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ON n-NORMED SPACES

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ABSTRACT. Given an n-normed space with $n \geq 2$, we offer a simple way to derive an $(n-1)$-norm from the n-norm and realize that any n-normed space is an $(n-1)$-normed space. We also show that, in certain cases, the $(n-1)$-norm can be derived from the n-norm in such a way that the convergence and completeness in the n-norm is equivalent to those in the derived $(n-1)$-norm. Using this fact, we prove a fixed point theorem for some n-Banach spaces.

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1. Introduction. Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $\|\cdot,\dots,\cdot\|$ on X^n satisfying the following four properties

(1) $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent;
(2) $\|x_1,\dots,x_n\|$ is invariant under permutation;
(3) $\|x_1,\dots,x_{n-1},\alpha x_n\| = |\alpha|\,\|x_1,\dots,x_{n-1},x_n\|$ for any $\alpha \in \mathbb{R}$;
(4) $\|x_1,\dots,x_{n-1},y+z\| \leq \|x_1,\dots,x_{n-1},y\| + \|x_1,\dots,x_{n-1},z\|$,

is called an *n-norm* on X and the pair $(X,\|\cdot,\dots,\cdot\|)$ is called an n-normed space.

A trivial example of an n-normed space is $X = \mathbb{R}^n$ equipped with the following n-norm:

$$\|x_1,\dots,x_n\|_E := \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right), \tag{1.1}$$

where $x_i = (x_{i1},\dots,x_{in}) \in \mathbb{R}^n$ for each $i = 1,\dots,n$. (The subscript E is for Euclidean.)

Note that in an n-normed space $(X,\|\cdot,\dots,\cdot\|)$, we have, for instance, $\|x_1,\dots,x_n\| \geq 0$ and $\|x_1,\dots,x_{n-1},x_n\| = \|x_1,\dots,x_{n-1},x_n + \alpha_1 x_1 + \cdots + \alpha_{n-1}x_{n-1}\|$ for all $x_1,\dots,x_n \in X$ and $\alpha_1,\dots,\alpha_{n-1} \in \mathbb{R}$.

The theory of 2-normed spaces was first developed by Gähler [3] in the mid 1960's, while that of n-normed spaces can be found in [11]. Recent results can be found, for example, in [9, 10]. Related works on n-metric spaces and n-inner product spaces may be found, for example, in [1, 2, 4, 5, 7, 6, 12].

In this note, we will show that every n-normed space with $n \geq 2$ is an $(n-1)$-normed space and hence, by induction, an $(n-r)$-normed space for all $r = 1,\dots,n-1$. In particular, given an n-normed space, we offer a simple way to derive an $(n-1)$-norm from the n-norm, different from that in [5].

We will also apply our result to study convergence and completeness in n-normed spaces, which will be defined later. This enables us to prove a fixed point theorem for some n-normed spaces.

The case $n = 2$ was previously studied in [8].

2. Preliminary results. Suppose hereafter that $n \geq 2$ and $(X, \|\cdot,\ldots,\cdot\|)$ is an n-normed space of dimension $d \geq n$. Take a linearly independent set $\{a_1,\ldots,a_n\}$ in X. With respect to $\{a_1,\ldots,a_n\}$, define the following function $\|\cdot,\ldots,\cdot\|_\infty$ on X^{n-1} by

$$\|x_1,\ldots,x_{n-1}\|_\infty := \max\{\|x_1,\ldots,x_{n-1},a_i\| : i = 1,\ldots,n\}. \tag{2.1}$$

Then we have the following result.

Theorem 2.1. *The function $\|\cdot,\ldots,\cdot\|_\infty$ defines an $(n-1)$-norm on X.*

Proof. We will verify that $\|\cdot,\ldots,\cdot\|_\infty$ satisfies the four properties of an $(n-1)$-norm.

(1) If $x_1,\ldots,x_{n-1}$ are linearly dependent, then $\|x_1,\ldots,x_{n-1}\| = 0$ for each $i = 1,\ldots,n$, and hence $\|x_1,\ldots,x_{n-1}\|_\infty = 0$. Conversely, if $\|x_1,\ldots,x_{n-1}\|_\infty = 0$, then $\|x_1,\ldots,x_{n-1},a_i\| = 0$ and accordingly $x_1,\ldots,x_{n-1}$, a_i are linearly dependent for each $i = 1,\ldots,n$. But this can only happen when $x_1,\ldots,x_{n-1}$ are linearly dependent.

(2) Since $\|x_1,\ldots,x_{n-1},a_i\|$ is invariant under any permutation of $\{x_1,\ldots,x_{n-1}\}$, we find that $\|x_1,\ldots,x_{n-1}\|_\infty$ is also invariant under any permutation.

(3) Observe that

$$\begin{aligned}
\|x_1,\ldots,x_{n-2},\alpha x_{n-1}\|_\infty &= \max\{\|x_1,\ldots,x_{n-2},\alpha x_{n-1},a_i\| : i = 1,\ldots,n\} \\
&= |\alpha| \max\{\|x_1,\ldots,x_{n-2},x_{n-1},a_i\| : i = 1,\ldots,n\} \\
&= |\alpha| \|x_1,\ldots,x_{n-2},x_{n-1}\|_\infty.
\end{aligned} \tag{2.2}$$

(4) Observe that

$$\begin{aligned}
\|x_1,\ldots,x_{n-2},y+z\|_\infty &= \max\{\|x_1,\ldots,x_{n-2},y+z,a_i\| : i = 1,\ldots,n\} \\
&\leq \max\{\|x_1,\ldots,x_{n-2},y,a_i\| : i = 1,\ldots,n\} \\
&\quad + \max\{\|x_1,\ldots,x_{n-2},z,a_i\| : i = 1,\ldots,n\} \\
&= \|x_1,\ldots,x_{n-2},y\|_\infty + \|x_1,\ldots,x_{n-2},z\|_\infty.
\end{aligned} \tag{2.3}$$

Therefore $\|\cdot,\ldots,\cdot\|_\infty$ defines an $(n-1)$-norm on X. □

Corollary 2.2. *Every n-normed space is an $(n-r)$-normed space for all $r = 1,\ldots,n-1$. In particular, every n-normed space is a normed space.*

Remark 2.3. Note that in general the function $\|x_1,\ldots,x_{n-1}\|_p := \{\sum_{i=1}^n \|x_1,\ldots, x_{n-1},a_i\|^p\}^{1/p}$, where $1 \leq p \leq \infty$, also defines an $(n-1)$-norm on X. These $(n-1)$-norms, however, are equivalent to $\|\cdot,\ldots,\cdot\|_\infty$, as long as we use the same set of n vectors $a_1,\ldots,a_n$. In certain cases, it is possible to get equivalent $(n-1)$-norms even if we use different sets of n vectors.

2.1. The standard case. Take a look at a standard example. Let X be a real inner product space of dimension $d \geq n$. Equip X with the standard n-norm

$$\|x_1,\dots,x_n\|_S := \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_1,x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,x_1\rangle & \cdots & \langle x_n,x_n\rangle \end{vmatrix}^{1/2}, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on X. (If $X = \mathbb{R}^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|\cdot,\dots,\cdot\|_E$ mentioned earlier.)

Notice that for $n = 1$, the above n-norm is the usual norm $\|x_1\|_S = \langle x_1,x_1\rangle^{1/2}$, which gives the length of x_1, while for $n = 2$, it defines the standard 2-norm $\|x_1,x_2\|_S = \{\|x_1\|_S^2\|x_2\|_S^2 - \langle x_1,x_2\rangle^2\}^{1/2}$, which represents the area of the parallelogram spanned by x_1 and x_2. Further, if $X = \mathbb{R}^3$, then $\|x_1,x_2,x_3\|_s = \|x_1,x_2,x_3\|_E$ is nothing but the volume of the parallelograms spanned by x_1, x_2, and x_3. In general, $\|x_1,\dots,x_n\|_S$ represents the volume of the n-dimensional parallelepiped spanned by $x_1,\dots,x_n$ in X.

Now let $\{e_1,\dots,e_n\}$ be an orthonormal set in X. Then, by Theorem 2.1, the following function

$$\|x_1,\dots,x_{n-1}\|_\infty := \max\{\|x_1,\dots,x_{n-1},e_i\|_S : i = 1,\dots,n\} \tag{2.5}$$

defines an $(n-1)$-norm on X. Further, we have the following fact.

Fact 2.4. *On a standard n-normed space X, the derived $(n-1)$-norm $\|\cdot,\dots,\cdot\|_\infty$, defined with respect to $\{e_1,\dots,e_n\}$, is equivalent to the standard $(n-1)$-norm $\|\cdot,\dots,\cdot\|_S$. Precisely, we have*

$$\|x_1,\dots,x_{n-1}\|_\infty \leq \|x_1,\dots,x_{n-1}\|_S \leq \sqrt{n}\,\|x_1,\dots,x_{n-1}\|_\infty \tag{2.6}$$

for all $x_1,\dots,x_{n-1} \in X$.

Proof. Assume that $x_1,\dots,x_{n-1}$ are linearly independent. For each $i = 1,\dots,n$, write $e_i = e_i^\circ + e_i^\perp$ where $e_i^\circ \in \operatorname{span}\{x_1,\dots,x_{n-1}\}$ and $e_i^\perp \perp \operatorname{span}\{x_1,\dots,x_{n-1}\}$. Then we have

$$\begin{aligned} \|x_1,\dots,x_{n-1},e_i\|_S &= \|x_1,\dots,x_{n-1},e_i^\perp\|_S \\ &= \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_1,x_{n-1}\rangle & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle x_{n-1},x_1\rangle & \cdots & \langle x_{n-1},x_{n-1}\rangle & 0 \\ 0 & \cdots & 0 & \langle e_i^\perp,e_i^\perp\rangle \end{vmatrix}^{1/2} \\ &\leq \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_1,x_{n-1}\rangle \\ \vdots & \ddots & \vdots \\ \langle x_{n-1},x_1\rangle & \cdots & \langle x_{n-1},x_{n-1}\rangle \end{vmatrix}^{1/2} \\ &= \|x_1,\dots,x_{n-1}\|_S. \end{aligned} \tag{2.7}$$

Hence we get $\|x_1,\dots,x_{n-1}\|_\infty \leq \|x_1,\dots,x_{n-1}\|_S$.

Next, take a unit vector $e = \alpha_1 e_1 + \cdots + \alpha_n e_n$ such that $e \perp \text{span}\{x_1, \ldots, x_{n-1}\}$. (Here we are still assuming that $x_1, \ldots, x_{n-1}$ are linearly independent.) Then, by properties (3) and (4) of the n-norm, we have

$$\begin{aligned} \|x_1, \ldots, x_{n-1}\|_S &= \|x_1, \ldots, x_{n-1}, e\|_S \\ &\le |\alpha_1| \|x_1, \ldots, x_{n-1}, e_1\|_S + \cdots + |\alpha_n| \|x_1, \ldots, x_{n-1}, e_n\|_S \\ &\le (|\alpha_1| + \cdots + |\alpha_n|) \|x_1, \ldots, x_{n-1}\|_\infty. \end{aligned} \tag{2.8}$$

But, by the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{n} |\alpha_i| \le \left\{ \sum_{i=1}^{n} 1^2 \right\}^{1/2} \left\{ \sum_{i=1}^{n} |\alpha_i|^2 \right\}^{1/2} = \sqrt{n}. \tag{2.9}$$

Hence we obtain

$$\|x_1, \ldots, x_{n-1}\|_S \le \sqrt{n} \|x_1, \ldots, x_{n-1}\|_\infty, \tag{2.10}$$

and this completes the proof. □

2.2. The finite-dimensional case. For finite-dimensional n-normed space $(X, \|\cdot, \ldots, \cdot\|)$, we can in general derive an $(n-1)$-norm from the n-norm in the following way. Take a linearly independent set $\{a_1, \ldots, a_m\}$ in X, with $n \le m \le d$. With respect to $\{a_1, \ldots, a_m\}$, define the following function $\|\cdot, \ldots, \cdot\|_\infty$ on X^{n-1} by

$$\|x_1, \ldots, x_{n-1}\|_\infty := \max\{\|x_1, \ldots, x_{n-1}, a_i\| : i = 1, \ldots, m\}. \tag{2.11}$$

Then, as in Theorem 2.1, the function $\|\cdot, \ldots, \cdot\|_\infty$ defines an $(n-1)$-norm on X.

As we will see later, we can obtain a better $(n-1)$-norm by using a set of d, rather than just n, linearly independent vectors in X (that is, by using a basis for X).

3. Applications and further results. Recall that a sequence $x(k)$ in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to *converge* to an $x \in X$ (in the n-norm) whenever

$$\lim_{k \to \infty} \|x_1, \ldots, x_{n-1}, x(k) - x\| = 0 \tag{3.1}$$

for every $x_1, \ldots, x_{n-1} \in X$.

The following proposition says that the convergence in the n-norm implies the convergence in the derived $(n-1)$-norm $\|\cdot, \ldots, \cdot\|_\infty$, defined with respect to an arbitrary linearly independent set $\{a_1, \ldots, a_n\}$ in X.

Proposition 3.1. *If $x(k)$ converges to an $x \in X$ in the n-norm, then $x(k)$ also converges to x in the derived $(n-1)$-norm $\|\cdot, \ldots, \cdot\|_\infty$, that is,*

$$\lim_{k \to \infty} \|x_1, \ldots, x_{n-2}, x(k) - x\|_\infty = 0 \tag{3.2}$$

for every $x_1, \ldots, x_{n-2} \in X$.

Proof. If $x(k)$ converges to an $x \in X$ in the n-norm, then

$$\lim_{k \to \infty} \|x_1, \ldots, x_{n-2}, x(k) - x, a_i\| = 0 \tag{3.3}$$

for every $x_1,\dots,x_{n-2} \in X$ and $i = 1,\dots,n$, and hence

$$\lim_{k\to\infty} \|x_1,\dots,x_{n-2},x(k)-x\|_\infty = 0 \tag{3.4}$$

for every $x_1,\dots,x_{n-2} \in X$, that is, $x(k)$ converges to x in the derived $(n-1)$-norm $\|\cdot,\dots,\cdot\|_\infty$. □

3.1. The standard case. In a standard n-normed space $(X,\|\cdot,\dots,\cdot\|_S)$, the converse of Proposition 3.1 is also true, especially when the derived $(n-1)$-norm $\|\cdot,\dots,\cdot\|_\infty$ is defined with respect to an orthonormal set $\{e_1,\dots,e_n\}$ in X as in Section 2.1.

Fact 3.2. *A sequence in a standard n-normed space X is convergent in the n-norm if and only if it is convergent in the derived $(n-1)$-norm $\|\cdot,\dots,\cdot\|_\infty$.*

Proof. Suppose that $x(k)$ converges to an $x \in X$ in the derived $(n-1)$-norm $\|\cdot,\dots,\cdot\|_\infty$. We want to show that $x(k)$ also converges to x in the n-norm. Take $x_1,\dots,x_{n-1} \in X$. Then one may observe that

$$\|x_1,\dots,x_{n-2},x_{n-1},x(k)-x\|_S \le \|x_1,\dots,x_{n-2},x(k)-x\|_S\|x_{n-1}\|_S, \tag{3.5}$$

where $\|\cdot,\dots,\cdot\|_S$ and $\|\cdot\|_S$ on the right-hand side denote the standard $(n-1)$-norm and the usual norm on X, respectively. By Fact 2.4, we have

$$\|x_1,\dots,x_{n-2},x_{n-1},x(k)-x\|_S \le \sqrt{n}\|x_1,\dots,x_{n-2},x(k)-x\|_\infty\|x_{n-1}\|_S. \tag{3.6}$$

But $\lim_{k\to\infty}\|x_1,\dots,x_{n-2},x(k)-x\|_\infty = 0$, and so we conclude that

$$\lim_{k\to\infty} \|x_1,\dots,x_{n-1},x(k)-x\|_S = 0, \tag{3.7}$$

that is, $x(k)$ converges to x in the n-norm. □

Corollary 3.3. *A sequence in a standard n-normed space is convergent in the n-norm if and only if it is convergent in the standard $(n-1)$-norm and, by induction, in the standard $(n-r)$-norm for all $r = 1,\dots,n-1$. In particular, a sequence in a standard n-normed space is convergent in the n-norm if and only if it is convergent in the usual norm $\|\cdot\|_S := \langle\cdot,\cdot\rangle^{1/2}$.*

3.2. The finite-dimensional case. We also have a similar result for finite-dimensional n-normed space $(X,\|\cdot,\dots,\cdot\|)$. Let $\{b_1,\dots,b_d\}$ be a basis for X. With respect to $\{b_1,\dots,b_d\}$, define the following function $\|\cdot,\dots,\cdot\|_{\bowtie}$ on X^{n-1} by

$$\|x_1,\dots,x_{n-1}\|_{\bowtie} := \max\{\|x_1,\dots,x_{n-1},b_i\| : i = 1,\dots,d\}. \tag{3.8}$$

Then, as mentioned before, the function $\|\cdot,\dots,\cdot\|_{\bowtie}$ defines an $(n-1)$-norm on X.

With this derived $(n-1)$-norm, we have the following result.

Proposition 3.4. *A sequence in the finite-dimensional n-normed space X is convergent in the n-norm if and only if it is convergent in the derived $(n-1)$-norm $\|\cdot,\dots,\cdot\|_{\bowtie}$.*

Proof. If a sequence in X is convergent in the n-norm, then it will certainly be convergent in the $(n-1)$-norm $\|\cdot,\dots,\cdot\|_{\bowtie}$. Conversely, suppose that $x(k)$ converges to an $x \in X$ in $\|\cdot,\dots,\cdot\|_{\bowtie}$. Take $x_1,\dots,x_{n-1} \in X$. Writing $x_{n-1} = \alpha_1 b_1 + \cdots + \alpha_d b_d$, we get

$$\begin{aligned} \|x_1,\dots,x_{n-2},x_{n-1},x(k)-x\| &\le |\alpha_1|\,\|x_1,\dots,x_{n-2},x(k)-x,b_1\| \\ &\quad + \cdots + |\alpha_d|\,\|x_1,\dots,x_{n-2},x(k)-x,b_d\| \\ &\le (|\alpha_1|+\cdots+|\alpha_d|)\|x_1,\dots,x_{n-2},x(k)-x\|_{\bowtie}. \end{aligned} \tag{3.9}$$

But $\lim_{k\to\infty}\|x_1,\dots,x_{n-2},x(k)-x\|_{\bowtie} = 0$, and so we obtain

$$\lim_{k\to\infty}\|x_1,\dots,x_{n-1},x(k)-x\| = 0, \tag{3.10}$$

that is, $x(k)$ converges to x in the n-norm. □

3.3. The standard, separable case. We go back to the standard case, where X is a real inner product space of dimension $d \ge n$ equipped with the standard n-norm $\|\cdot,\dots,\cdot\|_S$ as in Section 2.1. But suppose now that X is separable and that $\{e_i : i \in I_d\}$, where $I_d := \{1,\dots,d\}$ (if $d < \infty$) or $\mathbb{N}$ (if $d = \infty$), is an orthonormal basis for X. For every $x_1,\dots,x_{n-1} \in X$ and every basis vector e_i $(i \in I_d)$, we have

$$\|x_1,\dots,x_{n-1},e_i\|_S \le \|x_1,\dots,x_{n-1}\|_S, \tag{3.11}$$

where $\|\cdot,\dots,\cdot\|_S$ on the right-hand side denotes the standard $(n-1)$-norm on X. Hence, with respect to $\{e_i : i \in I_d\}$, we may define the function $\|\cdot,\dots,\cdot\|_{\bowtie}$ on X^{n-1} by

$$\|x_1,\dots,x_{n-1}\|_{\bowtie} := \sup\{\|x_1,\dots,x_{n-1},e_i\|_S : i \in I_d\} \tag{3.12}$$

and check that it also defines an $(n-1)$-norm on X. Moreover, we have the following relation between the two derived $(n-1)$-norms $\|\cdot,\dots,\cdot\|_{\bowtie}$ and $\|\cdot,\dots,\cdot\|_{\infty}$ (the latter being defined with respect to $\{e_1,\dots,e_n\}$ only):

$$\|x_1,\dots,x_{n-1}\|_{\infty} \le \|x_1,\dots,x_{n-1}\|_{\bowtie} \le \|x_1,\dots,x_{n-1}\|_S \le \sqrt{n}\|x_1,\dots,x_{n-1}\|_{\infty} \tag{3.13}$$

for every $x_1,\dots,x_{n-1} \in X$. Hence we conclude the following fact.

Fact 3.5. *On a standard n-normed space X, the two derived $(n-1)$-norms $\|\cdot,\dots,\cdot\|_{\infty}$ and $\|\cdot,\dots,\cdot\|_{\bowtie}$ and the standard $(n-1)$-norm $\|\cdot,\dots,\cdot\|_S$ are equivalent. Accordingly, a sequence in a standard n-normed space X is convergent in the n-norm if and only if it is convergent in one of the three $(n-1)$-norms.*

3.4. Cauchy sequences, completeness and fixed point theorem. Recall that a sequence $x(k)$ in an n-normed space $(X,\|\cdot,\dots,\cdot\|)$ is called *Cauchy* (with respect to the n-norm) if

$$\lim_{k,l\to\infty}\|x_1,\dots,x_{n-1},x(k)-x(l)\| = 0 \tag{3.14}$$

for every $x_1,\dots,x_{n-1} \in X$. If every Cauchy sequence in X converges to an $x \in X$, then X is said to be *complete* (with respect to the n-norm). A complete n-normed space is then called an *n-Banach space*.

By replacing the phrases "$x(k)$ converges to x" with "$x(k)$ is Cauchy" and "$x(k)-x$" with "$x(k)-x(l)$," we see that the analogues of Proposition 3.1, Fact 3.2, Corollary 3.3, Proposition 3.4, and Fact 3.5 hold for Cauchy sequences.

Hence, for the standard or finite-dimensional case, we have the following result.

Proposition 3.6. (a) *A standard n-normed space is complete if and only if it is complete with respect to one of the three $(n-1)$-norms $\|\cdot,\dots,\cdot\|_\infty$, $\|\cdot,\dots,\cdot\|_{\bowtie}$, or $\|\cdot,\dots,\cdot\|_S$. By induction, a standard n-normed space is complete if and only if it is complete with respect to the usual norm $\|\cdot\|_S := \langle\cdot,\cdot\rangle^{1/2}$.*

(b) *A finite-dimensional n-normed space is complete if and only if it is complete with respect to the derived $(n-1)$-norm $\|\cdot,\dots,\cdot\|_{\bowtie}$.*

Consequently, we have the following result.

Corollary 3.7 (fixed point theorem). *Let $(X,\|\cdot,\dots,\cdot\|)$ be a standard or finite-dimensional n-Banach space, and T a contractive mapping of X into itself, that is, there exists a constant $C \in (0,1)$ such that*

$$\|x_1,\dots,x_{n-1},Ty-Tz\| \le C\|x_1,\dots,x_{n-1},y-z\| \tag{3.15}$$

for all $x_1,\dots,x_{n-1},y,z$ in X. Then T has a unique fixed point in X.

Proof. First consider the case $n=2$ (see [8]). By Proposition 3.6, we know that X is a Banach space with respect to the derived norm $\|\cdot\|_\infty$ (for standard case) or $\|\cdot\|_{\bowtie}$ (for finite-dimensional case). Since the mapping T is also contractive with respect to $\|\cdot\|_\infty$ or $\|\cdot\|_{\bowtie}$, we conclude by the fixed point theorem for Banach spaces that T has a unique fixed point in X. For $n>2$, the result follows by induction. □

Remark 3.8. In the finite-dimensional case, it is actually enough to assume that X is an n-normed space because we know that all finite-dimensional normed spaces are complete and, by Proposition 3.6(b), so are all finite-dimensional n-normed spaces.

4. Concluding remark.

We have shown that an n-normed space with $n \ge 2$ is an $(n-1)$-normed space and that, for the standard or finite-dimensional case, the $(n-1)$-norm can be derived from the n-norm in such a way that the convergence and completeness in the n-norm is equivalent to those in the derived $(n-1)$-norm.

Below is an example of a non-standard, infinite-dimensional 2-normed space for which we can derive a norm from the 2-norm such that the convergence and completeness in the 2-norm is equivalent to those in the derived norm.

Let $X = l^\infty$, the space of bounded sequences of real numbers. Equip X with the following 2-norm

$$\|x,y\| := \sup_{i\in\mathbb{N}}\sup_{j\in\mathbb{N}} |x_iy_j - x_jy_i|, \tag{4.1}$$

where $x=(x_1,x_2,x_3,\dots)$ and $y=(y_1,y_2,y_3,\dots)$. Let $a_1=(1,0,0,\dots)$ and $a_2=(0,1,0,\dots)$.

With respect to $\{a_1,a_2\}$, we derive the norm $\|\cdot\|_\infty$ via

$$\|x\|_\infty := \max\{\|x,a_1\|,\|x,a_2\|\}. \tag{4.2}$$

But $\|x,a_1\| = \sup_{i\in\mathbb{N}\setminus\{1\}}|x_i|$ and $\|x,a_2\| = \sup_{i\in\mathbb{N}\setminus\{2\}}|x_i|$, and so we obtain

$$\|x\|_\infty = \sup_{i\in\mathbb{N}}|x_i|, \tag{4.3}$$

the usual norm on l^∞.

Now suppose that $x(k)$ is a sequence in X that converges to x in the derived norm $\|\cdot\|_\infty$. For every $y\in X$, we have

$$\begin{aligned}\|x(k)-x,y\| &= \sup_{i\in\mathbb{N}}\sup_{j\in\mathbb{N}}\left|(x_i(k)-x_i)y_j-(x_j(k)-x_j)y_i\right| \\ &\le \sup_{i\in\mathbb{N}}\sup_{j\in\mathbb{N}}\left|x_i(k)-x_i\right|\left|y_j\right|+\left|x_j(k)-x_j\right|\left|y_i\right| \\ &\le 2\|x(k)-x\|_\infty\|y\|_\infty,\end{aligned} \tag{4.4}$$

whence $\lim_{k\to\infty}\|x(k)-x,y\| = 0$. Hence $x(k)$ converges to x in the 2-norm $\|\cdot,\cdot\|$.

Thus, for this particular example, we see that the convergence in the 2-norm is equivalent to that in the derived norm. By similar arguments, we can also verify that the completeness in the 2-norm is equivalent to that in the derived norm.

For general non-standard, infinite-dimensional n-normed spaces, however, it is unknown whether we can always derive an $(n-1)$-norm from the n-norm such that the convergence and completeness in the n-norm is equivalent to those in the derived $(n-1)$-norm.

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