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STRICTLY WEBBED SPACES AND REGULARITY PROPERTIES OF INDUCTIVE LIMITS

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ABSTRACT. Sequentially complete, locally complete, locally Baire, and bornivorously webbed are equivalent for strictly webbed spaces. For inductive limits of strictly webbed spaces these properties are equivalent. Moreover, they imply regularity.

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1. Introduction. Throughout this note E is a locally convex space and $E_1 \subset E_2 \subset \cdots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps $\mathrm{id}:(E_n,\tau_n) \to (E_{n+1},\tau_{n+1})$, $n \in \mathbb{N}$ where τ_n is the topology of E_n. Their locally convex inductive limit is denoted by $\operatorname{ind} E_n$. A *web* W in a locally convex space E is a countable family of absolutely convex subsets of E, arranged in *layers*. The first layer of the web consists of a sequence $(A_p : p = 1,2,\ldots)$ whose union absorbs each point of E. For each set A_p of the first layer there is a sequence $(A_{pq} : q = 1,2,\ldots)$ of sets, called the sequence determined by A_p, such that

$$\begin{aligned} A_{pq} + A_{pq} \subset A_p \quad &\text{for each } q, \\ \bigcup \{A_{pq} : q = 1,2,\ldots\} \quad &\text{absorbs each point of } A_p. \end{aligned} \tag{1.1}$$

Further layers are made up in a corresponding way so that each set of the kth layer is indexed by a finite row of k integers and at each step the above mentioned two conditions are satisfied. Suppose that we choose a set A_p from the first layer, then a set A_{pq} of the sequence determined by A_p and so on. The resulting sequence $S = (A_p, A_{pq}, A_{pqr}, \ldots)$ is called a strand. Whenever we are dealing with only one strand we can simplify the notation by writing $W_1 = A_p$, $W_2 = A_{pq}$, and so forth, thus $S = (W_k)$ is a strand where for each k, W_k is a set of the kth layer.

Let $S = (W_k)$ be a strand. Consider $x_k \in W_k$ and the series $\sum_{k=1}^{\infty} x_k$. The space E is *webbed* if the series $\sum_{k=1}^{\infty} x_k$ is convergent for any choice of $x_k \in W_k$; E is *strictly webbed* if $\sum_{k=n+1}^{\infty} x_k$ converges to some element in W_n for every $n \in \mathbb{N}$ and for any choice of $x_k \in W_k$; E is *bornivorously webbed* if it is strictly webbed and for every bounded set $A \subset E$, there exist a strand $(W_k)_k$ and a sequence $(\alpha_k)_k \subset \mathbb{C}$ such that $A \subset \alpha_k W_k$, for every $k \in \mathbb{N}$ [2, 6, 7, 9].

A *disk* $A \subset E$ is an absolutely convex, bounded and closed set. Let E_A denote the linear span of A endowed with the normed topology generated by the Minkowski functional ρ of A. This topology is finer than the topology inherited from E. If (E_A, ρ_A) is a Banach (Baire) space, A is a *Banach* (*Baire*) *disk*. A locally convex space is *locally* *complete* (*locally Baire*) if every bounded subset is contained in a Banach (Baire) disk. E is a *quasi-locally complete space* if for each bounded subset B in (E,τ) there exists a weaker locally convex topology $\varsigma = \varsigma(B)$ on E and a Banach disk A in (E,ς) such that $B \subset A$ [8]. Note that locally complete implies quasi-locally complete.

E satisfies the *Mackey convergence condition* if for every null sequence $(x_n)_n \subset E$, there exists a disk A such that $(x_n)_n$ is a ρ_A-null sequence. Finally, E satisfies *property K* if each null sequence has a series convergent subsequence.

2. Bornivorously webbed

Lemma 2.1. *Let (E,τ) be a bornivorously webbed space. Then for every bounded set $A \subset E$ there exists a Fréchet space (F,γ) such that A is contained and bounded in F.*

Proof. Let $A \subset E$ be a bounded set. Then there exist a strand $(W_k)_k \subset W$ and a sequence $(\alpha_k)_k \subset \mathbb{C}$ such that $A \subset \alpha_k W_k$, for every $k \in \mathbb{N}$. Consider $E_{W_k} = \operatorname{span}(W_k)$ and $F = \bigcap_{k\in\mathbb{N}} E_{W_k}$. Let $\{F \cap (1/k)W_k : k \in \mathbb{N}\}$ be a fundamental system of neighborhoods of zero in F. This topology is metrizable and finer than τ. We will see that it is complete. Let $(x_k)_k \subset F$ be a Cauchy sequence, and take $(y_k)_k \subset (x_k)_k$ such that $(y_{k+1} - y_k) \in W_k/k$. Then $\sum_{k=1}^{\infty}(y_{k+1} - y_k) \xrightarrow{\tau} u$, for some u in E. $\sum_{k=p+1}^{\infty}(y_{k+1} - y_k) \in W_p/p$, for every $p \in \mathbb{N}$, so $\sum_{k=1}^{\infty}(y_{k+1} - y_k) \in F$. Hence $\sum_{k=1}^{\infty}(y_{k+1} - y_k) \xrightarrow{F} u$ and if $x = u + y_1$, we have $y_k \xrightarrow{F} x$ and $x_k \xrightarrow{F} x$. □

If $F = E$, and $\{(1/k)W_k : k \in \mathbb{N}\}$ is a fundamental system of neighborhoods, then E with the topology γ generated by this family is a Fréchet space. This topology is finer than the original one.

Theorem 2.2. *Let (E,τ) be a locally convex space. If E is strictly webbed, then the following properties are equivalent:*

(a) *E is sequentially complete.*
(b) *E is locally complete.*
(c) *E is locally Baire.*
(d) *E is bornivorously webbed.*

Proof. (a)⇒(b)⇒(c). The proof is obvious. (c)⇒(d). Let A be a bounded subset of E and $B \subset E$ be a Baire disk such that A is contained and bounded in B. By [6, Theorem 5.6.3] for $\mathrm{id} : E_B \to E$, there exists a strand $(W_k)_k$ such that $\mathrm{id}^{-1}(W_k) \in N_0(E_B)$. Hence, for every $k \in \mathbb{N}$ there exists $\alpha_k \in \mathbb{C}$ such that $A \subset \alpha_k \mathrm{id}^{-1}(W_k) \subset E_B$ and $A \subset \alpha_k W_k \subset E$.

(d)⇒(a). The argument of the proof is taken from [1, Theorem 1]: let $(x_n)_n$ be a Cauchy sequence in E, and $B_n = \mathrm{cl}_E \operatorname{co} \bigcup \{x_n : m \geq n\}$, $n \in \mathbb{N}$. The set B_1 is bounded in E which is bornivorously webbed, hence there exists a strand (W_k) in E and a sequence $(\alpha_k)_k \subset \mathbb{C}$ such that $B_1 \subset \alpha_k W_k$ for each $k \in \mathbb{N}$. Denote by γ the topology on E generated by the subbasis $\{W_k : k \in \mathbb{N}\}$ and, for brevity, by F the space (E,γ).

The set $B_1 \subset E$ is closed in E, and by the preceding lemma, it is closed in the locally convex space F. Since B_1 is convex, it is also weakly closed in F.

By lemma, F is a Fréchet space. Hence the canonical imbedding $F \to F''$, where F'' is the second dual of F equipped with the strong topology, is a topological isomorphism into F''. Since F is complete, it is closed in F'' and each functional from the strong dual F' of F can be continuously extended to F''. Thus the $\sigma(F,F')$-closed set B_1 is also $\sigma(F'',F')$-closed in F''.

Further, since B_1 is bounded in F'', it is equicontinuous in F'. Hence by Alaoglu theorem, the set B_1 is relatively $\sigma(F'',F')$-compact. This, together with the $\sigma(F'',F')$-closedness, implies that B_1 is $\sigma(F'',F')$-compact in F''.

Similarly, all sets B_n, $n \in \mathbb{N}$, are $\sigma(F'',F')$-compact. Every finite intersection $\bigcap\{B_n : 1 \le n \le m\} = B_m$, $m \in \mathbb{N}$, is nonempty. Hence there exists $x_0 \in \bigcap\{B_n : n \in \mathbb{N}\} \subset B_1 \subset E$. This implies the existence of an upper triangular matrix $\Lambda = (\lambda_{nm})$ with all entries $\lambda_{nm} \ge 0$, only finite number of nonzeros in each row, and the sum of all entries in each row is equal to 1, such that the sequence $\{y_n = \sum_{m=n}^{\infty} \lambda_{nm} x_m\}_n$ converges to x_0 in the topology γ. Then the continuity of the identity map $F \to E$ implies the convergence $y_k \to x_0$ in E.

Take a balanced, convex, zero neighborhood V in E. Then there exist $p,q \in \mathbb{N}$ such that $y_n - x_0 \in V$ for $n \ge p$ and $x_m - x_n \in V$ for $m \ge n \ge q$. Then for $n \ge \max(p,q)$, we have

$$x_0 - x_n = (x_0 - y_n) + (y_n - x_n) = (x_0 - y_n) + \sum_{m=n}^{\infty} \lambda_{nm}(x_m - x_n) \in V + V. \tag{2.1}$$

This implies $x_n \to x_0$ in the space E. □

Since property K implies locally Baire (see [4, Theorem 2]), this theorem proves that for strictly webbed spaces, property K implies local completeness. This answers Gilsdorf's question 3.2 in [4] in a negative way. Moreover, these different additional properties for strictly webbed spaces, which appear in [1, 4, 5], are proved to be all equivalent.

3. Inductive limits. Let $(E_n,\tau_n)_n$ be an inductive sequence of locally convex spaces, and let $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$ be its inductive limit. The space (E,τ) is *regular* if for each bounded subset B in (E,τ), there exists $n = n(B) \in \mathbb{N}$ such that B is contained and bounded in (E_n,τ_n). (E,τ) is *sequentially retractive* if for each convergent sequence $(x_k)_k$ in (E,τ) there exists $n = n((x_k)_k) \in \mathbb{N}$ such that the sequence converges to the same limit in (E_n,τ_n). Equivalently, each null sequence in (E,τ) is a null sequence in some (E_n,τ_n).

Sequentially retractive inductive limits were introduced and studied by Floret [3, 7]. He proved that sequential retractivity implies regularity. In order to get more information about the relation between regularity and sequential retractivity, we will prove the following proposition.

Proposition 3.1. *Let $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$ be a regular inductive limit. If E satisfies the Mackey convergence condition, then it is sequentially retractive.*

Proof. Let $(x_k)_k$ be a null sequence in E. Since the Mackey convergence condition holds, there exists a bounded disk $B \subset E$ such that $(x_k)_k$ is a ρ_B-null sequence. Now E is regular, so B is contained and bounded in some E_n. So, the topology ρ_B in $E_B \subset E_n$, is finer than that inherited from E_n. Hence $(x_k)_k$ is an E_n-null sequence. □

In the next propositions, we present other relations between these properties and regularity for strictly webbed spaces.

Following Floret [3] and the proof of Theorem 1 in [1], if $(E,\tau) = \text{ind}(E_n,\tau_n)$ is an inductive limit of an inductive sequence of bornivorously webbed spaces note that we have:

E sequentially retractive it follows that E is regular and implies that if $x_k \xrightarrow{\tau} x_0$, then there exists $n_0 \in \mathbb{N}$ and a sequence $\{y_k : y_k \in \text{conv}\{x_m\}_{m=k}^{\infty}\}_{k=1}^{\infty}$ such that $y_k \xrightarrow{\tau_{n_0}} x_0$.

Theorem 3.2. *Let $(E,\tau) = \text{ind}(E_n,\tau_n)$ be the inductive limit of an inductive sequence of strictly webbed locally convex spaces. Consider the conditions:*

(a) *E satisfies property K*
(b) *E is locally Baire*
(c) *E is bornivorously webbed*
(d) *E is sequentially complete*
(e) *E is locally complete*
(f) *E is quasi-locally complete*
(g) *E is regular.*

Then (a)⇒(b)⇔(c)⇔(d)⇔(e) *and* (e)⇒(f)⇒(g).

Proof. (a)⇒(b). [4, Theorem 2].

(b)⇔(c)⇔(d)⇔(e). By Theorem 2.2, since the inductive limit of strictly webbed spaces is strictly webbed.

(e)⇒(f). It is clear.

(f)⇒(g). [8, Theorem 1]. □

Proposition 3.3. *Let $(E,\tau) = \text{ind}(E_n,\tau_n)$ be the inductive limit of an inductive sequence of strictly webbed locally convex spaces such that every (E_n,τ_n) satisfies property K. If E is sequentially retractive then E satisfies property K.*

Proof. Let $(x_m)_m$ be a null sequence in E. Then there exist $n \in \mathbb{N}$, with $x_m \xrightarrow{E_n} 0$ and a subsequence $(x_{m_k})_k \subset (x_m)_m$ such that $\sum_{k=1}^{\infty} x_{m_k} \xrightarrow{E_n} x$. Therefore $\sum_{k=1}^{\infty} x_{m_k} \xrightarrow{E} x$. □

Note that combining the results of this section, and under the hypothesis of Proposition 3.3 we have (a)⇒(b)⇔(c)⇔(d)⇔(e)⇒(f)⇒(g). Moreover if E satisfies the Mackey convergence condition they are all equivalent.

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