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SPACES WHOSE ONLY FINITE-SHEETED COVERS ARE THEMSELVES. PART I

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This is a survey of results and open questions related to the topology of spaces that have no nontrivial finite-sheeted covers.

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1. Introduction. This paper focuses on those connected topological spaces that have the property that all of their finite-sheeted connected covering spaces have total space homeomorphic to the base space. Note that there are two ways that a space M can satisfy this property: either (1) M has no nontrivial finite-sheeted covers or (2) M has a k-fold connected cover $p : X \to M$ for some $k \geq 2$, and the total space of every connected finite-sheeted cover $p : X \to M$ is such that X is homeomorphic to M. In this paper, we consider those spaces that satisfy the first condition, that is, those that have no nontrivial finite-sheeted covers. In the sequel we will survey items related to spaces satisfying the second condition. A reader with a current interest in the second type of space can consult [34, 39].

Section 2 of this paper presents basic terminology and some elementary examples. Section 3 presents what is known about such metric continua. Section 4 presents what is known in the context of spaces with more structure such as low-dimensional manifolds and cell complexes. The development includes: examples, means to construct additional examples, and statements of interesting problems that, to the knowledge of the author, are unsolved as of this date. It is of interest to note that spaces with no nontrivial finite-sheeted covers are related to two problems of significant historical interest: the question of whether every nonseparating planar continuum has the fixed point property and the question of whether every compact 3-manifold can be decomposed into finitely many geometric pieces. See Scott [35].

It has been attempted to make the paper as self-contained as possible. Unfortunately, omissions have no doubt occurred. For more complete treatments of the general topology, including the topology of inverse limit spaces, consult Engelking [8]. A good reference for the algebraic topology is Spanier [36]. The books by Hempel [16] and Jaco [21] and the survey paper by Scott [35] are good references on 3-manifold topology. For the more specialized group theory, refer to the books by Robinson [33] and Magnus et al. [27] while the more elementary group theory can be found in Hungerford [20]. Also refer to Kirby [25] for the most recent version of his problem list. Kirby's paper, (which contains an extensive bibliography) can also be thought of as a crash course in the topology of low-dimensional manifolds.

2. Terminology and elementary examples. In this paper, a space M is a metric space in the metric topology, perhaps with additional structure imposed on it. Usually M is compact. Unless it is specified to the contrary, it is assumed that all spaces are connected. That is, we usually assume that M is a metric *continuum*, a compact connected metric space. A *neighborhood* of $x \in M$ is an open subset $U(x)$ of M containing x. A *map* $f : X \to M$ between the (connected) metric spaces X and M is a continuous function from X to M. If $S \subset X$, the *restriction* of f to S is denoted by $(f \mid S)$.

Let the space M be given. A space X, or more precisely, a pair (X,p), is a *covering space* of M if the map $p : X \to M$ is a surjective map, such that for each $y \in M$, there is a neighborhood $V(y)$ about y and a collection of pairwise disjoint neighborhoods $\{U(x) : x \in p^{-1}(y)\}$, such that for all $x \in p^{-1}(y)$ the restriction $(p \mid U(x)) : U(x) \to V(y)$ is a homeomorphism of $U(x)$ onto $V(y)$. The map p of the covering space (X,p) is called a *covering* projection. The covering space (X,p) is a k-sheeted, k-fold, or k-to-1 covering space of M if for all $y \in M$, $|p^{-1}(y)| = k$. In this case we write $|p| = k$. The covering space (X,p) of M is *trivial* if $p : X \to M$ is a homeomorphism onto M, that is, if $|p| = 1$. A self-homeomorphism $h : X \to X$ is a *deck transformation* of the covering space (X,p) over M if $p \circ h = p$. The set of all deck transformations $\mathrm{Aut}_M(X,p)$ of X over M is a group under function composition. We usually suppress the base point in a pointed space (X,x_0) and denote the fundamental group of X by $\pi_1(X)$. A covering space (X,p) of M is a *regular* covering space if for every $y \in M$ and every $x_1,x_2 \in p^{-1}(y)$, there is an $h \in \mathrm{Aut}_M(X,p)$ such that $h(x_1) = x_2$. A regular covering $p : X \to M$ is an *Abelian* cover if $\mathrm{Aut}_M(X,p)$ is an Abelian group. A regular cover (X,p) of M is a cyclic cover if $\mathrm{Aut}_M(X,p)$ is a cyclic group. Note that, for manifolds and cell complexes, the last three conditions are equivalent to, respectively, the conditions that $p_*(\pi_1(X))$ is a normal subgroup of $\pi_1(M)$, $\pi_1(M)/p_*(\pi_1(X))$ is an Abelian group, and $\pi_1(M)/p_*(\pi_1(X))$ is a cyclic group.

Recall that if M is connected, locally path connected, and semi-locally 1-connected, for example, M is a compact-connected manifold or finite-connected complex, then there is, in essence, a one-to-one correspondence between the subgroups of $\pi_1(M)$ and covering spaces of M. Much of what follows exploits this correspondence and accordingly much of what follows is stated in group theoretic terms.

An inverse sequence is a triple $\mathscr{X} = (X_n, f_n, N)$ where for each $n \in N$, X_n is a metric space and $f_n : X_{n+1} \to X_n$ is a continuous map. The maps are called *bonding* maps and we typically assumed that all bonding maps are surjective. The inverse limit of the inverse sequence, denoted either by X_∞ or $\varprojlim \mathscr{X}$, is the subspace of the infinite product $\prod_{n=1}^{\infty} X_n$ defined by $(x_n)_{n=1}^{\infty} \in X_\infty$ if and only if $x_n = f_n(x_{n+1})$. A continuum which is an inverse limit of an inverse sequence of finite trees, that is, finite-connected graphs, is said to be *tree-like*.

In the literature, groups that have no nontrivial finite quotients are said to be profinitely trivial. We dualize this notion to obtain a name for the sorts of spaces in which we are interested.

Definition 2.1. Let M be a connected metric space. Then M is *co-finitely self-similar* if whenever $p : X \to M$ is a finite-to-one connected covering space, it follows

that X is homeomorphic to M; M is *co-finitely trivial* if and only if it has no nontrivial finite-sheeted covers.

Interest in spaces whose only finite-sheeted covers are themselves was initially prompted after reading Jungck [22]. One of the things done in Jungck's paper is the development, in the context of spaces that lack the structure to do classical algebraic topology, of analogs of some of the classical covering space theory. While the next section contains more of these results, one that is of interest in this introductory section is that we may recognize finite-sheeted covers of a space M, via some purely point-set topology. A map $p : X \to M$ is proper if for every compact $K \subset M$, $p^{-1}(K)$ is compact. As a corollary of Jungck [22, Theorem 2.1, Corollary 2.7], we have the following theorem.

Theorem 2.2. *Any proper local homeomorphism $p : X \to M$ of a metric space X onto the connected metric space M is a k-to-1 cover for some $k < \infty$. (M is not assumed to be compact and X is not assumed to be connected.)*

Co-finitely trivial spaces are also called trivially h-connected spaces and H-connected spaces in the literature. Note that this "h-connected" terminology is used to emphasize that these notions are generalizations of simple connectivity that are dependent on the existence of some homeomorphism. In the context of this paper, we are going to focus on the fact that the spaces in question have no nontrivial finite-sheeted covers and are not so concerned with the fact that this is a generalization of simple connectivity.

Example 2.3. Note that, co-finitely trivial spaces abound since any simply connected space has this property. There are other interesting examples. By Jungck [22, Corollary 3.9], the topologist's sine curve

$$S = \{(x, \sin(x)) : x \in (0,1]\} \cup \{(0, y) : y \in [-1,1]\} \tag{2.1}$$

has no nontrivial finite-sheeted covers. In fact, by Lau [26, 3.5] any continuum that is either tree-like or a nonseparating planar continuum is co-finitely trivial since such spaces possess no nontrivial finite- or infinite-sheeted covers. The spaces constructed in Griffiths [10] of spaces X with nontrivial $\pi_1(X)$, are examples of co-finitely trivial spaces. They are also interesting because they are examples of spaces for which the correspondence between subgroups of the fundamental group and covering spaces is broken. The most well known of these "Griffiths type" examples is probably the join of the cone on a pair of Hawaiian Earrings given in Spanier [36, Example 2.5.18].

There are also examples of nicer nonsimply connected spaces, for example, finite-cell complexes, with nontrivial fundamental group that are co-finitely trivial. In these examples, the correspondence between subgroups of the fundamental group and covering spaces of the space is maintained, and so, the fundamental groups of these spaces must have no finite index subgroups of index greater than 1. Specifically, let $G \neq 1$ be a finitely presented group that has no proper finite index subgroups. We could, for example, choose a finitely presented infinite simple groups, see Higman [19], or Higman's [18] group $G = \langle a_0, \ldots, a_3 : a_{i+1}^{-2} a_i^{-1} a_{i+1} a_i \rangle$ (the addition in the subscripts

is mod 4). Build M, a compact manifold or cell complex with $\pi_1(M) \cong G$, see for example [38]. Then M is co-finitely trivial since $\pi_1(M)$ has no proper finite index subgroups.

3. Constructing co-finitely trivial spaces. Given a collection of metric continua with no nontrivial finite-sheeted covers, one may construct additional co-finitely trivial spaces by applying the "calculus" of H-connectedness (or calculus of noncovering) developed in Jungck [22, Section 3]. Specifically, we have the following results. Recall that if X is metric and $A_n \subset X$, then $\limsup(A_n)$ is defined to be

$$\{x \in X : \forall \epsilon > 0 \text{ there is } N \text{ such that if } n \geq N, \text{ then } B(x,\epsilon) \cap A_n \neq \emptyset\}. \tag{3.1}$$

Theorem 3.1 (A calculus of noncovering). (a) *If M_1 and M_2 are co-finitely trivial spaces and are closed subsets of a larger metric space M, such that $M = M_1 \cup M_2$, $M_1 \cap M_2 \neq \emptyset$, and $M_1 \cap M_2$ is connected, then M has no nontrivial finite-sheeted covers.* (b) *If M and N are metric spaces with no nontrivial finite-sheeted covers, at least one of which is compact, then $M \times N$ has no nontrivial finite-sheeted covers.* (c) *Assume that $M = A \cup C$ is a metric continuum such that A and C have no nontrivial finite-sheeted covers, $A \cap C = \emptyset$, A is open, C is closed, and for each $n \in N$, A_n is a connected subset of M such that $A - A_n$ is connected. If $\limsup(A_n) \subset C$, then M has no nontrivial finite-sheeted covers.* (d) *If $Y = \bigcup_{n=1}^{\infty} A_n$, $A_n \subset \operatorname{Int}(A_{n+1})$, and A_n has no nontrivial finite-sheeted covers, then Y has no nontrivial finite-sheeted covers.*

This calculus allows us to easily see that some of the examples above have no nontrivial finite-sheeted covers. For example, if M is the join of the cones on the pair of Hawaiian Earrings and C_1 and C_2 are the two cones in M, then the fact that M has no nontrivial finite-sheeted covers follows from Theorem 3.1(a). The fact that the topologist sine curve has no nontrivial finite-sheeted covers follows from Theorem 3.1(c).

Group theoretic applications of the above are also possible. For example, let G and H be finitely presented groups that have no proper finite index subgroups. Construct compact 2-complexes M and N whose fundamental groups are, respectively, G and H. By Theorem 3.1(a) and (b) the one point joint $M \vee N$ (formed by picking $x \in M$ and $y \in N$ and identifying x and y) and the product $M \times N$ both have no nontrivial finite-sheeted covers. So, by some basic algebraic topology, in particular, from the correspondence of subgroups of $\pi_1(M \vee N) \cong G * H$ and $\pi_1(M \times N) \cong G \times H$ with, respectively, covering spaces of $M \vee N$ and $M \times N$, it follows that $G * H$ and $G \times H$ have no proper finite index subgroups.

One may add to Jungck's calculus the main result of Lau [26, Theorem 2]. A metric space (M,d) is d-compressible if there is an $x_0 \in M$ and a map $G : M \times I \to M$ such that $G(x,1) = x$, $G(x,0) = x_0$ for all $x \in M$, $G(x_0,t) = x_0$ for all $t \in I$, and for all $x,y \in M$ and all $t \in I$, $d(G(x,t),G(y,t)) \leq d(x,y)$.

Theorem 3.2 [26, Theorem 2]. *If $p : X \to M$ is a surjective local homeomorphism and M is the inverse limit of the inverse sequence (M_n, f_n, N) in which the bonding maps $f_n : M_{n+1} \to M_n$ are onto and the M_n are d-compressible metric continua, then p is a homeomorphism.*

Lau's result, with its hypothesis of d-compressibility on the M_n, is requiring that the spaces in the inverse sequence be contractible and, since the connected spaces M_n can be homotoped to a point, they must satisfy a limited local connectivity hypothesis at least at one point. The next result is a generalization of Lau's theorem that removes both of these requirements for the case where the bonding maps are assumed to be open. It is also a recent addition to Jungck's calculus and has an interesting corollary. Note that in the sense that the notion of inverse limit generalizes that of product, it, its corollary, and Lau's result are all generalizations of Theorem 3.1(b).

Theorem 3.3 (Jungck and Timm [24, Theorem 3.1]). *If M_∞ is the inverse limit of the inverse sequence (M_n, f_n, N) of compact co-finitely trivial metric spaces M_n in which the bonding maps $f_n : M_{n+1} \to M_n$ are open surjective maps, then M_∞ is co-finitely trivial.*

Corollary 3.4. *If X is a countable product of co-finitely trivial metric compacta, then X is co-finitely trivial.*

The proof of Theorem 3.3 given in [24] makes explicit use of the openness of the bonding maps in the inverse sequence (M_n, f_n, N). We do not have an example showing that the openness of the f_n is necessary, though it seems likely that this is the case. Also, Lau's result does not contain this requirement. In particular, when the M_n are finite trees, an application of Lau's result shows that the analog of Theorem 3.3 is true without the openness hypothesis. This prompts the following question.

Question 3.5. If M_∞ is the inverse limit of the inverse sequence (M_n, f_n, N) of co-finitely trivial compact metric spaces M_n and the bonding maps $f_n : M_{n+1} \to M_n$ are surjective maps, must M_∞ be co-finitely trivial? What if the index set in the inverse limit defining M_∞ is allowed to be an arbitrary partially ordered set? Note the observations after Theorem 3.6, below, for further refinement of this question.

There are two other situations relating inverse limits to co-finitely trivial spaces in the context of continua with fairly little addition structure. We first explore their relationship to spaces that have the fixed point property. Consider the following result of Tominaga [41]. Recall that a space X has the fixed point property if every self-map $f : X \to X$ has a fixed point, that is, a point $x \in X$ such that $f(x) = x$. The space X has the fixed point property for homeomorphisms if every self-homeomorphism of X has a fixed point.

Theorem 3.6. *Let X and M be metric continua and $f : X \to M$ a local homeomorphism. If X is the inverse limit of an inverse sequence with onto bonding maps of connected, simply connected, locally connected metric continua and X has the fixed point property for homeomorphisms, then f is a homeomorphism.*

While Tominaga's result is a condition on domains of local homeomorphisms, it can be combined with Lau's result in [24] to give another test that can be used to determine that a metric continuum is co-finitely trivial. To see this, suppose M is a compact metric continuum and there exist a pair (X, f) satisfying the hypotheses of Theorem 3.6. In addition, assume that the spaces in the inverse limit that give X are

either d-compressible or that the bonding maps in the inverse sequence are open. Then let $p : X \to M$ be a finite-sheeted covering. By Theorem 3.6, M is homeomorphic to X and so there is an $|p|$-fold covering $p' : Y \to X$. But, as X is either the inverse limit of d-compressible spaces or the inverse limit of an inverse sequence in which the bonding maps are open, it is, by Lau [26] or [24], respectively, a space that has no nontrivial finite-sheeted covers. Therefore, M is co-finitely trivial.

Tominaga's result provides another test for the nonexistence of finite-sheeted covers and is dependent upon the space X having the fixed point property. Also, it is easy to see that a space X cannot be a nontrivial regular covering space of any space M, if X has the fixed point property. These observations provide motivation for further investigation of the exact relationship between the two concepts. A little thought shows some restriction on, or change in, the problem is necessary because the n-sphere, $n \geq 2$, being simply connected, has no nontrivial covers and yet the antipodal map on the n-sphere, $n \geq 2$, has no fixed point.

One line of investigation is suggested by restricting to 1-dimensional or planar continua. There are a couple of reasons why this line of investigation is interesting. First, the question of which continua, or more specifically, which nonseparating planar continua have the fixed point property has been of interest for 70 years or so. Also, Hagopian [13] has recently shown that every simply connected planar continuum has the fixed point property and the idea that a space is co-finitely trivial is a generalization of simple connectivity. Second, by a result of Jungck and Timm [23], a planar continuum is separating if and only if it is co-finitely trivial. Also contained in [23] are additional results relating the notion of co-finite triviality and Theorem 3.1(c) above to the fixed point property. In particular, every nonseparating planar continuum has the fixed point property if and only if every co-finitely trivial planar continuum has the fixed point property. This allows the translation of the this classical fixed point problem into a problem about the existence or nonexistence of finite-sheeted covering spaces of planar continua. If one attempts to exploit this logical equivalence and then attempts to prove that every co-finitely trivial planar continuum has the fixed point property via a proof by contradiction we are lead to the following interesting question.

Question 3.7. Let M be a continuum. Suppose that the self-map $f : M \to M$ has no fixed point. Is there a way to use f to construct a space X and define a map $\widetilde{f} : X \to M$ such that X is connected and $(X, \widetilde{f})$ is a covering space of M? For a warm-up, try this with $M = S^1$ and assume that f is the standard 2-fold covering map.

For a fairly current summary of what is known about the fixed point property for planar continua, refer to Hagopian [13].

For higher-dimensional situations or even just the nonplanar situation the correct version of the problem seems to be more difficult to formulate. First, as suggested by case of the n-sphere and its antipodal map or Bellamy's example [1] of a tree-like continuum with no fixed point, it appears that the correct version of the problem is to wonder if every self-map of a co-finitely trivial continuum must have a periodic point. Unfortunately, there is Minc's example [29] of a tree-like continuum with a self-map that has no periodic point.

Finally, note that the next example, see also [9, 24], shows that in certain instances the basic structure of an inverse sequence can force the nonexistence of certain finite-sheeted covers in the inverse limit.

Example 3.8. Let $f,g : X \to X$ be self-covers of the connected compact metric space X, such that $f \circ g = g \circ f$. Consider the inverse sequence $\mathscr{X} = (X_n = X, f_n = f, N)$ given by

$$X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} \cdots \xleftarrow{f} X \xleftarrow{f} \cdots \tag{3.2}$$

and consider the generalized solenoid $X_\infty = \varprojlim (X_n = X, f_n = f, N)$. Perhaps the most obvious way to hope to obtain a nontrivial covering of X_∞ is to consider the limit map $\lambda_\infty : X_\infty \to X_\infty$ of the self-map $\lambda = (\mathrm{id}, \lambda_n = g) : \mathscr{X} \to \mathscr{X}$ of the inverse system $\mathscr{X}$ given by $\lambda_n = g : X_n \to X_n$. While it is the case that the limit map is always a finite-to-1 cover, it is interesting that it is not always the case that λ_∞ is a nontrivial self cover. For example, when $\lambda_n = g = f$, the limit map λ_∞ is a homeomorphism that is in essence a coordinate shift. Actually, one may prove the following stronger result. The proof is a nice exercise.

Fact 3.9. Assume that $\mathscr{X} = (X, f_n, N)$ is an inverse sequence of metric continua with each $f_n : X_n = X \to X_{n-1} = X$ an m_n-fold covering projection. Assume that $f : X \to X$ is finite-to-one cover and $\lambda = (\mathrm{id}, \lambda_n, N) : \mathscr{X} \to \mathscr{X}$ is a self-map of the sequence such that for all $n \in N$, $\lambda_n = f$. Assume there is a subsequence $(n_k)_{k\in N}$ such that $f_{n_k} = \lambda_{n_k} = f$. Then $\lambda_\infty : X_\infty \to X_\infty$ is a homeomorphism.

Example 3.10. Let $\mathscr{X} = (S^1, f_n, N)$ be an inverse sequence of circles such that each $f_n : S^1 \to S^1$ is a prime order covering projection of order p_k. Assume that for each prime p there are infinitely many $k \in N$ for which $p_k = p$. Fact 3.9 implies that for each prime p and p-fold covering projection $f_p : S^1 \to S^1$ the limit map $\lambda_\infty : X_\infty \to X_\infty$ induced by the self-map $\lambda : \mathscr{X} \to \mathscr{X}$ of the inverse sequence that is defined by $\lambda_n = f_p$ must be a homeomorphism. By a result of Mardešić and Matijević [28], these sorts of maps are the only possible finite-sheeted covers of the solenoid X_∞. Hence X_∞ is co-finitely trivial.

These observations prompt a more general version of Question 3.5.

Problem 3.11. Determine conditions on the inverse system $\mathscr{X} = (X_\alpha, f_\alpha^\beta, A)$ such that the limit space $X_\infty = \varprojlim \mathscr{X}$ has no nontrivial finite-sheeted covers.

Before closing this section and turning to the situation of spaces with additional structure, we note that asking when a space has no nontrivial finite-sheeted covers is a version of the problem of determining when a local homeomorphism on a space is a global homeomorphism. This is the point of view that motivated Jungck's use of the term H-connected and his version of its definition. Thought of this way, the problem is almost a hundred year old with the first reference to it that this author is aware of appearing in Hadamard [11, 12] in 1906. Refer to Jungck [22] and the papers in its bibliography for some of the history of this problem in the intervening years. Also refer

to the survey papers by Heath [14, 15]. They are good sources of information on what is known about the general question of when a continuum can be either the domain or range of an exactly k-to-1 function (that has up to finitely many discontinuities). They also contain lists of interesting open problems. The bibliography in [14] is quite extensive.

4. Manifolds with no nontrivial finite-sheeted covers. For spaces with additional structure, in particular, manifolds, there are several situations that are related to the question of existence or nonexistence of finite-sheeted self-covers. The first of these treated here involves some work of Daverman.

Recall that an n-manifold M is closed if it is compact and $\partial M = \emptyset$. One of the problems that is of current interest in the topology of manifolds is that of determining when a closed n-manifold is a codimension k approximate fibrator. While one should consult Daverman [4, 5, 6, 7] for the precise definitions and additional background, the idea is that a closed n-manifold N is a codimension k fibrator if given any $(n+k)$-manifold M, a k-manifold B, and a proper surjective map $p : M \to B$ such that for each $b \in B$, $p^{-1}(b)$ has the homotopy type of N, it then follows that the map $p : M \to B$ has the approximate homotopy lifting property. The benefit of knowing that for a particular N, the map $p : M \to B$ has the approximate homotopy lifting property is that we may then apply results of Coram and Duval [2, 3] that give an exact sequence relating the homotopy groups of N, M, and B and do algebraic topology. By results of Daverman, a necessary condition for such an N to be a codimension k fibrator is that N not regularly cyclically cover itself. As no compact connected manifold can be an infinite cover of any manifold, a closed co-finitely trivial n-manifold is a reasonable candidate to be codimension k fibrators. We wonder if there is a more intimate connection between the two concepts.

That such a relationship may exist is suggested by the following. A finitely presented group G is said to be hyperhopfian if every homomorphism $\psi : G \to G$ with $\psi(G)$ normal and $G/\psi(G)$ cyclic is necessarily an automorphism of G. For a space N as in the preceding paragraph, having a hyperhopfian fundamental group is a sufficient (though not necessary) condition for obtaining the fibrator condition. Any (closed) manifold N that has no nontrivial finite-sheeted covers has a fundamental group $G = \pi_1(N)$ with no proper finite index subgroups and so such a group almost satisfies the hyperhopfian condition: what remains to show is that G is hopfian, that is we must show that if G is a finitely presented group with no proper finite index subgroups and $\psi : G \to G$ is an epimorphism, then ψ is an isomorphism. There is some hope that this is true. For example, if $\pi_1(N)$ is a finitely presented infinite simple group then it has no proper finite index subgroups and so is hopfian. Thus, we wonder if all finitely presented groups that have no proper finite-index subgroups are hopfian. For those co-finitely trivial n-manifolds N that are hopfian and aspherical it follows, by Daverman [7, Corollary 5.5], that N is a codimension 2-fibrator. (In order to apply Daverman's result note that a co-finitely trivial n-manifold is necessarily orientable since a nonorientable n-manifold has a 2-fold connected cover.)

There is a very large volume of work that shows that the groups that can be a 3-manifold group, that is, the fundamental group of a 3-manifold, must satisfy quite

restrictive criteria. Three such interesting criteria and appropriate terminology relating to the discussion at hand follow.

Theorem 4.1 (The Scott-Shalen Theorem [2, Corollary V.16]). *If G is a finitely generated group that is a 3-manifold group then G is finitely presented.*

Theorem 4.2 [21, Example V.8]. *If G is a finitely generated Abelian 3-manifold group, then G is isomorphic to one of 1, Z, $Z \times Z$, $Z \times Z \times Z$, $Z \times Z_2$, or Z_p for some $p \geq 2$.*

The connect sum $M\#N$ of two 3-manifolds is obtained by deleting the interiors of two 3-balls $B^3 \subset M$ and $D^3 \subset N$ (with bi-collared boundaries) and then gluing the resulting 3-manifolds together via a homeomorphism $\varphi : \partial B^3 \to \partial D^3$. A closed 3-manifold X is prime if whenever $X = M\#N$ then either M or N is homeomorphic to S^3. A 3-manifold M is *irreducible* if every $S^2 \subset M$ bounds a 3-ball in M. With the exception of S^3 and $S^1 \times S^2$, a closed orientable 3-manifold is prime if and only if it is irreducible. A surface T properly embedded in a 3-manifold M is compressible in M if either $T = S^2$ and T bounds a 3-cell in M or $T \neq S^2$ and there is a disk $D \subset M$ such that $D \cap T$ is a curve that cannot be contracted in T. Otherwise, T is incompressible. A 3-manifold is sufficiently large if it contains a 2-sided incompressible surface. A 3-manifold is a Haken manifold if it is a compact orientable, irreducible manifold that contains a 2-sided incompressible surface. A 3-manifold is virtually Haken if it has a finite-sheeted covering space that is a Haken manifold. A group G is residually finite if $\cap\{H : H < G \text{ and } |G : H| < \infty\} = 1$.

Theorem 4.3 [17, Theorem 1.2]. *If G is the fundamental group of a compact 3-manifold whose prime factors are either virtually Haken or have infinite cyclic fundamental group, then G is residually finite.*

As mentioned above, given any finitely presented group G, there are standard constructions that yield higher dimensional manifolds M^n, $n > 3$, with $\pi_1(M) \cong G$. If G has no finite index subgroups then M^n is a co-finitely trivial manifold. Thus, there are an abundance of high dimensional co-finitely trivial compact manifolds. This does not appear to be the case for dimension 3. It is conjectured, Thurston [37], Hempel [17], and Kirby [25, Problem 3.5], that all compact 3-manifolds can be written as a connect sum in which the prime factors are as stated in Theorem 4.3. As a consequence of this conjecture and Hempel's result, a reasonable conjecture is that all compact 3-manifolds have residually finite fundamental group. Therefore, since the fundamental group of a co-finitely trivial n-manifold has no proper finite index subgroups, a co-finitely trivial manifold M can have residually finite fundamental group if and only if $\pi_1(M) = 1$. Thus, for 3-manifolds, the most interesting question about co-finitely trivial spaces is the existence problem. In the light of the preceding, and in conjunction with the above conjecture and the Poincare conjecture, we state this problem as a conjecture.

Conjecture 4.4. (a) *There do not exist any nonsimply connected co-finitely trivial compact 3-manifolds.* (b) *The only compact co-finitely trivial 3-manifolds are the closed 3-ball, $S^2 \times I$, and S^3.*

The topological dual to the notion of a co-finitely trivial space should be a space that cannot non-trivially cover any space. This situation and variants of it have been analyzed by a number of people. For results relating to general metric continua that cannot be the domain of any finite-to-one map other than a homeomorphism, the reader should consult the survey by Heath [15]. As a consequence of Tominaga [41] and Jungck and Timm [23] nonseparating-planar continua cannot be the domain of any nontrivial covering projection and are co-finitely trivial. Also, it is easy to see that co-finitely trivial spaces, like the join of the cone on the pair of Hawaiian Earrings, that have a unique point with especially bad local topology cannot be the domain of a nontrivial-covering projection. For 3-manifolds, Myers [30] has recently given a method that provides specific examples of open simply connected 3-dimensional Whitehead manifolds that cannot cover compact 3-manifolds. Other papers, for example, Myers [30, 31, 32], Wright [42], Tinsley and Wright [40], also investigate the topology of these sorts of spaces. However S^3 is co-finitely trivial and yet finitely covers infinitely many different spaces. This collection of observations prompts our last question.

Question 4.5. Which co-finitely trivial spaces cannot be the domain of a nontrivial covering projection?

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