\hat{X}_{n,k}^{\omega}-\mu)\| > \epsilon n\} < \infty$ for almost every $\omega \in \Omega$. It suffices to note that since $\bar{X}_n(\omega) \to \mu$ a.s., for almost every $\omega \in \Omega$, there exists an integer $N = N(\omega)$ such that for all $n > N$ we have $\|\bar{X}_n(\omega) - \mu\| < \epsilon/2$. Let $n > N$, then

$$\begin{aligned} P\left\{\left\|\sum_{k=1}^{n}\left(\hat{X}_{n,k}^{\omega}-\mu\right)\right\|>\epsilon n\right\} &= P\left\{\left\|\left[\frac{1}{n}\sum_{k=1}^{n}\left(\hat{X}_{n,k}^{\omega}-\bar{X}_n(\omega)\right)\right]+\left[\bar{X}_n(\omega)-\mu\right]\right\|>\epsilon\right\} \\ &\leq P\left\{\left\|\sum_{k=1}^{n}\left(\hat{X}_{n,k}^{\omega}-\bar{X}_n(\omega)\right)\right\|>\frac{\epsilon n}{2}\right\}. \end{aligned} \tag{3.12}$$

□

Remark 3.4. (1) Theorems 2.1, 2.2, and 3.2 are trivial in case $\sum_{n=1}^{\infty} c_n < \infty$.

(2) We do not consider the case $q = 1$ in the corollary since our proof will not cover this situation.

(3) Finally, we are not able to compare Theorem 3.2 of this paper and Theorem 2.1 of Li et al. [4] since their assumptions concern convergence of partial sums, whereas we use only boundedness of partial sums.

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