γ -SETS AND γ -CONTINUOUS FUNCTIONS

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We introduce a new class of sets, called γ -sets, and the notion of γ -continuity and investigate some properties and characterizations. In particular, γ -sets and γ -continuity are used to extend known results for semi-open sets and semi-continuity.

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1. Introduction. Let X , Y , and Z be topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X . The closure (resp., interior) of S will be denoted by $\text{cl } S$ (resp., $\text{int } S$). A subset S of X is called a semi-open set [2] (resp., α -set [4]) if $S \subset \text{cl}(\text{int}(S))$ (resp., $S \subset \text{int}(\text{cl}(\text{int}(S)))$). The complement of a semi-open set (resp., α -set) is called semi-closed set (resp., α -closed set). The family of all semi-open sets (resp., α -sets) in X will be denoted by $\text{SO}(X)$ (resp., $\alpha(X)$). A function $f : X \rightarrow Y$ is called semi-continuous [2] (resp., α -continuous [3]) if $f^{-1}(V) \in \text{SO}(X)$ (resp., $f^{-1}(V) \in \alpha(X)$) for each open set V of Y . A function $f : X \rightarrow Y$ is called semi-open [2] (resp., α -open [3]) if for every semi-open (resp., α -open) set U in X , $f(U)$ is semi-open (resp., α -open) in Y .

A subset $M(x)$ of a space X is called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subset M(x)$. In [1], Latif introduced the notion of semi-convergence of filters and investigated some characterizations related to semi-open continuous functions. Now, we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in \text{SO}(X) : x \in A\}$ and let $S_x = \{A \subset X : \exists \mu \subset S(x) \text{ such that } \mu \text{ is finite and } \cap \mu \subset A\}$. Then, S_x is called the semi-neighborhood filter at x . For any filter F on X , we say that F semi-converges to x if and only if F is finer than the semi-neighborhood filter at x .

2. γ -sets

DEFINITION 2.1. Let (X, τ) be a topological space. A subset U of X is called a γ -set in X if whenever a filter F semi-converges to x and $x \in U$, $U \in F$.

The class of all γ -sets in X will be denoted by $\gamma(X)$. In particular, the class of all γ -sets induced by the topology τ will be denoted by γ_τ .

REMARK 2.2. From the definition of semi-neighborhood filter and γ -set, we can easily say that every semi-open set is a γ -set, but the converse is always not true.

EXAMPLE 2.3. Let X be the real number set with the usual topology. For each $x \in X$, since both $(a, x]$ and $[x, b)$ are semi-open sets containing x , where $a < x < b$, $\{x\}$ is

an element of S_x . For any filter F on X , if F semi-converges to x and since F includes S_x , then x is a γ -set. But it is not semi-open.

REMARK 2.4. In a topological space (X, τ) , it is always true that

$$\tau \subset \alpha(X) \subset \text{SO}(X) \subset \gamma(X). \quad (2.1)$$

THEOREM 2.5. Let (X, τ) be a topological space. The intersection of finitely many semi-open subsets in X is a γ -set.

PROOF. Let U_1 and U_2 be semi-open sets in X . For each $x \in U_1 \cap U_2$, we get $U_1 \cap U_2 \in S_x$. Thus, from the concept of the semi-convergence of filters, whenever every filter F semi-converges to x , $U_1 \cap U_2 \in F$. $\square$

DEFINITION 2.6. Let (X, τ) be a topological space. The γ -interior of a set A in X , denoted by $\text{int}_\gamma(A)$, is the union of all γ -sets contained in A .

THEOREM 2.7. Let (X, τ) be a topological space and $A \subset X$.

- (a) $\text{int}_\gamma(A) = \{x \in A : A \in S_x\}$.
- (b) A is γ -set if and only if $A = \text{int}_\gamma(A)$.

PROOF. (a) For each $x \in \text{int}_\gamma(A)$, there exists a γ -set U such that $x \in U$ and $U \subset A$. From the notion of γ -set, the subset U is in the semi-neighborhood filter S_x . Since S_x is a filter, $A \in S_x$. Conversely, let $x \in A$ and $A \in S_x$, then there exist $U_1 \cdots U_n \in S(x)$ such that $U = U_1 \cap \cdots \cap U_n \subset A$. By Theorem 2.5, U is a γ -set and $U \subset A$. Thus $x \in \text{int}_\gamma(A)$.

- (b) The proof is obvious. $\square$

THEOREM 2.8. Let (X, τ) be a topological space. Then, the class $\gamma(X)$ of all γ -subsets in X is a topology on X .

PROOF. Since $\emptyset$ and X are semi-open, they are also γ -sets in X . Let $A, B \in \gamma(X)$, $x \in A \cap B$, and let F be a filter. Suppose the filter F semi-converges to x . Then $A, B \in F$ and since F is a filter, $(A \cap B) \in F$. Thus, $A \cap B$ is a γ -set.

For each $\alpha \in I$ let $A_\alpha \in \gamma(X)$ and $U = \cup A_\alpha$. For each $x \in U$ and for a filter F semi-converging to x there exists a subset A_α of U such that $x \in A_\alpha$, and since A_α is γ -set, it is obvious that $A_\alpha \in F$. Since F is a filter, U is an element of the filter F and thus $U = \cup A_\alpha$ is a γ -set.

In a topological space (X, τ) , the class of all γ -sets induced by the topology τ will be denoted by (X, γ_τ) . A subset B of X is called a γ -closed set if the complement of B is a γ -set. Thus, the intersection of any family of γ -closed sets is a γ -closed set and the union of finitely many γ -closed sets is a γ -closed set. $\square$

Obviously, we obtain the following theorem by definition of the γ -set.

THEOREM 2.9. Let (X, τ) be a topological space. A set G is γ -closed if and only if whenever F semi-converges to x and $A \in F$, $x \in A$.

DEFINITION 2.10. Let (X, τ) be a topological space and $A \subset X$,

$$\text{cl}_\gamma(A) = \{x \in X : A \cap U \neq \emptyset \ \forall U \in S_x\}. \quad (2.2)$$

We call $\text{cl}_\gamma(A)$ the γ -closure of the set A .

Now we can get the following theorem.

THEOREM 2.11. *Let (X, τ) be a topological space and let A be a subset of X . Then the following properties hold:*

- (1) $A \subset \text{cl}_\gamma(A)$;
- (2) A is γ -closed if and only if $A = \text{cl}_\gamma A$;
- (3) $\text{int}_\gamma(A) = X - \text{cl}_\gamma(X - A)$;
- (4) $\text{cl}_\gamma(A) = X - \text{int}_\gamma(X - A)$.

3. γ -continuous and γ -irresolute functions

DEFINITION 3.1. Let (X, τ) and (Y, μ) be topological spaces. A function $f : X \rightarrow Y$ is called γ -continuous if the inverse image of each open set of Y is a γ -set in X .

Since the class of all γ -sets in a given topological space is also a topology, we get the following equivalent statements.

THEOREM 3.2. *Let (X, τ) and (Y, μ) be topological spaces. If $f : (X, \tau) \rightarrow (Y, \mu)$ is a function, then the following statements are equivalent:*

- (1) f is γ -continuous;
- (2) the inverse image of each closed set in Y is γ -closed;
- (3) $\text{cl}_\gamma(f^{-1}(B)) \subset f^{-1}(\text{cl}(B))$ for every $B \subset Y$;
- (4) $f(\text{cl}_\gamma(A)) \subset \text{cl}(f(A))$ for every $A \subset X$;
- (5) $f^{-1}(\text{int}(B)) \subset \text{int}_\gamma(f^{-1}(B))$ for every $B \subset Y$.

THEOREM 3.3. *Let $f : (X, \tau) \rightarrow (Y, \mu)$ be a function between topological spaces. Then the following statements are equivalent:*

- (1) f is γ -continuous at x ;
- (2) if a filter F semi-converges to x , then $f(F)$ converges to $f(x)$;
- (3) for $x \in X$ and for each neighborhood U of $f(x)$, there is a subset $V \in S_x$ such that $f(V) \subset U$.

PROOF. (1) $\Rightarrow$ (2). Let V be any open neighborhood of $f(x)$ in Y . Then $f^{-1}(V)$ is a γ -set containing x . Thus $f^{-1}(V)$ is an element in S_x . Since F semi-converges to x and $f(F)$ is a filter, $V \in f(F)$. Consequently, $f(F)$ converges to $f(x)$.

(2) $\Rightarrow$ (3). Let U be any γ -neighborhood of $f(x)$. Since always S_x semi-converges to x , from the hypothesis $S_{f(x)} \subset f(S_x)$, and so $U \in f(S_x)$. Thus, there is a subset $V \in S_x$ such that $f(V) \subset U$.

(3) $\Rightarrow$ (1). The proof is obvious. □

We can easily verify the following result.

COROLLARY 3.4. *Let $f : (X, \tau) \rightarrow (Y, \mu)$ be a function. If f is semi-continuous at $x \in X$, then whenever a filter F semi-converges to x in X , $f(F)$ converges to $f(x)$ in Y .*

REMARK 3.5. The following example shows that the converse of [Corollary 3.4](#) may not be true. And we say that every γ -continuous function is semi-continuous.

EXAMPLE 3.6. Let $\mathbb{R}$ be the set of real numbers with the usual topology. We define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 0$, if $x \in Q$ and otherwise, $f(x) = \sqrt{2}$. Clearly, a filter F semi-

converges to x if and only if $\dot{x} \subset F$. Thus $\dot{f}(x) \subset f(F)$ and so $f(F)$ converges to $f(x)$. For an open interval $(-1, 1)$ containing 0, $f^{-1}\{(-1, 1)\} = Q$. Since Q is not semi-open in $\mathbb{R}$, f is not semi-continuous.

DEFINITION 3.7. Let (X, τ) and (Y, μ) be topological spaces. A function $f : X \rightarrow Y$ is called γ -irresolute if the inverse image of each γ set of Y is a γ -set in X .

The following theorems are obtained by [Definition 3.7](#).

THEOREM 3.8. Let $f : (X, \tau) \rightarrow (Y, \mu)$ be a function between topological spaces. Then the following statements are equivalent:

- (1) f is γ -irresolute;
- (2) the inverse image of each γ -closed set in Y is a γ -closed set;
- (3) $\text{cl}_{Y\tau}(f^{-1}(V)) \subset f^{-1}(\text{cl}_{Y\mu}(V))$ for every $V \subset Y$;
- (4) $f(\text{cl}_{Y\tau}(U)) \subset \text{cl}_{Y\mu}(f(U))$ for every $U \subset X$;
- (5) $f^{-1}(\text{int}_{Y\mu}(B)) \subset \text{int}_{Y\tau}(f^{-1}(B))$ for every $B \subset Y$.

THEOREM 3.9. Let $f : (X, \tau) \rightarrow (Y, \mu)$ be a function between topological spaces. Then the following statements are equivalent:

- (1) f is γ -irresolute;
- (2) for $x \in X$ and for each $V \in S_{f(x)}$, there exists an element U in the semi-neighborhood filter S_x such that $f(U) \subset V$;
- (3) for each $x \in X$, if a filter F semi-converges to x , then $f(F)$ semi-converges to $f(x)$ in Y .

PROOF. (1) $\Rightarrow$ (2). The proof is obvious.

(2) $\Rightarrow$ (3). Let V be an element of the semi-neighborhood filter of $S_{f(x)}$ and U be an element of S_x and let F be a filter on X semi-converging to x . Then $f(S_x) \subset f(F)$. Since U is an element in S_x and $f(F)$ is a filter, we can say that $V \in f(F)$. Consequently, $f(F)$ semi-converges to $f(x)$.

(3) $\Rightarrow$ (1). Let V be any γ -set in Y and suppose $f^{-1}(V)$ is not empty. For each $x \in f^{-1}(V)$, since the semi-neighborhood filter S_x semi-converges to x and the hypothesis, clearly, $f(S_x)$ semi-converges to x . And since V is γ -set containing $f(x)$ and $S_{f(x)} \subset f(S_x)$, $V \in f(S_x)$. Now we can take some γ -set U in S_x such that $f(U) \subset V$. Thus, $U \subset f^{-1}(V)$ and since S_x is a filter, so $f^{-1}(V)$ is an element of S_x . And $f^{-1}(V)$ is a γ -set in X from [Theorem 2.7\(b\)](#). $\square$

COROLLARY 3.10. Let $f : (X, \tau) \rightarrow (Y, \mu)$ be a function. If f is irresolute, then whenever a filter F semi-converges to x in X , $f(F)$ semi-converges to $f(x)$ in Y .

REMARK 3.11. We can get the following diagrams:

$$\begin{aligned} \text{continuity} &\Rightarrow \alpha\text{-continuity} \Rightarrow \text{semi-continuity} \Rightarrow \gamma\text{-continuity;} \\ \alpha\text{-irresolute} &\Rightarrow \text{irresolute} \Rightarrow \gamma\text{-irresolute.} \end{aligned} \quad (3.1)$$

DEFINITION 3.12. For two topological spaces (X, τ) and (Y, μ) , a function $f : (X, \tau) \rightarrow (Y, \mu)$ is γ -open if for every open set G in X , $f(G)$ is a γ -set in Y .

THEOREM 3.13. Let $f : (X, \tau) \rightarrow (Y, \mu)$ be a function between topological spaces. Then, f is γ -open if and only if $\text{int}(f^{-1}(B)) \subset f^{-1}(\text{int}_{Y\mu}(B))$, for each $B \subset Y$.

PROOF. Let $B \subset Y$ and $x \in \text{int}(f^{-1}(B))$. Then, $f(\text{int}(f^{-1}(B)))$ is a γ -set containing $f(x)$. Since $f(\text{int}(f^{-1}(B))) \in S_{f(x)}$ and $S_{f(x)}$ is a filter, $B \in S_{f(x)}$. Thus, $f(x) \in \text{int}_{\gamma\mu}(B)$ and so $x \in f^{-1}(\text{int}_{\gamma\mu}(B))$.

Conversely, let A be an open in X and $y \in f(A)$. Then,

$$A \subset \text{int}(f^{-1}f(A)) \subset f^{-1}(\text{int}_{\gamma\mu}(f(A))). \quad (3.2)$$

Let $x \in A$ be such that $f(x) = y$, then $x \in f^{-1}(\text{int}_{\gamma\mu}(f(A)))$. Then, $y \in \text{int}_{\gamma\mu}(f(A))$, and from Theorem 2.7(b) $f(A)$ is a γ -set. $\square$

REMARK 3.14. If any function is semi-open, then it is also γ -open. But the converse may not hold. Consider a function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 0$ for all $x \in \mathbb{R}$, where the real number set $\mathbb{R}$ with the usual topology. Then f is γ -open. For any semi-open set G , $f(G) = \{0\}$ and $\{0\}$ is not semi-open set, thus f is not semi-open.

THEOREM 3.15. Let $f: (X, \tau) \rightarrow (Y, \mu)$ be a function between topological spaces. The function f is γ -open if and only if for each $x \in X$ and for each neighborhood G of x , $f(G)$ is also an element of semi-neighborhood filter $S_{f(x)}$ in Y .

PROOF. Let G be a neighborhood of x , then there exists an open set U such that $x \in U \subset G$. Since f is γ -open, $f(x) \in f(U) = \text{int}_{\gamma\mu}(f(U))$, and so $f(U) \in S_{f(x)}$. Since $S_{f(x)}$ is a filter, $f(G) \in S_{f(x)}$.

Conversely, let $B \subset Y$ and $x \in \text{int}(f^{-1}(B))$, then since $\text{int}(f^{-1}(B))$ is an element of S_x and S_x is a filter, $f^{-1}(B) \in S_x$. By the hypothesis $f(f^{-1}(B)) \in S_{f(x)}$, and since $S_{f(x)}$ is a filter, B is also an element of $S_{f(x)}$. By Definition 2.6, $f(x) \in \text{int}_{\gamma\mu}(B)$ and by Theorem 3.13, the function f is γ -open. $\square$

REMARK 3.16. Now we get the following diagram:

$$\text{open function} \Rightarrow \alpha\text{-open function} \Rightarrow \text{semi-open function} \Rightarrow \gamma\text{-open function.} \quad (3.3)$$

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