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APPROXIMATE CONTROLLABILITY OF NEUTRAL FUNCTIONAL DIFFERENTIAL SYSTEM WITH UNBOUNDED DELAY

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Abstract. We consider a class of control systems governed by the neutral functional differential equation with unbounded delay and study the approximate controllability of the system. An example is given to illustrate the result.

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1. Introduction. Let $\mathcal{B}$ be an abstract phase space. Consider the following nonlinear control equation:

$$\frac{d}{dt}\{x(t)+F(t,x_t)\}=Ax(t)+G(t,x_t)+Bv(t),\quad 0<t<T,\ x_0=\varphi\in\Omega,\tag{1.1}$$

where $F,G:[0,T]\times\mathcal{B}\to X$ are continuous functions, A is the infinitesimal generator of an analytic semigroup $S(\cdot)$ of bounded linear operators on a Banach space X, the state function $x(t)$, $0\le t\le T$, takes values in X, and the control function $v(\cdot)$ is given in $L^2(0,T:V)$, which is a Banach space of admissible control functions, with V as a Banach space. Also, B is a bounded linear operator from $L^2(0,T:V)$ into $L^2(0,T:X)$.

The theory of functional differential equations with unbounded delay has been studied by many authors. Hale and Kato [1] have established the local existence and continuation of solutions for retarded equations with infinite delay with initial values in an abstract phase space. Henríquez [2] proved the existence of solutions and the periodic solutions of a class of partial functional differential equations. Recently, Hernández and Henríquez [3] have studied the existence problem for partial neutral functional differential equations with initial values in phase space.

In this paper, we study the approximate controllability of system (1.1) by using the results of Hernández and Henríquez [3]. Similar results on controllability and approximate controllability of linear and nonlinear control systems have been studied in [5, 6, 8].

To study the nonlinear system (1.1), we assume that the histories $x_t:(-\infty,0]\to X$, $x_t(\theta):=x(t+\theta)$, belong to some abstract phase space $\mathcal{B}$, that is, a phase space defined axiomatically. Here, $\mathcal{B}$ is a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and $\mathcal{B}$ satisfies the following axioms (see [1]):

(A_1) If $x:(-\infty,\sigma+a)\to X$, $a>0$, is continuous on $[\sigma,\sigma+a)$, σ is fixed, and $x_\sigma\in\mathcal{B}$, then for every t $\in[\sigma,\sigma+a)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\|x(t)\|\le H\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$, where $H \ge 0$ is a constant; $K, M : [0,\infty) \to [0,\infty)$, K is continuous and M is locally bounded, and H, K, and M are independent of $x(\cdot)$.

(A_2) For the function $x(\cdot)$ in (A_1), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma+a)$.

(A_3) The space $\mathcal{B}$ is complete.

Denote by $\hat{\mathcal{B}}$ the quotient Banach space $\mathcal{B}/\|\cdot\|_{\mathcal{B}}$ and if $\varphi \in \mathcal{B}$, we write $\hat{\varphi}$ for the coset determined by φ. Examples of phase space satisfying the above axioms can be found in [3, 4].

2. Preliminaries. Let the norm of the space X be denoted by $\|\cdot\|$ and for the other spaces we use $\|\cdot\|_{L^2(0,T:X)}$, $\|\cdot\|_{L^2(0,T:V)}$, $\|\cdot\|_\infty$, and so on.

We assume the following hypotheses:

(H_1) $-A$ is the infinitesimal generator of an analytic semigroup $S(\cdot)$ of bounded linear operator on X, where the semigroup $S(t)$ is uniformly bounded, $\|S(t)\| \le M$ for some constant $M \ge 1$ and for every $t \ge 0$, and $0 \in \rho(A)$.

(H_2) There exist constants $\beta \in (0,1)$ and $L_1 \ge 0$, such that the function $F : [0,T]\times\mathcal{B} \to X$ is X_β-valued and satisfies the Lipschitz condition

$$||(-A)^\beta F(t,\psi_1) - (-A)^\beta F(s,\psi_2)|| \le L_1\{|t-s| + ||\psi_1-\psi_2||_{\mathcal{B}}\}, \tag{2.1}$$

for every $0 \le s,\ t \le T$, and $\psi_1, \psi_2 \in \mathcal{B}$, and

$$\mu = 1 - L_1||(-A)^{-\beta}|| \cdot \|K\|_\infty \tag{2.2}$$

is positive.

(H_3) The nonlinear operator $G : [0,T]\times\mathcal{B} \to X$ satisfies the Lipschitz condition

$$||G(s,\psi_1) - G(s,\psi_2)|| \le L_2\{||\psi_1-\psi_2||_{\mathcal{B}}\}, \tag{2.3}$$

for every $0 \le s \le T$, and $\psi_1, \psi_2 \in \mathcal{B}$,

(H_4) Let $\varphi \in \mathcal{B}$ be a function such that $\varphi(0) \in D(A)$ and $F([0,T)\times\mathcal{B}) \subseteq D(A)$, a.e. $t \in [0,T)$ and

$$\Lambda(t) = \int_0^t (-A)S(t-s)F(s,x_s)\,ds \tag{2.4}$$

is differentiable a.e. on $[0,T)$, that is, $\Lambda(t) \in D(A)$.

(H_5) The operator B is a bounded linear operator from $L^2(0,T:V)$ to $L^2(0,T:X)$.

Under the above hypotheses it is well known [3] that for each $u \in L^2(0,T:X)$ there exists a unique mild solution

$$\begin{aligned} x_t(u) = {} & S(t)\{\phi(0) + F(0,\phi)\} - F(t,x_t(u)) - \int_0^t AS(t-s)F(s,x_s(u))\,ds \\ & + \int_0^t S(t-s)G(s,x_s(u))\,ds + \int_0^t S(t-s)u(s)\,ds. \end{aligned} \tag{2.5}$$

The solution mapping W from $L^2(0,T:X)$ to $C(0,T:X)$ can be defined by

$$W(u)(t) = x_t(u)(\cdot). \tag{2.6}$$

We also define the continuous linear operator Φ from $L^2(0,T:X)$ to X by

$$\Phi p = \int_0^T S(T-s)p(s)ds, \quad \text{for } p \in L^2(0,T:X). \tag{2.7}$$

Definition 2.1. Let the reachable set of the system (1.1) at time T be

$$K_T(G) = \{x_T(Bv);\ v \in L^2(0,T:V)\}, \tag{2.8}$$

where $x_t(Bv)$ is a mild solution which satisfies (2.5) with $u = Bv$.

Definition 2.2. The system (1.1) is said to be approximate controllable on the interval $[0,T]$ if $\overline{K_T(G)} = X$, that is, for every $\epsilon > 0$ and $\xi \in D(A)$ there exists a control $v \in L^2(0,T:V)$ such that

$$\begin{aligned} \Big| \xi - S(t)\{\phi(0) + F(0,\phi)\} + F(T, x_T(Bv)) \\ + \int_0^T AS(t-s)F(s,x_s(Bv))ds - \Phi\{G(s,x_s(Bv)) - Bv(s)\} \Big| < \epsilon, \end{aligned} \tag{2.9}$$

where $x_t(Bv)$ is a solution of (1.1) associated with the nonlinear term G and control Bv at the time t.

To simplify our task we consider the linear case of F. We introduce the following assumptions.

For any given $\epsilon > 0$ and $p(\cdot) \in L^2(0,T:X)$, there exists some $v(\cdot) \in L^2(0,T:V)$ such that

(P_1) $\|\Phi p - \Phi Bv\|_X < \epsilon$,

(P_2) $\|Bv(\cdot)\|_{L^2(0,T:X)} \le q_1 \|p(\cdot)\|_{L^2(0,T:X)}$, where q_1 is a positive constant independent of $p(\cdot)$,

(P_3) the constant q_1 satisfies

$$\mu^{-1} q_1 L_2 \|K\|_\infty MT \exp\left\{\left(\frac{C_\alpha L_1}{\beta}T^\beta + ML_2T\right)\|K\|_\infty \mu^{-1}\right\} < 1. \tag{2.10}$$

3. Approximate controllability.

First, we show the approximate controllability of the corresponding system with $G \equiv 0$.

Lemma 3.1. *Under hypotheses (H_1), (H_2), and (P_1), $\overline{K_T(0)} = X$.*

Proof. Since the domain $D(A)$ of the operator A is dense in X (see [7]), it is sufficient to prove that $D(A) \subset \overline{K_T(0)}$, that is, for any given $\epsilon > 0$ and $\xi \in D(A)$ there exists a $v(\cdot) \in L^2(0,T:V)$ such that

$$\begin{gathered} |\xi - h(T,\varphi) - \Phi Bv| < \epsilon, \\ h(T,\varphi) = S(T)\{\varphi(0) + F(0,\varphi)\} - F(T,x_T(Bv)) - \int_0^T AS(T-s)F(s,x_s(Bv))ds. \end{gathered} \tag{3.1}$$

Let $\xi \in D(A)$, then $\xi - h(T,\varphi) \in D(A)$. So there exists some $p \in C^1(0,T:X)$ such that

$$\eta = \int_0^T S(T-s)p(s)ds, \tag{3.2}$$

where $\eta = \xi - h(T,\varphi)$. For instance, if we take $p(s) = \{1-sA\}\{\xi - h(T,\varphi)\}/T$, then the first equality of (3.3) holds, and by hypothesis (P_1) there exists a function $v(\cdot) \in L^2(0,T:V)$ such that

$$\eta = \int_0^T S(T-s)p(s)ds = \int_0^T S(T-s)Bv(s)ds. \tag{3.3}$$

Since $\eta = \xi - h(T,\varphi)$, then $\xi = h(T,\varphi) + \int_0^T S(T-s)Bv(s)ds$.

The denseness of the domain $D(A)$ in X implies the approximate controllability of the corresponding system with $G \equiv 0$. $\square$

To prove the approximate controllability of system (1.1), we need the following lemma.

Lemma 3.2. *Let v_1 and v_2 be in $L^2(0,T:V)$. Then under hypotheses (H_1), (H_2), (H_3), and (H_5), the solution mapping $W(Bv)(t) = x_t(Bv)$ of (1.1) satisfies*

$$\begin{aligned} &||x_t(Bv_1) - x_t(Bv_2)||_\infty \\ &\quad \le \mu^{-1}M\sqrt{T}\exp\left\{\left(\frac{C_\alpha L_1}{\beta}T^\beta + ML_2T\right)\|K\|_\infty \mu^{-1}\right\}||Bv_1 - Bv_2||_{L^2(0,T:X)}. \end{aligned} \tag{3.4}$$

Proof. Let $y(\cdot,\varphi)$; $(-\infty,T] \to X$ be the function defined by

$$y(t,\varphi) := \begin{cases} \varphi(t), & -\infty < t < 0, \\ T(t)\varphi(0), & t \ge 0. \end{cases} \tag{3.5}$$

Denote $y(t,\varphi)$ by $y(t)$ with the continuous map $t \to y_t$.

Next, for each $z \in C(0,T:X)$, $z(0) = 0$, we denote by $\tilde{z}$ the function defined by $\tilde{z}(\theta) = 0$, for $\theta \le 0$, and $\tilde{z}(t) := z(t)$, for $0 \le t \le T$.

So if $x(u)(t)$ satisfies (2.5), we can decompose it as $x(u)(t) = z(u)(t) + y(t)$, for $0 \le t \le T$, which implies that $x_t(u) = \tilde{z}_t(u) + y_t$, for $0 \le t \le T$ and for each $u \in L^2(0,T:X)$ and that the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) = S(t)F(0,\phi) - F(t,\tilde{z}_t(u)+y_t) - \int_0^t AS(t-s)F(s,\tilde{z}_s(u)+y_s)ds \\ + \int_0^t S(t-s)G(s,\tilde{z}_s(u)+y_s)ds + \int_0^t S(t-s)u(s)ds. \end{aligned} \tag{3.6}$$

Thus for each $v_1, v_2 \in L^2(0,T:V)$, it is clear that for $0 \le t \le T$,

$$||x_t(Bv_1) - x_t(Bv_2)|| = ||\{\tilde{z}_t(Bv_1)+y_t\} - \{\tilde{z}_t(Bv_2)+y_t\}||$$

$$
\begin{aligned}
&= ||\tilde{z}_t(Bv_1) - \tilde{z}_t(Bv_2)|| \\
&\leq ||F(t,\tilde{z}_t(Bv_1)+y_t) - F(t,\tilde{z}_t(Bv_2)+y_t)|| \\
&\quad + \left\| \int_0^t AS(t-s)\{F(s,\tilde{z}_s b(Bv_1)+y_s) - F(s,\tilde{z}_s(Bv_2)+y_s)\}ds \right\| \\
&\quad + \left\| \int_0^t S(t-s)\{G(s,\tilde{z}_s(Bv_1)+y_s) - G(s,\tilde{z}_s(Bv_2)+y_s)\}ds \right\| \\
&\quad + \left\| \int_0^t S(t-s)\{Bv_1(s) - Bv_2(s)\}ds \right\| \\
&\leq ||(-A)^{-\beta}|| \cdot L_1 \cdot \|K\|_\infty ||z(Bv_1) - z_t(Bv_2)||_\infty \\
&\quad + \left\{ \frac{C_\alpha L_1 T^\beta}{\beta} + ML_2T \right\} \|K\|_\infty ||z(Bv_1) - z(Bv_2)||_\infty \\
&\quad + M\sqrt{T}||Bv_1 - Bv_2||_{L^2(0,T:X)}.
\end{aligned}
\tag{3.7}
$$

By Gronwall's inequality, we have

$$
\begin{aligned}
&||x.(Bv_1) - x.(Bv_2)||_\infty \\
&\quad \leq \mu^{-1}M\sqrt{T}\exp\left\{ \left(\frac{C_\alpha L_1}{\beta}T^\beta + ML_2T \right) \|K\|_\infty \mu^{-1} \right\} ||Bv_1 - Bv_2||_{L^2(0,T:X)}.
\end{aligned}
\tag{3.8}
$$

□

Theorem 3.3. *Under hypotheses* (H_1), (H_2), (H_3), (H_4), (H_5), *and* (P_1), (P_2), (P_3), $\overline{K_T(G)} = X$, *that is, system* (1.1) *is approximately controllable.*

Proof. Since by Lemma 3.1, $\overline{K_T(0)} = X$, it is sufficient to show that $\overline{K_T(0)} \subset \overline{K_T(G)}$. Let $\xi \in \overline{K_T(0)}$. Then for any given $\epsilon > 0$, there exists $v \in L^2(0,T:V)$ such that

$$
\begin{gathered}
|\,\xi - h(T,\varphi) - \Phi Bv\,| < \frac{\epsilon}{2^3}, \\
h(T,\varphi) = S(T)\{\varphi(0)+F(0,\varphi)\} - F(T,x_T(Bv)) - \int_0^T AS(T-s)F(s,x_s(Bv))ds.
\end{gathered}
\tag{3.9}
$$

Assume $v_1 \in L^2(0,T:V)$ is arbitrarily given. By hypothesis (P_2), there exists some $v_2 \in L^2(0,T:V)$ such that

$$
|\,\Phi\{Bv - G(s,x_s(Bv_1))\} - \Phi Bv_2\,| < \frac{\epsilon}{2^3}. \tag{3.10}
$$

By (3.9) and (3.10), we obtain

$$
|\,\xi - h(T,\varphi) - \Phi G(s,x_s(Bv_1)) - \Phi Bv_2\,| < \frac{\epsilon}{2^2}. \tag{3.11}
$$

For $v_2 \in L^2(0,T:V)$ thus obtained, we determine $w_2 \in L^2(0,T:V)$ by hypotheses (P_1)

and (P_2) such that

$$|\Phi\{G(s,x_s(Bv_2))-G(s,x_s(Bv_1))\}-\Phi Bw_2| < \frac{\epsilon}{2^3}, \tag{3.12}$$

and so by (P_2) and Lemma 3.2,

$$\begin{aligned}
\|Bw_2\|_{L^2(0,T:X)} &\le q_1\|G(\cdot,x.(Bv_2))-G(\cdot,x.(Bv_1))\|_{L^2(0,T:X)}\\
&\le q_1L_2\sqrt{T}\|K\|_\infty\cdot\|x.(Bv_2)-x.(Bv_1)\|_\infty\\
&\le \mu^{-1}q_1L_2\|K\|_\infty MT\exp\left\{\left(\frac{C_\alpha L_1}{\beta}T^\beta+ML_2T\right)\|K\|_\infty\mu^{-1}\right\}\\
&\quad\times\|Bv_2-Bv_1\|_{L^2(0,T:X)}.
\end{aligned}\tag{3.13}$$

Thus we may define $v_3 = v_2-w_2$ in $L^2(0,T:V)$, which has the following property:

$$\begin{aligned}
&|\xi-h(T,\varphi)-\Phi G(s,x_s(Bv_2))-\Phi Bv_3|\\
&\quad=|\xi-h(T,\varphi)-\Phi G(s,x_s(Bv_1))-\Phi Bv_2+\Phi Bw_2\\
&\qquad-\Phi\{G(s,x_s(Bv_2))-G(s,x_s(Bv_1))\}| < \left(\frac{1}{2^2}+\frac{1}{2^3}\right)\epsilon.
\end{aligned}\tag{3.14}$$

By induction, it is proved that there exists a sequence v_n in $L^2(0,T:V)$ such that

$$\begin{aligned}
&|\xi-h(T,\varphi)-\Phi G(s,x_s(Bv_n))-\Phi Bv_{n+1}| < \left(\frac{1}{2^2}+\cdots+\frac{1}{2^{n+1}}\right)\epsilon,\quad n=1,2,\dots,\\
&\|Bv_{n+1}-Bv_n\|_{L^2(0,T:X)}\\
&\quad\le\mu^{-1}q_1L_2\|K\|_\infty MT\exp\left\{\left(\frac{C_\alpha L_1}{\beta}T^\beta+ML_2T\right)\|K\|_\infty\mu^{-1}\right\}\cdot\|Bv_n-Bv_{n-1}\|.
\end{aligned}\tag{3.15}$$

By hypothesis (P_3), the sequence $\{Bv_n : n = 1,2,\dots\}$ is a Cauchy sequence in the Banach space $L^2(0,T:X)$, and there exists some u in $L^2(0,T:X)$ such that $\lim_{n\to\infty} Bv_n = u$ in $L^2(0,T:X)$. Therefore, for any given $\epsilon > 0$, there exists some integer N_ϵ such that

$$\begin{aligned}
&|\Phi Bv_{N_\epsilon+1}-\Phi Bv_{N_\epsilon}| < \frac{\epsilon}{2},\\
&|\xi-h(T,\varphi)-\Phi G(s,x_s(Bv_{N_\epsilon}))-\Phi Bv_{N_\epsilon}|\\
&\quad\le|\xi-h(T,\varphi)-\Phi G(s,x_s(Bv_{N_\epsilon}))-\Phi Bv_{N_\epsilon+1}|+|\Phi(Bv_{N_\epsilon+1})-\Phi Bv_{N_\epsilon}|\\
&\quad<\left(\frac{1}{2^2}+\cdots+\frac{1}{2^{N_\epsilon}}\right)\epsilon+\frac{1}{2}\epsilon\le\epsilon.
\end{aligned}\tag{3.16}$$

This means that $\xi \in \overline{K_T(G)}$. Hence the nonlinear system (1.1) is approximately controllable on $[0,T]$. □

4. Example. We consider the boundary value problem

$$\frac{d}{dt}\left[z(t,\tau)+\int_{-\infty}^{t}\int_{0}^{\pi}b(s-t,\eta,\tau)z(s,\eta)\,d\eta\,ds\right]$$
$$=\frac{d^2}{d\tau^2}z(t,\tau)+\int_{-\infty}^{t}a(s-t)z(s,\tau)\,ds+Bv(t),\quad 0\le t\le T,\ 0\le\tau\le\pi, \tag{4.1}$$
$$z(t,0)=z(t,\pi)=0,$$
$$z(\theta,\tau)=\varphi(\theta,\tau),\quad \theta\le 0,\ 0\le\tau\le\pi.$$

To represent this problem as a Cauchy problem, we take $X=L^2([0,\pi])$ and define $x(t):=z(t,\cdot)$. Let $A:X\to X$ be defined by $Af(\tau):=f''(\tau)$ with the domain

$$D(A):=\{f(\cdot)\in L^2([0,\pi]):f''(\cdot)\in L^2([0,\pi]),\ f(0)=f(\pi)=0\}. \tag{4.2}$$

It is well known that A generates a strongly continuous semigroup $T(\cdot)$ which is compact, analytic, and selfadjoint. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2$, $n\in\mathbb{N}$, with corresponding normalized eigenvectors $e_n(\tau):=(2/\pi)^{1/2}\sin(n\tau)$. These eigenvectors satisfy the properties stated in [3].

Define an infinite-dimensional space V by

$$V=\left\{v\mid v=\sum_{n=2}^{\infty}v_ne_n\text{ with }\sum_{n=2}^{\infty}{v_n}^2<+\infty\right\}. \tag{4.3}$$

The norm in V is defined by $\|v\|_V=(\sum_{n=2}^{\infty}{v_n}^2)^{1/2}$. Define a mapping $B\in\mathcal{L}(V\to X)$ as follows:

$$Bv=2v_2e_1+\sum_{n=2}^{\infty}v_ne_n,\quad\text{for } v=\sum_{n=2}^{\infty}v_ne_n\in V. \tag{4.4}$$

Obviously, $\|B\|_{\mathcal{L}(V\to X)}\le\sqrt{5}$.

Then the operator B is well defined by $v(\cdot,\cdot)\in L^2((0,T)\times(0,\pi))$; and by [8], we know that B satisfies hypotheses (H_5), (P_1), (P_2), and (P_3).

Let $\mathcal{B}$ denote the space $C_r\times L^2(g;X)$ with $r=0$, as in [4]. To prove approximate controllability of the problem (4.1), we assume that conditions (i)–(v) of [4] hold. Consequently, equation (4.1) can be formulated abstractly as

$$\frac{d}{dt}\{x(t)+\Lambda_1(x_t)\}=Ax(t)+\Lambda_2(x_t)+Bu(t),\quad 0\le t\le\xi,$$
$$x_0=\varphi\in\mathcal{B}, \tag{4.5}$$

where Λ_1, Λ_2 are linear operators in $\mathcal{B}$. Using the assumptions stated in [4, pages 471–473], one can see that the system is approximately controllable.

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