

## ON WHITEHEAD'S INEQUALITY, $\text{nil}[X, G] \leq \text{cat} X$

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ABSTRACT. A new proof of Whitehead's inequality,  $\text{nil}[X, G] \leq \text{cat} X$ , is given.

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One of the beautiful theorems of elementary homotopy theory is the result that  $\text{nil}[X, G] \leq \text{cat} X$ . We begin by explaining the notation. Let  $X$  and  $G$  be based, connected topological spaces and let  $G$  be group-like. Thus there is a multiplication  $G \times G \rightarrow G$  on  $G$  which satisfies the group axioms up to homotopy [7, page 118]. Then the set  $[X, G]$  of based homotopy classes of maps from  $X$  to  $G$  inherits a group structure from  $G$ . For a nilpotent group  $\pi$ ,  $\text{nil} \pi$  is the nilpotency class of  $\pi$ . In particular,  $\text{nil} \pi = 0$  means that  $\pi$  is the trivial group and  $\text{nil} \pi \leq 1$  means that  $\pi$  is abelian. Finally,  $\text{cat} X$  denotes the Lusternik-Schnirelmann category of  $X$ , normalized so that contractible spaces have  $\text{cat} = 0$ .

**THEOREM 1** [7, page 464]. *With the above assumptions,  $\text{nil}[X, G] \leq \text{cat} X$ .*

The proof given in [7, pages 462-464] uses the following definition of category [7, page 458]:  $\text{cat} X$  is the smallest nonnegative integer  $l$  such that the diagonal map  $X \rightarrow X^{l+1}$  factors up to homotopy through the subspace of  $X^{l+1}$  with at least one coordinate equal to the base point. Recently, another equivalent definition of category given by the existence of cross-sections to certain fibrations, called Ganea fibrations, has been widely used.

The purpose of this paper is to give a new proof of Whitehead's theorem using this latter definition of category.

For a space  $X$ , we define the Ganea fibrations

$$F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X \quad (1)$$

inductively [3]: for  $n = 0$  the fibration is just  $\Omega X \rightarrow EX \rightarrow X$ , the standard path-space fibration. Assume  $F_{n-1}(X) \xrightarrow{i_{n-1}} G_{n-1}(X) \xrightarrow{p_{n-1}} X$  is defined and let  $G'_n(X) = G_{n-1}(X) \cup_{i_{n-1}} CF_{n-1}(X)$  be the mapping cone of  $i_{n-1}$ . Define  $p'_n : G'_n(X) \rightarrow X$  as  $p_{n-1}$  on  $G_{n-1}(X)$  and trivial on the cone  $CF_{n-1}(X)$ . Replacing  $p'_n$  by an equivalent fibre map, we obtain the fibre sequence  $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X$ . The connection of the Ganea fibrations to category is as follows (see [2, 4]):  $\text{cat} X \leq n$  if and only if  $p_n$  admits a cross-section.

We now start the proof of the theorem. We begin in Lemma 2 with a general result which is probably known (see [5, page 22] and [6]). Let  $f : A \rightarrow B$  be any map and

consider the mapping cone sequence of  $f$ ,

$$A \xrightarrow{f} B \xrightarrow{j} C_f \xrightarrow{q} \Sigma A, \tag{2}$$

where  $C_f$  is the mapping cone of  $f$  and  $\Sigma A$  is the suspension of  $A$ . If  $G$  is any group-like space, we obtain a homomorphism  $q^* : [\Sigma A, G] \rightarrow [C_f, G]$ .

**LEMMA 2.** *The image of  $q^*$  is contained in the center of  $[C_f, G]$ .*

**PROOF.** We sketch the proof which is based on the operation of  $[\Sigma A, G]$  on  $[C_f, G]$  [7, page 136]. We denote this operation by “ $\cdot$ ” and the group operation in  $[\Sigma A, G]$  and  $[C_f, G]$  by “ $+$ ”. Then for  $a, b \in [\Sigma A, G]$  and  $x, y \in [C_f, G]$ , it is easily seen (see [1] and also [5, page 5]) that

$$(a + b) \cdot (x + y) = (a \cdot x) + (b \cdot y). \tag{3}$$

Let  $e$  denote the homotopy class of the constant map. By taking  $b = e$  and  $x = e$ , we obtain

$$a \cdot y = q^*(a) + y. \tag{4}$$

By taking  $a = e$  and  $y = e$ , we obtain  $b \cdot x = x + q^*(b)$  which we write as

$$a \cdot y = y + q^*(a). \tag{5}$$

Thus Image  $q^*$  is in the center of  $[C_f, G]$ . □

**LEMMA 3.** *For any space  $X$  and group-like space  $G$ ,  $\text{nil}[G_k(X), G] \leq k$ .*

**PROOF.** This is proved by induction on  $k$ . Clearly,  $\text{nil}[G_0(X), G] = 0$  since  $G_0(X)$  is contractible. Suppose the result is true for  $k - 1$ . It suffices to show that  $\text{nil}[G'_k(X), G] \leq k$ . Consider the mapping cone sequence

$$F_{k-1}(X) \xrightarrow{i_{k-1}} G_{k-1}(X) \xrightarrow{j_{k-1}} G'_k(X) \xrightarrow{q_k} \Sigma F_{k-1}(X), \tag{6}$$

where  $G'_k(X)$  is the mapping cone of  $i_{k-1}$ ,  $j_{k-1}$  is the inclusion, and  $q_k$  is the projection. This gives an exact sequence of groups

$$[\Sigma F_{k-1}(X), G] \xrightarrow{q_k^*} [G'_k(X), G] \xrightarrow{j_{k-1}^*} [G_{k-1}(X), G]. \tag{7}$$

By Lemma 2, Image  $q_k^*$  is contained in the center of  $[G'_k(X), G]$ . By induction,  $\text{nil}[G_{k-1}(X), G] \leq k - 1$ . Therefore,  $\text{nil}[G_k(X), G] \leq k$ . □

Now we complete the proof of the theorem. Suppose  $\text{cat} X = n$ . Thus there is a section  $s : X \rightarrow G_n(X)$ , that is,  $p_n s$  is homotopic to the identity map. Hence  $s^* : [G_n(X), G] \rightarrow [X, G]$  is onto. Since  $\text{nil}[G_n(X), G] \leq n$  by Lemma 3, it follows that  $\text{nil}[X, G] \leq n$ .

**REMARK 4.** By dualizing the Ganea fibrations we obtain the Ganea cofibrations  $X \rightarrow C_n(X) \rightarrow Q_n(X)$  [2, Section 4]. Then the cocategory of  $X$  is defined to be the smallest integer  $n$  such that the cofibre map  $X \rightarrow C_n(X)$  has a retraction. If  $C$  is a co- $H$ -group, then an argument dual to the one above yields  $\text{nil}[C, Y] \leq \text{cocat} Y$ .

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