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# ON $n$-FOLD IMPLICATIVE FILTERS OF LATTICE IMPLICATION ALGEBRAS

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Abstract. We introduce the notion of $n$-fold implicative filters and $n$-fold implicative lattice implication algebras. We give characterizations of $n$-fold implicative filters and $n$-fold implicative lattice implication algebras. Finally, we construct an extension property for $n$-fold implicative filter.

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**1. Introduction.** In order to research the logical system whose propositional value is given in a lattice, Xu [2] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [3] introduced the notions of filter and implicative filter in a lattice implication algebra, and investigated their properties. The author of this paper [1] gave an equivalent condition of a filter, and provided some equivalent conditions for a filter to be an implicative filter in a lattice implication algebra. In this paper, we discuss the foldness of implicative filters in lattice implication algebras.

## 2. Preliminaries

**Definition 2.1** (see [2])**.** By a *lattice implication algebra* we mean a bounded lattice $(L,\vee,\wedge,0,1)$ with order-reversing involution "$'$" and a binary operation "$\to$" satisfying the following axioms:

$$x \longrightarrow (y \longrightarrow z) = y \longrightarrow (x \longrightarrow z), \tag{2.1}$$

$$x \longrightarrow x = 1, \tag{2.2}$$

$$x \longrightarrow y = y' \longrightarrow x', \tag{2.3}$$

$$x \longrightarrow y = y \longrightarrow x = 1 \Longrightarrow x = y, \tag{2.4}$$

$$(x \longrightarrow y) \longrightarrow y = (y \longrightarrow x) \longrightarrow x, \tag{2.5}$$

$$(x \vee y) \longrightarrow z = (x \longrightarrow z) \wedge (y \longrightarrow z), \tag{2.6}$$

$$(x \wedge y) \longrightarrow z = (x \longrightarrow z) \vee (y \longrightarrow z), \tag{2.7}$$

for all $x,y,z \in L$.

**Example 2.2** (see [3])**.** Let $L := \{0,a,b,c,1\}$. Define the partially-ordered relation on $L$ as $0 < a < b < c < 1$, and define

$$x \wedge y := \min\{x,y\}, \qquad x \vee y := \max\{x,y\}, \tag{2.8}$$

TABLE 2.1.

| $x$ | $x'$ |
|---|---|
| 0 | 1 |
| $a$ | $c$ |
| $b$ | $b$ |
| $c$ | $a$ |
| 1 | 0 |

| $\to$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $c$ | 1 | 1 | 1 | 1 |
| $b$ | $b$ | $c$ | 1 | 1 | 1 |
| $c$ | $a$ | $a$ | $c$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

for all $x, y \in L$ and "$'$" and "$\to$" as in Table 2.1. Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra.

In what follows, the binary operation "$\to$" will be denoted by juxtaposition. We can define a partial ordering "$\leq$" on a lattice implication algebra $L$ by $x \leq y$ if and only if $xy = 1$.

In a lattice implication algebra $L$, the following hold (see [2]):

$$0x = 1, \qquad 1x = x, \qquad x1 = 1, \tag{2.9}$$

$$xy \leq (yz)(xz), \tag{2.10}$$

$$x \leq y \text{ implies } yz \leq xz,\ zx \leq zy. \tag{2.11}$$

In what follows, $L$ will denote a lattice implication algebra, unless otherwise specified.

**Definition 2.3** (see [3]). A subset $F$ of $L$ is called a *filter* of $L$ if it satisfies for all $x, y \in L$ the following:

$$1 \in F, \tag{2.12}$$

$$x \in F,\ xy \in F \text{ imply } y \in F. \tag{2.13}$$

**Definition 2.4** (see [3]). A subset $F$ of $L$ is called an *implicative filter* of $L$ if it satisfies (2.12) and

$$x(yz) \in F,\ xy \in F \text{ imply } xz \in F, \quad \forall x, y, z \in L. \tag{2.14}$$

**Proposition 2.5** (see [1, Proposition 3.2]). *Every filter $F$ of $L$ has the property*

$$x \leq y,\ x \in F \textit{ imply } y \in F. \tag{2.15}$$

**3. $n$-fold implicative filters.** For any elements $x$ and $y$ of $L$ and any positive integer $n$, let $x^n y$ denote $x(\cdots(x(xy))\cdots)$ in which $x$ occurs $n$ times, and $x^0 y = y$.

**Definition 3.1.** Let $n$ be a positive integer. A subset $F$ of $L$ is called an *$n$-fold implicative filter* of $L$, if it satisfies (2.12) and

$$x^n(yz) \in F,\ x^n y \in F \text{ imply } x^n z \in F, \quad \forall x, y, z \in L. \tag{3.1}$$

Note that the 1-fold implicative filter is an implicative filter.

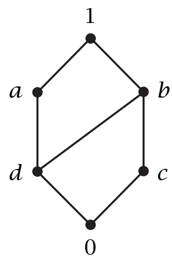


FIGURE 3.1.

TABLE 3.1.

| $x$ | $x'$ |
|---|---|
| 0 | 1 |
| $a$ | $c$ |
| $b$ | $d$ |
| $c$ | $a$ |
| $d$ | $b$ |
| 1 | 0 |

(a)

| | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $c$ | 1 | $b$ | $c$ | $b$ | 1 |
| $b$ | $d$ | $a$ | 1 | $b$ | $a$ | 1 |
| $c$ | $a$ | $a$ | 1 | 1 | $a$ | 1 |
| $d$ | $b$ | 1 | 1 | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

(b)

**EXAMPLE 3.2.** Let $L := \{0, a, b, c, d, 1\}$ be a set with Figure 3.1 as a partial ordering. Define a unary operation "$'$" and a binary operation denoted by juxtaposition on $L$ as in Tables 3.1a and 3.1b, respectively.

Define $\vee$- and $\wedge$-operations on $L$ as follows:

$$x \vee y := (xy)y, \quad x \wedge y := ((x'y')y')', \quad \forall x, y \in L. \tag{3.2}$$

Then $L$ is a lattice implication algebra. It is easy to check that $F := \{b, c, 1\}$ is an $n$-fold implicative filter of $L$.

**THEOREM 3.3.** *Every $n$-fold implicative filter of $L$ is a filter of $L$.*

**PROOF.** Let $F$ be an $n$-fold implicative filter of $L$. Taking $x = 1$ in (3.1) and using (2.9), we conclude that $yz \in F$ and $y \in F$ imply $z \in F$, that is, (2.13) holds. Hence $F$ is a filter of $L$. □

The converse of Theorem 3.3 is not true. For example, let $L$ be a lattice implication algebra in Example 3.2. Then $\{1\}$ is a filter of $L$, but $\{1\}$ is not a 1-fold implicative filter of $L$ because $d^1(bc) = db = 1$ and $d^1 b = 1$, but $d^1 c = b \neq 1$.

We give conditions for a filter to be an $n$-fold implicative filter.

**THEOREM 3.4.** *Let $F$ be a filter of $L$. Then the following statements are equivalent:*

(i) *$F$ is an $n$-fold implicative filter of $L$.*

(ii) *$x^{n+1}y \in F$ implies $x^n y \in F$.*

(iii) *$x^n(yz) \in F$ implies $(x^n y)(x^n z) \in F$.*

**Proof.** (i)⇒(ii). Assume that $F$ is an $n$-fold implicative filter of $L$ and let $x, y \in L$ be such that $x^{n+1}y \in F$. Then $x^n(xy) \in F$, and since $x^n x = 1 \in F$, it follows from (3.1) that $x^n y \in F$.

(ii)⇒(iii). Suppose (ii) holds and let $x, y, z \in L$ be such that $x^n(yz) \in F$. Since $x^n(yz) \leq x^n((x^n y)(x^n z))$, we have

$$x^{n+1}(x^{n-1}((x^n y)z)) = x^n(x^n((x^n y)z)) = x^n((x^n y)(x^n z)) \in F. \tag{3.3}$$

It follows from (ii) that $x^{n+1}(x^{n-2}((x^n y)z)) = x^n(x^{n-1}((x^n y)z)) \in F$. Using (ii) again, we get

$$x^{n+1}(x^{n-3}((x^n y)z)) = x^n(x^{n-2}((x^n y)z)) \in F. \tag{3.4}$$

Repeating this process, we conclude that $(x^n y)(x^n z) = x^n((x^n y)z) \in F$.

(iii)⇒(i). Let $x, y, z \in L$ be such that $x^n(yz) \in F$ and $x^n y \in F$. It follows from (iii) that $(x^n y)(x^n z) \in F$ and $x^n y \in F$, so from (2.13), we have $x^n z \in F$. Hence $F$ is an $n$-fold implicative filter of $L$. □

**Definition 3.5.** Let $n$ be a positive integer. A lattice implication algebra $L$ is said to be *$n$-fold implicative* if it satisfies the equality $x^{n+1}y = x^n y$ for all $x, y \in L$.

**Corollary 3.6.** *In an $n$-fold implicative lattice implication algebra, the notion of filters and $n$-fold implicative filters coincide.*

We give a characterization of an $n$-fold implicative lattice implication algebra.

**Theorem 3.7.** *A lattice implication algebra $L$ is $n$-fold implicative if and only if the filter $\{1\}$ of $L$ is $n$-fold implicative.*

**Proof.** Necessity is by Corollary 3.6. Assume that the filter $\{1\}$ of $L$ is $n$-fold implicative. Noticing that $x^n((xy)y) = 1$, and applying Theorem 3.4, we have

$$(x^{n+1}y)(x^n y) = (x^n(xy))(x^n y) = 1. \tag{3.5}$$

On the other hand, it is clear that $(x^n y)(x^{n+1}y) = 1$. Hence $x^{n+1}y = x^n y$, as desired. □

The following is a characterization of an $n$-fold implicative filter.

**Theorem 3.8.** *A nonempty subset $F$ of $L$ is an $n$-fold implicative filter of $L$ if and only if it satisfies (2.12) and*

$$x(y^{n+1}z) \in F,\ x \in F \text{ imply } y^n z \in F, \quad \forall x, y, z \in L. \tag{3.6}$$

**Proof.** Suppose that $F$ is an $n$-fold implicative filter of $L$ and let $x, y, z \in L$ be such that $x(y^{n+1}z) \in F$ and $x \in F$. Since $F$ is a filter of $L$ (see Theorem 3.3), it follows that $y^{n+1}z \in F$. Using Theorem 3.4, we know that $y^n z \in F$.

Conversely, assume that $F$ satisfies (2.12) and (3.6). Let $x, y \in L$ be such that $xy \in F$ and $x \in F$. Then $x(1^{n+1}y) = xy \in F$ and $x \in F$. Thus, by (3.6), we have $y = 1^n y \in F$. Hence $F$ is a filter of $L$. Now, if $x^{n+1}y \in F$ for all $x, y \in L$, then $1(x^{n+1}y) = x^{n+1}y \in F$ and $1 \in F$. It follows from (3.6) that $x^n y \in F$. Hence $F$ is an $n$-fold implicative filter of $L$ by Theorem 3.4. This completes the proof. □

**Theorem 3.9** (extension property for $n$-fold implicative filters)**.** *Let $F$ and $G$ be filters of $L$ such that $F \subseteq G$. If $F$ is $n$-fold implicative, then so is $G$.*

**Proof.** Let $x, y \in L$ be such that $x^{n+1}y \in G$. Since $x^{n+1}((x^{n+1}y)y) = 1 \in F$, it follows from (2.1) and Theorem 3.4(ii) that

$$(x^{n+1}y)(x^n y) = x^n((x^{n+1}y)y) \in F \subseteq G, \tag{3.7}$$

so that $x^n y \in G$ since $G$ is a filter. Using Theorem 3.4, we conclude that $G$ is an $n$-fold implicative filter of $L$. □

Using Theorems 3.7 and 3.9, we have the following theorem.

**Theorem 3.10.** *A lattice implication algebra is $n$-fold implicative if and only if every filter is $n$-fold implicative.*

## References


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