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THE WAVE EQUATION APPROACH TO AN INVERSE EIGENVALUE PROBLEM FOR AN ARBITRARY MULTIPLY CONNECTED DRUM IN $\mathbb{R}^2$ WITH ROBIN BOUNDARY CONDITIONS

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ABSTRACT. The spectral function $\hat{\mu}(t) = \sum_{j=1}^{\infty} \exp(-it\mu_j^{1/2})$, where $\{\mu_j\}_{j=1}^{\infty}$ are the eigenvalues of the two-dimensional negative Laplacian, is studied for small $|t|$ for a variety of domains, where $-\infty < t < \infty$ and $i = \sqrt{-1}$. The dependencies of $\hat{\mu}(t)$ on the connectivity of a domain and the Robin boundary conditions are analyzed. Particular attention is given to an arbitrary multiply-connected drum in $\mathbb{R}^2$ together with Robin boundary conditions on its boundaries.

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1. Introduction. The underlying inverse problem is to deduce some geometric quantities associated with a bounded domain from complete knowledge of the eigenvalues $\{\mu_j\}_{j=1}^{\infty}$ for the negative Laplace operator $-\Delta = -\sum_{n=1}^{2}(\partial/\partial x^n)^2$ in the (x^1, x^2)-plane.

Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected bounded domain with a smooth boundary $\partial\Omega$. Consider the Robin problem

$$(\Delta+\mu)u = 0 \quad \text{in } \Omega, \qquad \left(\frac{\partial}{\partial n}+\gamma\right)u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\partial/\partial n$ denotes the differentiation along the inward pointing normal to $\partial\Omega$, γ is a positive constant (impedance), and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Denote its eigenvalues, counted according to multiplicity, by

$$0 < \mu_1 \le \mu_2 \le \mu_3 \le \cdots \le \mu_j \le \cdots \longrightarrow \infty \quad \text{as } j \longrightarrow \infty. \tag{1.2}$$

The basic problem is to determine some geometrical properties of Ω from the knowledge of its eigenvalues (1.2).

At the beginning of this century the principal problem was that of investigating the asymptotic behavior of the eigenvalues (1.2). It is well known (see [1]) that if $N(\mu)$ is the number of these eigenvalues less than μ, then

$$\begin{aligned} N(\mu) &\sim \frac{|\Omega|}{4\pi}\mu \quad \text{as } \mu \longrightarrow \infty \quad \text{(Weyl, 1912)}, \\ N(\mu) &= \frac{|\Omega|}{4\pi}\mu + O(\mu^{1/2}\log\mu) \quad \text{as } \mu \longrightarrow \infty \quad \text{(Courant, 1920)}, \end{aligned} \tag{1.3}$$

where $|\Omega|$ is the area of the domain Ω.

In order to obtain further information about the geometry of Ω, one studies certain functions of the spectrum. The most useful to date comes from the heat equation or the wave equation. Accordingly, let $e^{-t\Delta}$ denote the heat operator, then we can construct the trace function

$$\Theta(t) = \operatorname{tr}\left(e^{-t\Delta}\right) = \sum_{j=1}^{\infty} e^{-t\mu_j} \tag{1.4}$$

which converges for all positive t.

Let $e^{-it\Delta^{1/2}}$ be the wave operator, then an alternative to (1.4) is to study the trace function

$$\hat{\mu}(t) = \operatorname{tr}\left(e^{-it\Delta^{1/2}}\right) = \sum_{j=1}^{\infty} e^{-it\mu_j^{1/2}} \tag{1.5}$$

which represents a tempered distribution for $-\infty < t < \infty$ and $i = \sqrt{-1}$.

In the present paper, we shall concentrate our efforts on a study of the tempered distribution $\hat{\mu}(t)$ for small $|t|$.

Zayed et al. [22] have recently discussed problem (1.1) for small/large impedance γ, by using the wave equation approach and have determined some geometrical properties of Ω from the asymptotic expansion of $\hat{\mu}(t)$ as $|t| \to 0$.

Zayed [14] has shown that if $\gamma = 0$ (Neumann problem), then

$$\hat{\mu}(t) = \frac{|\Omega|}{2\pi t} H(|t|) + \frac{|\partial\Omega|}{8} \operatorname{sign} t + a_0|t| + O\left(t^2 \operatorname{sign} t\right) \quad \text{as } |t| \longrightarrow 0, \tag{1.6}$$

while, if $\gamma \to \infty$ (Dirichlet problem), then

$$\hat{\mu}(t) = \frac{|\Omega|}{2\pi t} H(|t|) - \frac{|\partial\Omega|}{8} \operatorname{sign} t + a_0|t| + O\left(t^2 \operatorname{sign} t\right) \quad \text{as } |t| \longrightarrow 0, \tag{1.7}$$

where $H(|t|)$ is the Heaviside's unit function, and

$$\operatorname{sign} t = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases} \tag{1.8}$$

An examination of the results (1.6), (1.7) shows that the first term of $\hat{\mu}(t)$ determines the area $|\Omega|$ of Ω, and the second term determines the total length $|\partial\Omega|$ of $\partial\Omega$ while the sign $\pm$ of the second term of $\hat{\mu}(t)$ determines whether we have the Neumann or the Dirichlet problem. The coefficient a_0 has geometric significance, for example, if Ω is smooth and convex, then $a_0 = 1/6$, and if Ω is permitted to have a finite number "h" of smooth convex holes, then $a_0 = (1-h)/6$. Further, the order term $O(t^2 \operatorname{sign} t)$ in (1.6) and (1.7) is yet undetermined. So, in the present paper, we discuss what geometric quantities are contained in this order term.

Let Ω be an arbitrary multiply-connected drum in $\mathbb{R}^2$ which is bounded internally by simply connected holes Ω_J with smooth boundaries $\partial\Omega_J$ $(J = 1,\dots,m-1)$ and externally by a smooth boundary $\partial\Omega_m$. Suppose that the eigenvalues (1.2) are given for the Robin problem

$$(\Delta+\mu)u = 0 \quad \text{in } \Omega, \tag{1.9}$$

$$\left(\frac{\partial}{\partial n_J}+\gamma_J\right)u = 0 \quad \text{on } \partial\Omega_J \ (J = 1,\dots,m), \tag{1.10}$$

where $\partial/\partial n_J$ denote differentiations along the inward normal to $\partial\Omega_J$, and γ_J are positive constants. The basic problem is that of determining some geometric quantities associated with the general multiply connected drum Ω in $\mathbb{R}^2$, using the asymptotic expansions of the spectral function $\hat{\mu}(t)$ for small $|t|$.

Note that Zayed et al. [22, 24] have discussed problem (1.9), (1.10) when $J = 1$ (i.e., Ω is a simply connected bounded domain) while Zayed et al. [21] and Zayed [15] have discussed this problem when $J = 1,2$ (i.e., Ω is a general annular drum and also Ω is a circular annulus (r,θ) such that $a \le r \le b$, $0 \le \theta \le 2\pi$). Therefore, problem (1.9), (1.10) can be considered as a more general one of that obtained in [15, 21, 22, 24], which does not seem to have been investigated elsewhere.

2. Statement of the results. Suppose that the boundaries $\partial\Omega_J$ $(J = 1,\dots,m)$ of the region Ω are given locally by the equations $x^n = y^n(\sigma_J)$ $(n = 1,2)$ in which σ_J $(J = 1,\dots,m)$ are the arc-lengths of the counterclock-wise oriented boundaries $\partial\Omega_J$ and $y^n(\sigma_J) \in C^\infty(\partial\Omega_J)$. Let L_J and $K_J(\sigma_J)$ be the lengths and the curvatures of the boundaries $\partial\Omega_J$ $(J = 1,\dots,m)$, respectively. Then, the results of the main problem (1.9), (1.10) can be summarized in the following cases.

Case 2.1 ($0 < \gamma_J \ll 1$ $(J = 1,\dots,k)$ and $\gamma_J \gg 1$ $(J = k+1,\dots,m)$**).**

$$\begin{aligned}
\hat{\mu}(t) = {} & \frac{|\Omega|}{2\pi t}H(|t|)+\frac{1}{8}\left\{\sum_{J=1}^{k}L_J-\sum_{J=k+1}^{m}\left[L_J+2\pi\gamma_J^{-1}\right]\right\}\operatorname{sign}t \\
& +\left\{(2-m)+\frac{3}{\pi}\left(\sum_{J=1}^{\ell}\gamma_J L_J-\sum_{J=\ell+1}^{k}\gamma_J L_J\right)\right\}\frac{|t|}{6} \\
& +\frac{1}{512}\left\{7\sum_{J=1}^{k}\int_{\partial\Omega_J}\left[K_J^2(\sigma_J)-\frac{64}{7}\left(\frac{\pi\gamma_J}{L_J}-\gamma_J^2\right)\right]d\sigma_J\right. \\
& \qquad\left.+\sum_{J=k+1}^{m}\int_{\partial\Omega_J}\left[K_J^2(\sigma_J)-\left(\frac{2\pi}{L_J}\right)^3\gamma_J^{-1}\right]d\sigma_J\right\}t^2\operatorname{sign}t \\
& +O\left(t^3\operatorname{sign}t\right) \quad \text{as } |t|\longrightarrow 0.
\end{aligned} \tag{2.1}$$

Remark 2.2. On setting $\gamma_J = 0$ $(J = 1,\dots,k)$ and $\gamma_J \to \infty$ $(J = k+1,\dots,m)$ in (2.1), we obtain the results of Neumann boundary conditions on $\partial\Omega_J$ $(J = 1,\dots,k)$ and Dirichlet boundary conditions on $\partial\Omega_J$ $(J = k+1,\dots,m)$.

Case 2.3 ($\gamma_J \gg 1$ $(J = 1,\dots,k)$ and $0 < \gamma_J \ll 1$ $(J = k+1,\dots,m)$**).** In this case, the asymptotic expansion of $\hat{\mu}(t)$ as $|t| \to 0$ has the same form (2.1) with the interchanges $\partial\Omega_J$ $(J = 1,\dots,k) \leftrightarrow \partial\Omega_J$ $(J = k+1,\dots,m)$ and γ_J $(J = 1,\dots,k) \leftrightarrow \gamma_J$ $(J = k+1,\dots,m)$.

Case 2.4 ($\gamma_J \gg 1$ $(J = 1,\dots,m)$).

$$\begin{aligned}\hat{\mu}(t) = {} & \frac{|\Omega|}{2\pi t}H(|t|) - \frac{1}{8}\left[\sum_{J=1}^{m}(L_J + 2\pi\gamma_J^{-1})\right]\operatorname{sign}t + (2-m)\frac{|t|}{6} \\ & + \frac{1}{512}\left\{\sum_{J=1}^{m}\int_{\partial\Omega_J}\left[K_J^2(\sigma_J) - \left(\frac{2\pi}{L_J}\right)^3\gamma_J^{-1}\right]d\sigma_J\right\}t^2\operatorname{sign}t \\ & + O(t^3\operatorname{sign}t) \quad \text{as } |t| \longrightarrow 0.\end{aligned} \tag{2.2}$$

Remark 2.5. On setting $\gamma_J \to \infty$ $(J = 1,\dots,m)$ in (2.2), we obtain the results of Dirichlet boundary conditions on $\partial\Omega_J$ $(J = 1,\dots,m)$.

Case 2.6 ($0 < \gamma_J \ll 1$ $(J = 1,\dots,m)$).

$$\begin{aligned}\hat{\mu}(t) = {} & \frac{|\Omega|}{2\pi t}H(|t|) + \frac{1}{8}\left(\sum_{J=1}^{m}L_J\right)\operatorname{sign}t \\ & + \left\{(2-m) + \frac{3}{\pi}\left(\sum_{J=1}^{\ell}\gamma_J L_J - \sum_{J=\ell+1}^{m}\gamma_J L_J\right)\right\}\frac{|t|}{6} \\ & + \frac{7}{512}\left\{\sum_{J=1}^{m}\int_{\partial\Omega_J}\left[K_J^2(\sigma_J) - \frac{64}{7}\left(\frac{\pi\gamma_J}{L_J} - \gamma_J^2\right)\right]d\sigma_J\right\}t^2\operatorname{sign}t \\ & + O(t^3\operatorname{sign}t) \quad \text{as } |t| \longrightarrow 0.\end{aligned} \tag{2.3}$$

Remark 2.7. On setting $\gamma_J = 0$ in (2.3), we obtain the results of Neumann boundary conditions on $\partial\Omega_J$ $(J = 1,\dots,m)$.

With reference to formulae (1.6) and (1.7) and to [15, 22], the asymptotic expansions (2.1), (2.2), and (2.3) may be interpreted as follows:

(i) Ω is an arbitrary multiply connected drum in $\mathbb{R}^2$ and we have the Robin boundary conditions (1.10) with small/large impedances γ_J $(J = 1,\dots,m)$ as indicated in the specifications of the four respective cases.

(ii) For the first four terms, Ω is an arbitrary multiply connected drum in $\mathbb{R}^2$ of area $|\Omega|$.

In Case 2.1, it has $h = \{(m-1) - (3/\pi)(\sum_{J=1}^{\ell}\gamma_J L_J - \sum_{J=\ell+1}^{k}\gamma_J L_J)\}$ holes, the boundaries $\partial\Omega_J$ $(J = 1,\dots,k)$ are of the lengths $\sum_{J=1}^{k}L_J$ and of curvatures $[K_J^2(\sigma_J) - (64/7)(\pi\gamma_J/L_J - \gamma_J^2)]^{1/2}$ $(J = 1,\dots,k)$ together with the Neumann boundary conditions, while the boundaries $\partial\Omega_J$ $(J = k+1,\dots,m)$ are of lengths $[\sum_{J=k+1}^{m}(L_J + 2\pi\gamma_J^{-1})]$ and of curvatures $[K_J^2(\sigma_J) - (2\pi/L_J)^3\gamma_J^{-1}]^{1/2}$ together with the Dirichlet boundary conditions, provided h is a positive integer.

In Case 2.4, it has $h = (m-1)$ holes, the boundaries $\partial\Omega_J$ $(J = 1,\dots,m)$ are of the lengths $\sum_{J=1}^{m}(L_J + 2\pi\gamma_J^{-1})$ and of curvatures $[K_J^2(\sigma_J) - (2\pi/L_J)^3\gamma_J^{-1}]^{1/2}$ together with the Dirichlet boundary conditions.

In Case 2.6, it has $h = \{(m-1) - (3/\pi)(\sum_{J=1}^{\ell}\gamma_J L_J - \sum_{J=\ell+1}^{m}\gamma_J L_J)\}$ holes, and the boundaries $\partial\Omega_J$ $(J = 1,\dots,m)$ are of the lengths $\sum_{J=1}^{m}L_J$ and of curvatures $[K_J^2(\sigma_J) -$

$(64/7)(\pi\gamma_J/L_J-\gamma_J^2)]^{1/2}$ together with the Neumann boundary conditions, provided h is a positive integer.

We close this section with the following interesting question: what is the interpretation of Ω if "h" is not integer? The answer of this question is still open, which has been left for the interested readers.

3. Formulation of the mathematical problem. With reference to [14, 15, 21, 22, 24], it can be easily seen that the spectral function $\hat{\mu}(t)$ is given by

$$\hat{\mu}(t)=\iint_{\Omega}G\left(\underset{\sim}{x},\underset{\sim}{x};t\right)d\underset{\sim}{x}, \tag{3.1}$$

where $G(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t)$ is the Green's function for the wave equation

$$\left(\Delta-\frac{\partial^2}{\partial t^2}\right)G\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t\right)=0 \quad \text{in } \Omega\times\{-\infty<t<\infty\}, \tag{3.2}$$

subject to the Robin boundary conditions (1.10) and the initial conditions

$$\lim_{t\to 0}G\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t\right)=0, \qquad \lim_{t\to 0}\frac{\partial G\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t\right)}{\partial t}=\delta\left(\underset{\sim}{x}_1-\underset{\sim}{x}_2\right), \tag{3.3}$$

where $\delta(\underset{\sim}{x}_1-\underset{\sim}{x}_2)$ is the Dirac delta function located at the source point $\underset{\sim}{x}_1=\underset{\sim}{x}_2$. The points $\underset{\sim}{x}_1=(x_1^1,x_1^2)$ and $\underset{\sim}{x}_2=(x_2^1,x_2^2)$ belong to the arbitrary multiply-connected drum Ω. Write

$$G\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t\right)=G_0\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t\right)+\varkappa\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t\right), \tag{3.4}$$

where

$$G_0\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t\right)=\frac{H\left(|t|-\left|\underset{\sim}{x}_1-\underset{\sim}{x}_2\right|\right)}{2\pi\sqrt{t^2-\left|\underset{\sim}{x}_1-\underset{\sim}{x}_2\right|^2}} \tag{3.5}$$

is the "fundamental solution" of the wave equation (3.2) while $\varkappa(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t)$ is the "regular solution" chosen in such a way that $G(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t)$ satisfies the Robin boundary conditions (1.10).

From (3.1), (3.4), and (3.5), we find that

$$\hat{\mu}(t)=\frac{|\Omega|}{2\pi t}H(|t|)+K(t), \tag{3.6}$$

where

$$K(t)=\iint_{\Omega}\varkappa\left(\underset{\sim}{x},\underset{\sim}{x};t\right)d\underset{\sim}{x}. \tag{3.7}$$

In what follows, we will use Fourier transforms with respect to $-\infty<t<\infty$ and use $-\infty<\eta<\infty$ as the Fourier transform parameter.

Thus, we define

$$\hat{G}\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta\right)=\int_{-\infty}^{+\infty}e^{-2\pi i\eta t}G\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;t\right)dt. \tag{3.8}$$

An application of the Fourier transform to the wave equation (3.2) shows that $\hat{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta)$ satisfies the reduced wave equation

$$(\Delta+4\pi^2\eta^2)\hat{G}\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta\right)=-\delta\left(\underset{\sim}{x}_1-\underset{\sim}{x}_2\right)\quad\text{in }\Omega, \tag{3.9}$$

together with the Robin boundary conditions (1.10).

The asymptotic expansion of $K(t)$, for small $|t|$, may then be deduced directly from the asymptotic expansion of $\hat{K}(\eta)$, for large $|\eta|$, where

$$\hat{K}(\eta)=\iint_{\Omega}\hat{\varkappa}\left(\underset{\sim}{x},\underset{\sim}{x};\eta\right)d\underset{\sim}{x}. \tag{3.10}$$

4. Derivation of our results. It is well known (see [15, 22]) that (3.9) has the fundamental solution

$$\hat{G}_0\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta\right)=-\frac{1}{2}Y_0\left(2\pi\eta r_{\underset{\sim}{x}_1\underset{\sim}{x}_2}\right), \tag{4.1}$$

where $r_{\underset{\sim}{x}_1\underset{\sim}{x}_2}=|\underset{\sim}{x}_1-\underset{\sim}{x}_2|$ is the distance between the points $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ of the region Ω, while Y_0 is the Bessel function of the second kind and of zero order. The existence of (4.1) enables us to construct integral equations for $\hat{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta)$ satisfying the Robin boundary conditions (1.10) for small/large impedances γ_J $(J=1,\dots,m)$. Therefore, if we consider the main problem (1.9), (1.10) with the case $0<\gamma_J\ll 1$ $(J=1,\dots,k)$ and $\gamma_J\gg 1$ $(J=k+1,\dots,m)$, then Green's theorem gives the following integral equation:

$$\begin{aligned}\hat{G}\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta\right)=&-\frac{1}{2}Y_0\left(2\pi\eta r_{\underset{\sim}{x}_1\underset{\sim}{x}_2}\right)\\&-\frac{1}{2}\sum_{J=1}^{k}\int_{\partial\Omega_J}\hat{G}\left(\underset{\sim}{x}_1,\underset{\sim}{y};\eta\right)\left[\left(\frac{\partial}{\partial n_{J\underset{\sim}{y}}}+\gamma_J\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}\underset{\sim}{x}_2}\right)\right]d\underset{\sim}{y}\\&-\frac{1}{2}\sum_{J=k+1}^{m}\int_{\partial\Omega_J}\frac{\partial}{\partial n_{J\underset{\sim}{y}}}\hat{G}\left(\underset{\sim}{x}_1,\underset{\sim}{y};\eta\right)\left[\left(1+\gamma_J^{-1}\frac{\partial}{\partial n_{J\underset{\sim}{y}}}\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}\underset{\sim}{x}_2}\right)\right]d\underset{\sim}{y}.\end{aligned} \tag{4.2}$$

On applying the iteration method (see [17, 22]) to the integral equation (4.2), we obtain the Green's function $\hat{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta)$, which has the following regular part:

$$\begin{aligned}\hat{\varkappa}\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta\right)=&\frac{1}{4}\sum_{J=1}^{k}\int_{\partial\Omega_J}Y_0\left(2\pi\eta r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)\left[\left(\frac{\partial}{\partial n_{J\underset{\sim}{y}}}+\gamma_J\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}\underset{\sim}{x}_2}\right)\right]d\underset{\sim}{y}\\&+\frac{1}{4}\sum_{J=k+1}^{m}\int_{\partial\Omega_J}\frac{\partial}{\partial n_{J\underset{\sim}{y}}}Y_0\left(2\pi\eta r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)\left[\left(1+\gamma_J^{-1}\frac{\partial}{\partial n_{J\underset{\sim}{y}}}\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}\underset{\sim}{x}_2}\right)\right]d\underset{\sim}{y}\\&-\frac{1}{4}\sum_{J=1}^{k}\int_{\partial\Omega_J}\int_{\partial\Omega_J}Y_0\left(2\pi\eta r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)M_J\left(\underset{\sim}{y},\underset{\sim}{y}'\right)\left[\left(\frac{\partial}{\partial n_{J\underset{\sim}{y}'}}+\gamma_J\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}'\underset{\sim}{x}_2}\right)\right]d\underset{\sim}{y}\,d\underset{\sim}{y}'\\&-\frac{1}{4}\sum_{J=k+1}^{m}\int_{\partial\Omega_J}\int_{\partial\Omega_J}\frac{\partial}{\partial n_{J\underset{\sim}{y}}}Y_0\left(2\pi\eta r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)L_J\left(\underset{\sim}{y},\underset{\sim}{y}'\right)\end{aligned}$$

$$
\begin{aligned}
&\times\left[\left(1+\gamma_J^{-1}\frac{\partial}{\partial n_{J\underset{\sim}{y}'}}\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}'\underset{\sim}{x}_2}\right)\right]d\underset{\sim}{y}\,d\underset{\sim}{y}'\\
&-\frac{1}{4}\sum_{J=k+1}^{m}\int_{\partial\Omega_J}\left\{\sum_{J=1}^{k}\int_{\partial\Omega_J}Y_0\left(2\pi\eta r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)M_J^{\star}\left(\underset{\sim}{y},\underset{\sim}{y}'\right)d\underset{\sim}{y}\right\}\\
&\times\left\{\left(1+\gamma_J^{-1}\frac{\partial}{\partial n_{J\underset{\sim}{y}'}}\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}'\underset{\sim}{x}_2}\right)\right\}d\underset{\sim}{y}'\\
&-\frac{1}{4}\sum_{J=1}^{k}\int_{\partial\Omega_J}\left\{\sum_{J=k+1}^{m}\int_{\partial\Omega_J}\frac{\partial}{\partial n_{J\underset{\sim}{y}}}Y_0\left(2\pi\eta r_{\underset{\sim}{x}_1\underset{\sim}{y}}\right)L_J^{\star}\left(\underset{\sim}{y},\underset{\sim}{y}'\right)d\underset{\sim}{y}\right\}\\
&\times\left\{\left(\frac{\partial}{\partial n_{J\underset{\sim}{y}'}}+\gamma_J\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}'\underset{\sim}{x}_2}\right)\right\}d\underset{\sim}{y}',
\end{aligned}
\tag{4.3}
$$

where for $J=1,\dots,k$, we find that

$$
\begin{aligned}
M_J\left(\underset{\sim}{y},\underset{\sim}{y}'\right)&=\sum_{\upsilon=0}^{\infty}(-1)^{\upsilon}K_{\gamma_J}^{(\upsilon)}\left(\underset{\sim}{y}',\underset{\sim}{y}\right),\\
M_J^{\star}\left(\underset{\sim}{y},\underset{\sim}{y}'\right)&=\sum_{\upsilon=0}^{\infty}(-1)^{\upsilon}\overset{\star}{K}{}_{\gamma_J}^{(\upsilon)}\left(\underset{\sim}{y}',\underset{\sim}{y}\right),\\
K_{\gamma_J}\left(\underset{\sim}{y}',\underset{\sim}{y}\right)&=\frac{1}{2}\left[\left(\frac{\partial}{\partial n_{J\underset{\sim}{y}}}+\gamma_J\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}\underset{\sim}{y}'}\right)\right],\\
\overset{\star}{K}_{\gamma_J}\left(\underset{\sim}{y}',\underset{\sim}{y}\right)&=\frac{1}{2}\left[\left(\frac{\partial^2}{\partial n_{J\underset{\sim}{y}}\partial n_{J\underset{\sim}{y}'}}+\gamma_J\frac{\partial}{\partial n_{J\underset{\sim}{y}'}}\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}\underset{\sim}{y}'}\right)\right],
\end{aligned}
\tag{4.4}
$$

while for $J=k+1,\dots,m$, we find that

$$
\begin{aligned}
L_J\left(\underset{\sim}{y},\underset{\sim}{y}'\right)&=\sum_{\upsilon=0}^{\infty}(-1)^{\upsilon}K_{\gamma_J^{-1}}^{(\upsilon)}\left(\underset{\sim}{y}',\underset{\sim}{y}\right),\\
L_J^{\star}\left(\underset{\sim}{y},\underset{\sim}{y}'\right)&=\sum_{\upsilon=0}^{\infty}(-1)^{\upsilon}\overset{\star}{K}{}_{\gamma_J^{-1}}^{(\upsilon)}\left(\underset{\sim}{y}',\underset{\sim}{y}\right),\\
K_{\gamma_J^{-1}}\left(\underset{\sim}{y}',\underset{\sim}{y}\right)&=\frac{1}{2}\left[\left(\frac{\partial}{\partial n_{J\underset{\sim}{y}'}}+\gamma_J^{-1}\frac{\partial^2}{\partial n_{J\underset{\sim}{y}}\partial n_{J\underset{\sim}{y}'}}\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}\underset{\sim}{y}'}\right)\right],\\
\overset{\star}{K}_{\gamma_J^{-1}}\left(\underset{\sim}{y}',\underset{\sim}{y}\right)&=\frac{1}{2}\left[\left(1+\gamma_J^{-1}\frac{\partial}{\partial n_{J\underset{\sim}{y}'}}\right)Y_0\left(2\pi\eta r_{\underset{\sim}{y}\underset{\sim}{y}'}\right)\right].
\end{aligned}
\tag{4.5}
$$

On using argument similar to that obtained in [10, 20, 21, 22, 24], we deduce, after some mathematical analysis, that the asymptotic expansion of $\hat{\varkappa}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta)$ for small/large impedances γ_J $(J=1,\dots,m)$ has the form

$$
\hat{\varkappa}\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta\right)=\sum_{J=1}^{m}\hat{\varkappa}_J\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta\right),
\tag{4.6}
$$

where

(a) if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to sufficiently small domains $\mathfrak{D}(I_J)$ $(J=1,\dots,k)$ then

$$\hat{\varkappa}_J\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta\right)=\frac{1}{4}\left\{1-\gamma_J\left(\frac{\partial}{\partial\xi_1^2}\right)^{-1}\right\}Y_0(2\pi\eta\rho_{12})+O\{\eta^{-1}\exp(-A_J\eta\rho_{12})\},\tag{4.7}$$

(b) if $\underset{\sim}{x}_1,\underset{\sim}{x}_2$ belong to sufficiently small domains $\mathfrak{D}(I_J)$ $(J=k+1,\dots,m)$, then

$$\hat{\varkappa}_J\left(\underset{\sim}{x}_1,\underset{\sim}{x}_2;\eta\right)=\frac{1}{4}\left\{1-\gamma_J^{-1}\left(\frac{\partial}{\partial\xi_1^2}\right)\right\}Y_0(2\pi\eta\rho_{12})+O\{\eta^{-1}\exp(-A_J\eta\rho_{12})\},\tag{4.8}$$

where A_J are positive constants, while ρ_{12} is the distance between the points $\underset{\sim}{\xi}_1=(\xi_1^1,\xi_1^2)$ and $\underset{\sim}{\xi}_2=(\xi_2^1,-\xi_2^2)$ of the upper half plane $\xi^2>0$ (see [22]).

From [20, 22], it can be seen that for $\xi^2\geqslant h_J>0$ $(J=1,\dots,m)$ the functions $\hat{\varkappa}_J$ $(\underset{\sim}{x},\underset{\sim}{x};\eta)$ are of order $O\{\exp(-2\eta A_Jh_J)\}$ $(J=1,\dots,m)$, and the integral of the function $\hat{\varkappa}(\underset{\sim}{x},\underset{\sim}{x};\eta)$ over the multiply connected drum Ω can be approximated in the following way (see (3.10)):

$$\begin{aligned}\hat{K}(\eta)={}&\sum_{J=k+1}^{m}\int_{\xi^2=0}^{h_J}\int_{\xi^1=0}^{L_J}\hat{\varkappa}_J\left(\underset{\sim}{x},\underset{\sim}{x};\eta\right)\{1-K_J(\xi^1)\xi^2\}\,d\xi^1\,d\xi^2\\&-\sum_{J=1}^{k}\int_{\xi^2=0}^{h_J}\int_{\xi^1=0}^{L_J}\hat{\varkappa}_J\left(\underset{\sim}{x},\underset{\sim}{x};\eta\right)\{1+K_J(\xi^1)\xi^2\}\,d\xi^1\,d\xi^2\\&+\sum_{J=1}^{m}O\{\exp(-2\eta A_Jh_J)\}\quad\text{as }|\eta|\longrightarrow\infty,\end{aligned}\tag{4.9}$$

where L_J and K_J $(J=1,\dots,m)$ are, respectively, the total lengths and the curvatures of the boundaries $\partial\Omega_J$ $(J=1,\dots,m)$ of the multiply connected drum Ω.

If the e^{λ}-expansions of $\hat{\varkappa}_J(\underset{\sim}{x},\underset{\sim}{x};\eta)$, (cf. [22]) are introduced into (4.9), one obtains an asymptotic series of the form

$$\hat{K}(\eta)=\sum_{n=1}^{p}a_n\eta^{-n}+O(\eta^{-p-1})\quad\text{as }|\eta|\longrightarrow\infty,\tag{4.10}$$

where the coefficients a_n in (4.10) are calculated from the e^{λ}-expansions with the help of the formula (11.3) in [22, Section 11].

On inverting Fourier transforms to both sides of (4.10) and using (3.6), we arrive at the result (2.1). Similarly, we can prove the other results (2.2) and (2.3).

5. Discussions and conclusions. The problem of determining some geometric quantities of the multiply connected bounded drum Ω in $\mathbb{R}^2$ from a complete knowledge of its eigenvalues, has been discussed in the present paper by using the wave equation approach. It is well known that the wave equation methods have given very strong results; the definitive one is that of Hormander [6]. He has studied the distribution $\operatorname{tr}(e^{-itP})$ near $t = 0$ for an elliptic positive semidefinite pseudodifferential operator P in $\mathbb{R}^n$ of order m. Recently, the wave equation methods in solving particular problems have been discussed by Zayed [14, 15] and Zayed et al. [21, 22, 24] who have studied the spectral distribution $\hat{\mu}(t)$ for small $|t|$ for some bounded domains with certain boundary conditions. On the other hand, the applications of the heat kernel $\Theta(t)$ for small positive t to problem (1.1) and to more general ones can be found in Kac [8], Pleijel [10], Stewartson and Waechter [13], Sleeman and Zayed [11], Gottlieb [2, 3, 4], Hsu [7], McKean and Singer [9], Smith [12], Greiner [5], Zayed and Younis [23], and Zayed [16, 17, 18, 19, 20]. In these references, one can ask a question, is it possible just by listening with a perfect ear to hear the shape of Ω? This question has been put nicely by Kac [8], who simply asked, can one hear the shape of a drum? From these references, one can see the differences between the two different methods in solving the inverse problems. Of course, the asymptotic expansions of $\hat{\mu}(t)$ for small $|t|$ are different from the asymptotic expansions of $\Theta(t)$ for small positive t, but they both give the same information about the geometry of the domain Ω. In particular, the present paper provides a useful technique to inverse problem methods via the spectral distribution of the Laplacian.

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