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DISCRETIZING A BACKWARD STOCHASTIC DIFFERENTIAL EQUATION

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We show a simple method to discretize Pardoux-Peng's nonlinear backward stochastic differential equation. This discretization scheme also gives a numerical method to solve a class of semi-linear PDEs.

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1. Introduction. Given a probability space (Ω, F, P). Let W_t be a standard Brownian motion with $(F_t) \subset F$ as its natural filtration. Given any positive constant $T < \infty$ and a random variable $\xi \in F_T$. A backward stochastic differential equation is the equation [9]

$$Y_t = \xi - \int_t^T f(Y_s, Z_s, s)\, ds - \int_t^T Z_s\, dW_s, \tag{1.1}$$

where (Y_t, Z_t) are unknown predictable processes. We will assume that f is a Lipschitz function with respect to its arguments throughout this paper. Since this equation has its important applications into control theory and mathematical finance, many mathematicians are not satisfied merely by descriptive existence theorems. They are also interested in constructing the numerical solutions. In order to make real construction, Antonelli [1] solved in short time the coupled forward-backward stochastic differential equations, in which it is assumed that $\xi = g(V_T)$ where $\{V_t\}_t$ is the solution to a forward stochastic differential equation

$$V_t = x - \int_0^t b(V_s, Y_s, Z_s)\, ds - \int_0^t \sigma(V_s, Y_s, Z_s)\, dW_s. \tag{1.2}$$

Later, Hu and Peng [4] and Yong [11] proved the long time existence of the coupled forward-backward stochastic differential equations. Moreover, a four step scheme was suggested by Ma et al. in [7] to solve (1.1) and (1.2) jointly. However, their scheme is related to solving a high dimensional semi-linear partial differential system, which is nontrivial and numerically difficult as noticed by Zhang [12]. Bally [2] has a random time discretization scheme, which requires to approximate integrals of dimension as high as the partition size. Chevance [3] has a scheme under higher regularity condition (C^4). Ma et al. have a quite general numerical method which converge weakly to the true solution [6].

Zhang [12] studied a numerical scheme to solve a coupled forward-backward stochastic differential equations, which converge strongly to the real solution. His approach is still quite different than ours. The main difference is that Zhang's finite difference equation is a little less natural than ours (3.4), but certainly his condition is more general.

The purpose of this paper is to develop the idea appeared in [7] and to give a simpler numerical scheme to solve (1.1) completely. Our method also gives a numerical probability scheme to solve a semi-linear partial differential equation related to a backward stochastic differential equation [10]

$$\frac{\partial}{\partial t}u = \frac{1}{2}\Delta_x u - h(u, u'_x, t), \quad \forall t > \beta \tag{1.3}$$

$$u(\beta, x) = \phi(x), \tag{1.4}$$

where we assume that h is Lipschitz with respect to its arguments, $\phi \in H_{2+\alpha}$ and $\phi(x) \to 0$ $(|x| \to \infty)$. It is well known [5, page 306] that (1.3) has a unique solution $u \in H_{2+\alpha}$ and it is easy to see (e.g., using Feyman-Kac formula) that $u(x,t) \to 0$ $(|x| \to \infty)$.

For simplifying our notation, we only consider the one-dimensional case. However, it is easy to see that our argument is still valid for multidimensional case as well.

2. A PDE approach. It is well known [8] that the solution to (1.1) is stable when we perturb the given value ξ. Given $\xi \in \sigma(W_s,$ for all $s \le T)$ and $\epsilon > 0$, we can always find an integer N and a compact supported smooth function $g(x_1, \dots, x_M)$ with bounded partial derivatives up to the third order such that

$$E\left|g(W_{t_1}, \dots, W_{t_M}) - \xi\right|^2 < \epsilon, \tag{2.1}$$

where $0 < t_1 < \cdots < t_M = T$. We will take $g(W_{t_1}, \dots, W_{t_M})$ as the given value ξ, the error caused by this replacement of the given value is controlled by Ma et al. [8, Chapter 1, Theorem 4.4].

Set now $h(\cdot,\cdot,t) = f(\cdot,\cdot,T-t)$ in (1.3). Taking $\phi(x) = g(y_1, \dots, y_{M-1}, x)$ with $(y_1, \dots, y_{M-1})$ fixed as parameters, we know from the assumptions imposed on ϕ that (1.3) has its solution (see the introduction) on the time interval $(0, T - t_{M-1})$ and it is denoted as $u(t, y_1, \dots, y_{M-1}, x)$. It is easy to check that $u(T - t_{M-1}, y_1, \dots, y_{M-2}, x, x)$, as a function of x, is still in $H_{2+\alpha}$ and $u(T - t_{M-1}, y_1, \dots, y_{M-2}, x, x) \to 0$ $(|x| \to 0)$. Next, we use $\phi(x) = u(T - t_{M-1}, y_1, \dots, y_{M-2}, x, x)$ as the initial value to solve (1.3) on the time interval $(T - t_{M-1}, T - t_{M-2})$ and denote the solution as $u(t, y_1, \dots, y_{M-2}, x)$. The term $u(T - t_{M-1}, y_1, \dots, y_{M-3}, x, x)$ as a function of x is still in $H_{2+\alpha}$ and $u(T - t_{M-1}, y_1, \dots, y_{M-2}, x, x) \to 0$ $(|x| \to 0)$. Iteratedly, we get the final equation

$$\begin{gathered}\frac{\partial}{\partial t}u = \frac{1}{2}\Delta_x u - f(u, u_x, T-t), \quad T - t_1 < t < T, \\ u(T - t_1, x) = u(T - t_1, x, x).\end{gathered} \tag{2.2}$$

Denote

$$\begin{aligned}
&Y_t = u(T-t, W_{t_1}, \dots, W_{t_{M-1}}, W_t) \quad \forall t_{M-1} \le t \le T;\\
&Y_t = u(T-t, W_{t_1}, \dots, W_{t_{M-2}}, W_t) \quad \forall t_{M-2} \le t \le t_{M-1};\\
&\ \vdots\\
&Y_t = u(T-t, W_t) \quad \forall t \le t_1,\\
&Z_t = u'_x(T-t, W_{t_1}, \dots, W_{t_{M-1}}, W_t) \quad \forall t_{N-1} \le t \le T;\\
&Z_t = u'_x(T-t, W_{t_1}, \dots, W_{t_{M-2}}, W_t) \quad \forall t_{N-2} \le t \le t_{N-1};\\
&\ \vdots\\
&Z_t = u'_x(T-t, W_t) \quad \forall t \le t_1.
\end{aligned} \tag{2.3}$$

Then it is easy to see that through Ito's formula (see [13] for details) (Y_t, Z_t) satisfy (1.1).

3. Discretization. In this section, we consider the problem of discretization. From the above discussion, it is sufficient to discuss the case where $\xi = g(W_T)$. The more general case that $\xi = g(W_{t_1}, \dots, W_T)$ follows from the fact that we can patch the pieces corresponding to different intervals (t_i, t_{i+1}) together as in Section 2. Since there is no closed form solution to (1.3), our first goal is to solve it numerically by a discrete probabilistic approach.

Denote by P_t the standard semigroup of Gaussian operators, that is,

$$P_t g(x) = \int \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(x-y)^2}{2t} \right\} g(y)\, dy. \tag{3.1}$$

Given an integer $N > 0$. We first consider the following backward equation on discrete time $\{kT/N\}_{k=0,1,\dots,N}$. Set $X_T^{(N)} = g(W_T)$ and define $X_{kT/N}^{(N)}$ inductively by the following equation:

$$\begin{aligned}
X_{kT/N}(W_{kT/N}) &= P_{T/N} X_{(k+1)T/N}(W_{kT/N})\\
&\quad - \int_{kT/N}^{(k+1)T/N} f\left(X_{kT/N}(W_{kT/N}), \frac{\partial}{\partial x} P_{T/N} X_{(k+1)T/N}(W_{kT/N}), s \right) ds,
\end{aligned} \tag{3.2}$$

where we used the fact that $X_{(k+1)T/N}^{(N)}$ is a function and still denoted as $X_{(k+1)T/N}^{(N)}$ of $W_{(k+1)T/N}$, so $P_t X_{(k+1)T/N}^{(N)}$ is defined as the result of P_t operating on $X_{(k+1)T/N}^{(N)}$. The same type of notation will be carried out through this paper. Since f is Lipschitz, by implicit function theorem, when N is large enough, (3.2) has a unique solution $X_{kT/N}^{(N)}$ as a function of $W_{kT/N}$.

Next, define

$$Q_{kT/N}^{(N)} = \frac{\partial}{\partial x} P_{T/N} X_{(k+1)T/N}^{(N)}(W_{kT/N}). \tag{3.3}$$

Finally, we consider the forward equation

$$\underline{Y}^{(N)}_{(k+1)T/N}-\underline{Y}^{(N)}_{kT/N}=\frac{1}{N}f\left(\underline{Y}^{(N)}_{kT/N},Q^{(N)}_{kT/N},\frac{kT}{N}\right)+Q^{(N)}_{kT/N}(W_{(k+1)T/N}-W_{kT/N}) \tag{3.4}$$

with the initial condition that $\underline{Y}^{(N)}_0=X^{(N)}_0$.

Consider several useful facts. We may also get $X^{(N)}$ through the following equation:

$$X^{(N)}_{kT/N}=X^{(N)}_{(k+1)T/N}-\int_{kT/N}^{(k+1)T/N}f(X^{(N)}_{kT/N},E[Q^{(N)}_s\,|\,W_{kT/N}],s)\,ds-\int_{kT/N}^{(k+1)T/N}Q^{(N)}_s\,dW_s \tag{3.5}$$

with the initial condition $X^{(N)}_T=g(W_T)$. In fact, since $(\partial/\partial s)P_{T-s}=-(1/2)\Delta P_{T-s}$, we easily have, from Ito's formula, that

$$P_{T/N}g(W_{(N-1)T/N})=g(W_T)-\int_{(N-1)T/N}^{T}\frac{\partial}{\partial x}P_{T-s}g(W_s)\,dW_s. \tag{3.6}$$

Define $Q_s=(\partial/\partial x)P_{T-s}g(W_s)$ (for all $s\ge(N-1)T/N$) and take conditional expectation with respect to $W_{(N-1)T/N}$ on both sides of (3.5),

$$\begin{aligned}
&P_{T/N}g(W_{(N-1)T/N})\\
&\quad=X_{(N-1)T/N}+\int_{(N-1)T/N}^{T}f\left(X_{(N-1)T/N},E\left[\frac{\partial}{\partial x}P_{T-s}g(W_s)\,|\,W_{(N-1)T/N}\right],s\right)ds\\
&\quad=X_{(N-1)T/N}+\int_{(N-1)T/N}^{T}f(X_{(N-1)T/N},P_{T/N}g_x,s)(W_{(N-1)T/N})\,ds,
\end{aligned} \tag{3.7}$$

where we used the exchangeability between P_{t-s} and $\partial/\partial x$. Then we get the solution to (3.5) for $k=N-1$. Repeating this procedure, we get the solution to (3.5) for all k by mathematical induction.

It is also easy to check that

$$\begin{aligned}
Q^{(N)}_{kT/N}&=\frac{\partial}{\partial x}P_{T/N}X^{(N)}_{(k+1)T/N}(W_{k/N})\\
&=NE\left[\int_{kT/N}^{(k+1)T/N}\frac{\partial}{\partial x}P_{(k+1)T/N-s}X^{(N)}_{(k+1)T/N}(W_s)\,ds|W_{k/N}\right]\\
&=NE\left[\int_{kT/N}^{(k+1)T/N}Q^{(N)}_s\,ds|W_{k/N}\right].
\end{aligned} \tag{3.8}$$

We will show in Section 5 the rate of convergence of $\underline{X}^{(N)}$ towards Y and in Section 6 the rate of convergence of $Q^{(N)}$ towards Z. Moreover, we will show in Section 6 that Z_s is Hölder continuous. Thus we may compare two forward stochastic differential equations, (3.4) and (3.5), and easily see that $\underline{Y}^{(N)}-X^{(N)}\to 0$. Hence $\underline{Y}^{(N)}\to Y$ and $Q^{(N)}\to Z$.

4. Stability of difference equation. Let $X_{iT/N}$ and $\tilde{X}_{iT/N}$ be two solutions to (3.5), that is,

$$\begin{aligned}
X_{kT/N} &= X_{(k+1)T/N} - \int_{kT/N}^{(k+1)T/N} f\left(X_{kT/N}, E\left[Q_s \mid W_{kT/N}\right], s\right) ds - \int_{kT/N}^{(k+1)T/N} Q_s \, dW_s, \\
\tilde{X}_{kT/N} &= \tilde{X}_{(k+1)T/N} - \int_{kT/N}^{(k+1)T/N} f\left(\tilde{X}_{kT/N}, E\left[\tilde{Q}_s \mid W_{kT/N}\right], s\right) ds - \int_{kT/N}^{(k+1)T/N} \tilde{Q}_s \, dW_s,
\end{aligned} \tag{4.1}$$

with the initial condition that $X_T = g(W_T)$ and $\tilde{X}_T = \tilde{g}(W_T)$. Then we have the following stability result, which is the discretized version of Pardoux-Peng's remarkable result [9].

Theorem 4.1. *Suppose that $|f(x,y,t) - f(\tilde{x},\tilde{y},s)| \le L(|x-\tilde{x}| + |y-\tilde{y}| + |t-s|)$. Then there is a constant C such that when $i+k \le N$,*

$$E\left[\int_{jT/N}^{(j+1)T/N} \left(Q_s - \tilde{Q}_s\right)^2 ds\right] + E\left(X_{iT/N} - \tilde{X}_{iT/N}\right)^2 \le CE\left\{\left(X_{(j+k)T/N} - \tilde{X}_{(j+k)T/N}\right)^2\right\}. \tag{4.2}$$

Proof. We have

$$\begin{aligned}
&\left(X_{(j+1)T/N} - \tilde{X}_{(j+1)T/N}\right) - \left(X_{jT/N} - \tilde{X}_{jT/N}\right) \\
&\quad = \int_{jT/N}^{(j+1)T/N} \left\{f\left(X_{jT/N}, E\left[Q_s \mid W_{jT/N}\right], s\right) - f\left(\tilde{X}_{jT/N}, E\left[\tilde{Q}_s \mid W_{jT/N}\right], s\right)\right\} ds \\
&\qquad + \int_{jT/N}^{(j+1)T/N} \left(Q_s - \tilde{Q}_s\right) dW_s.
\end{aligned} \tag{4.3}$$

The two parts on the right-hand side are orthogonal,

$$\begin{aligned}
&E\left[\left|\left(X_{(j+1)T/N} - \tilde{X}_{(j+1)T/N}\right) - \left(X_{jT/N} - \tilde{X}_{jT/N}\right)\right|^2\right] \\
&\quad = E\left[\left|\int_{jT/N}^{(j+1)T/N} \left\{f\left(X_{jT/N}, E\left[Q_s \mid W_{jT/N}\right], s\right) - f\left(\tilde{X}_{jT/N}, E\left[\tilde{Q}_s \mid W_{jT/N}\right], s\right)\right\} ds\right|^2\right] \\
&\qquad + E\left[\left|\int_{jT/N}^{(j+1)T/N} \left(Q_s - \tilde{Q}_s\right) dW_s\right|^2\right] \\
&\quad \ge E\left[\left|\int_{jT/N}^{(j+1)T/N} \left(Q_s - \tilde{Q}_s\right) dW_s\right|^2\right] \\
&\quad = E\left[\int_{jT/N}^{(j+1)T/N} \left|Q_s - \tilde{Q}_s\right|^2 ds\right].
\end{aligned} \tag{4.4}$$

Since

$$
\begin{aligned}
&(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})^2-(X_{jT/N}-\tilde{X}_{jT/N})^2\\
&\quad=[(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})-(X_{jT/N}-\tilde{X}_{jT/N})]^2\\
&\qquad+2(X_{jT/N}-\tilde{X}_{jT/N})[(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})-(X_{jT/N}-\tilde{X}_{jT/N})]\\
&\quad=[(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})-(X_{jT/N}-\tilde{X}_{jT/N})]^2\\
&\qquad+2(X_{jT/N}-\tilde{X}_{jT/N})\int_{jT/N}^{(j+1)T/N}\{f(X_{jT/N},E[Q_s\,|\,W_{jT/N}],s)\\
&\qquad\qquad-f(\tilde{X}_{jT/N},E[\tilde{Q}_s\,|\,W_{jT/N}],s)\}\,ds\\
&\qquad+2(X_{jT/N}-\tilde{X}_{jT/N})\int_{jT/N}^{(j+1)T/N}(Q_s-\tilde{Q}_s)\,dW_s,
\end{aligned}
\tag{4.5}
$$

and the expectation of the last term vanishes, we get

$$
\begin{aligned}
&E\Big[(X_{jT/N}-\tilde{X}_{jT/N})^2\Big]\\
&\quad=E\Big[(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})^2\Big]-E\Big\{[(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})-(X_{jT/N}-\tilde{X}_{jT/N})]^2\Big\}\\
&\qquad-2E\Big\{(X_{jT/N}-\tilde{X}_{jT/N})\int_{jT/N}^{(j+1)T/N}\{f(X_{jT/N},E[Q_s\,|\,W_{jT/N}],s)\\
&\qquad\qquad-f(\tilde{X}_{jT/N},E[\tilde{Q}_s\,|\,W_{jT/N}],s)\}\,ds\Big\}.
\end{aligned}
\tag{4.6}
$$

By (4.4) and (4.6),

$$
\begin{aligned}
&E(X_{jT/N}-\tilde{X}_{jT/N})^2\\
&\quad\le E(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})^2-E\Big[\int_{jT/N}^{(j+1)T/N}(Q_s-\tilde{Q}_s)^2\,ds\Big]\\
&\qquad+2\int_{jT/N}^{(j+1)T/N}E\{(X_{jT/N}-\tilde{X}_{jT/N})[f(X_{jT/N},E[Q_s\,|\,W_{jT/N}],s)\\
&\qquad\qquad-f(\tilde{X}_{jT/N},E[\tilde{Q}_s\,|\,W_{jT/N}],s)]\}\,ds.
\end{aligned}
\tag{4.7}
$$

However,

$$
\begin{aligned}
&\int_{jT/N}^{(j+1)T/N}E\{(X_{jT/N}-\tilde{X}_{jT/N})[f(X_{jT/N},E[Q_s\,|\,W_{jT/N}],s)-f(\tilde{X}_{jT/N},E[\tilde{Q}_s\,|\,W_{jT/N}],s)]\}\,ds\\
&\quad\le L\int_{jT/N}^{(j+1)T/N}E\Big[(X_{jT/N}-\tilde{X}_{jT/N})^2\Big]\\
&\qquad+L\int_{jT/N}^{(j+1)T/N}E[\,|X_{jT/N}-\tilde{X}_{jT/N}|\,|E[Q_s\,|\,W_{jT/N}]-E[\tilde{Q}_s\,|\,W_{jT/N}]|\,]\,ds\\
&\quad\le\frac{LT}{N}E\Big[\,|X_{jT/N}-\tilde{X}_{jT/N}|^2\Big]\\
&\qquad+L\int_{jT/N}^{(j+1)T/N}E[\,|X_{jT/N}-\tilde{X}_{jT/N}|\,|E[Q_s\,|\,W_{jT/N}]-E[\tilde{Q}_s\,|\,W_{jT/N}]|\,]\,ds.
\end{aligned}
\tag{4.8}
$$

Therefore

$$\begin{aligned}
&E(X_{jT/N}-\tilde{X}_{jT/N})^2\\
&\quad\le E(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})^2-E\left[\int_{jT/N}^{(j+1)T/N}(Q_s-\tilde{Q}_s)^2\,ds\right]+2\frac{LT}{N}E\left[|X_{jT/N}-\tilde{X}_{jT/N}|^2\right]\\
&\qquad+2L\int_{jT/N}^{(j+1)T/N}E\{|X_{jT/N}-\tilde{X}_{jT/N}|\,|E[Q_s\,|\,W_{jT/N}]-E[\tilde{Q}_s\,|\,W_{jT/N}]|\}\,ds.
\end{aligned}\tag{4.9}$$

For $a>0$, $b>0$, and $\lambda>0$ we have $ab=(\lambda a)(b/\lambda)\le(1/2)[(\lambda a)^2+(b/\lambda)^2]$. Therefore (4.9) becomes

$$\begin{aligned}
&E(X_{jT/N}-\tilde{X}_{jT/N})^2\\
&\quad\le E(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})^2-E\left[\int_{jT/N}^{(j+1)T/N}(Q_s-\tilde{Q}_s)^2\,ds\right]\\
&\qquad+2\frac{LT}{N}E\left[|X_{jT/N}-\tilde{X}_{jT/N}|^2\right]+\lambda^2L\int_{jT/N}^{(j+1)T/N}E\left[|X_{jT/N}-\tilde{X}_{jT/N}|^2\right]ds\\
&\qquad+\frac{L}{\lambda^2}\int_{jT/N}^{(j+1)T/N}E\left[|Q_s-\tilde{Q}_s|^2\right]ds.
\end{aligned}\tag{4.10}$$

Taking $\lambda=\sqrt{2L}$, then

$$\begin{aligned}
&E\left[(X_{jT/N}-\tilde{X}_{jT/N})^2\right]+\frac{1}{2}E\left[\int_{jT/N}^{(j+1)T/N}(Q_s-\tilde{Q}_s)^2\,ds\right]\\
&\qquad\le E\left[(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})^2\right]+\frac{\kappa}{N}E\left(|X_{jT/N}-\tilde{X}_{jT/N}|^2\right),
\end{aligned}\tag{4.11}$$

where $\kappa=2LT+2L^2T$. When N is large, $\kappa/N<1$, hence

$$E(X_{jT/N}-\tilde{X}_{jT/N})^2\le\left(1-\frac{\kappa}{N}\right)^{-1}E(X_{(j+1)T/N}-\tilde{X}_{(j+1)T/N})^2.\tag{4.12}$$

Therefore

$$\begin{aligned}
E(X_{jT/N}-\tilde{X}_{jT/N})^2&\le\left(1-\frac{\kappa}{N}\right)^{-k}E(X_{(j+k)T/N}-\tilde{X}_{(j+k)T/N})^2\\
&\le\left(1-\frac{\kappa}{N}\right)^{-N}E(X_{(j+k)T/N}-\tilde{X}_{(j+k)T/N})^2.
\end{aligned}\tag{4.13}$$

Since $(1-\kappa/N)^{-N}\to e^{-\kappa}$, we get the conclusion. □

5. Convergence of $X^{(N)}$ to Y. We are going to estimate the error between $X^{(N)}$ and Y in this section.

Theorem 5.1. *There is a constant C such that*

$$\sup_i E^{1/2}\left[\,|X^{(N)}_{iT/N}-Y_{iT/N}|^2\right]\le CN^{-1/2}. \tag{5.1}$$

Proof. We introduce for each $n\le N$ a discrete time process $\{X^{(n,N)}_{iT/N}\}_{i=0,1,\dots,N}$ as follows: define $X^{(n,N)}_{iT/N}=Y_{iT/N}$ (for all $i\ge n$) and for each $i\le n-1$ we define the process repeatedly by

$$\begin{aligned}X^{(n,N)}_{iT/N}&=X^{(n,N)}_{(i+1)T/N}-\int_{iT/N}^{(i+1)T/N} f\left(X^{(n,N)}_{iT/N},\frac{\partial}{\partial x}P_{T/N}X^{(n,N)}_{(i+1)T/N}(W_{iT/N}),s\right)ds\\&\quad-\int_{iT/N}^{(i+1)T/N}\frac{\partial}{\partial x}P_{(i+1)T/N-s}X^{(n,N)}_{(i+1)T/N}(W_s)\,dW_s.\end{aligned}\tag{5.2}$$

It is easy to see that $X^{(0,N)}_{iT/N}=Y_{iT/N}$ and $X^{(N,N)}_{iT/N}=X^{(N)}_{iT/N}$. The difference $X^{(n,N)}_{nT/N}-X^{(n+1,N)}_{nT/N}$ is given by

$$\begin{aligned}&X^{(n,N)}_{nT/N}-X^{(n+1,N)}_{nT/N}\\&\quad=\int_{nT/N}^{(n+1)T/N}\Big\{f\left(X^{(n+1,N)}_{nT/N},\frac{\partial}{\partial x}P_{T/N}u\left(T-\frac{(n+1)T}{N},\cdot\right)(W_{nT/N}),s\right)\\&\qquad\qquad-f(u(T-s,W_s),u'_x(T-s,W_s),s)\Big\}\,ds\\&\qquad+\int_{nT/N}^{(n+1)T/N}\left\{\frac{\partial}{\partial x}P_{(n+1)T/N-s}X^{(n,N)}_{(n+1)T/N}(W_s)-u'_x(T-s,W_s)\right\}dW_s\\&\quad=\int_{nT/N}^{(n+1)T/N}\Big\{f\left(X^{(n+1,N)}_{nT/N},P_{T/N}u'_x\left(T-\frac{(n+1)T}{N},\cdot\right)(W_{nT/N}),s\right)\\&\qquad\qquad-f(u(T-s,W_s),u'_x(T-s,W_s),s)\Big\}\,ds\\&\qquad+\int_{nT/N}^{(n+1)T/N}\left\{P_{(n+1)T/N-s}u'_x\left(T-\frac{(n+1)T}{N},W_s\right)-u'_x(T-s,W_s)\right\}dW_s,\end{aligned}\tag{5.3}$$

that is,

$$\begin{aligned}&(X^{(n,N)}_{nT/N}-X^{(n+1,N)}_{nT/N})-\int_{nT/N}^{(n+1)T/N}\left\{P_{(n+1)T/N-s}u'_x\left(T-\frac{(n+1)T}{N},W_s\right)-u'_x(T-s,W_s)\right\}dW_s\\&\quad=\int_{nT/N}^{(n+1)T/N}\Big\{f\left(X^{(n+1,N)}_{nT/N},P_{T/N}u'_x\left(T-\frac{(n+1)T}{N},\cdot\right)(W_{nT/N}),s\right)\\&\qquad\qquad-f(u(T-s,W_s),u'_x(T-s,W_s),s)\Big\}\,ds.\end{aligned}\tag{5.4}$$

Since the two terms on the left-hand side of (5.4) are orthogonal, we get

$$
\begin{aligned}
&E\,|\,X^{(n,N)}_{nT/N}-X^{(n+1,N)}_{nT/N}\,|^2\\
&\quad\le E\bigg[\bigg|\int_{nT/N}^{(n+1)T/N}\Big\{f\Big(X^{(n+1,N)}_{nT/N},P_{T/N}u_x\Big(T-\frac{(n+1)T}{N},\cdot\Big)(W_{nT/N}),s\Big)\\
&\qquad\qquad -f(u(T-s,W_s),u_x(T-s,W_s),s)\Big\}\,ds\bigg|^2\bigg]\\
&\quad\le \frac{1}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|f\Big(X^{(n+1,N)}_{nT/N},P_{T/N}u_x\Big(T-\frac{(n+1)T}{N},\cdot\Big)(W_{nT/N}),s\Big)\\
&\qquad\qquad -f(u(T-s,W_s),u_x(T-s,W_s),s)\Big|^2\,ds\bigg]\\
&\quad\le \frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}|\,X^{(n+1,N)}_{nT/N}-u(T-s,W_s)\,|^2\,ds\bigg]\\
&\qquad +\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|P_{T/N}u_x\Big(T-\frac{(n+1)T}{N}\Big)(W_{nT/N})-u_x(T-s,W_s)\Big|^2\,ds\bigg]\\
&\quad\le \frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|X^{(n+1,N)}_{nT/N}-X^{(n,N)}_{nT/N}\Big|^2\,ds\bigg]\\
&\qquad +\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}|\,X^{(n,N)}_{nT/N}-u(T-s,W_s)\,|^2\,ds\bigg]\\
&\qquad +\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|P_{T/N}u_x\Big(T-\frac{(n+1)T}{N}\Big)(W_{nT/N})-u_x(T-s,W_s)\Big|^2\,ds\bigg],
\end{aligned}
\tag{5.5}
$$

where we used Lipschitz condition on f. We use C to denote an arbitrary constant, whose value may change according to the context but not depending on the given variables.

We estimate the last two terms separately

$$
\begin{aligned}
&\frac{1}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}|\,X^{(n,N)}_{nT/N}-u(T-s,W_s)\,|^2\,ds\bigg]\\
&=\frac{1}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|u\Big(T-\frac{nT}{N},W_{nT/N}\Big)-u(T-s,W_s)\Big|^2\,ds\bigg]\\
&\le\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|u\Big(T-\frac{nT}{N},W_s\Big)-u(T-s,W_s)\Big|^2\,ds\bigg]\\
&+\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|u\Big(T-\frac{nT}{N},W_{nT/N}\Big)-u\Big(T-\frac{nT}{N},W_s\Big)\Big|^2\,ds\bigg]\\
&\le\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|\frac{1}{N}\Big|^2\,ds\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|u\Big(T-\frac{nT}{N},W_{nT/N}\Big)-u\Big(T-\frac{nT}{N},W_s\Big)\Big|^2ds\bigg]\\
&=\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|\frac{1}{N}\Big|^2ds\bigg]\\
&+\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|\frac{1}{2}\int_{nT/N}^{s}\Delta u\Big(T-\frac{nT}{N},W_t\Big)dt+\int_{nT/N}^{s}u'_x\Big(T-\frac{nT}{N},W_t\Big)dW_t\Big|^2ds\bigg]\\
&\le\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|\frac{1}{N}\Big|^2ds\bigg]+\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|\int_{nT/N}^{s}\Delta u\Big(T-\frac{nT}{N},W_t\Big)dt\Big|^2ds\bigg]\\
&+\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|\int_{nT/N}^{s}u'_x\Big(T-\frac{nT}{N},W_t\Big)dW_t\Big|^2ds\bigg]\\
&=\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|\frac{1}{N}\Big|^2ds\bigg]+\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|\int_{nT/N}^{s}\Delta u\Big(T-\frac{nT}{N},W_t\Big)dt\Big|^2ds\bigg]\\
&+\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\int_{nT/N}^{s}\Big|u'_x\Big(T-\frac{nT}{N},W_t\Big)\Big|^2dt\,ds\bigg]\\
&=O(N^{-3}).
\end{aligned}
\tag{5.6}
$$

In the deduction (5.2), (5.3), (5.4), (5.5), and (5.6), we used repeatedly the fact that $u\in H_{2+\alpha}$ so that u and its derivatives up to the second order are bounded.

For the last term of (5.5), we have

$$
\begin{aligned}
&\frac{1}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|P_{T/N}u'_x\Big(T-\frac{(n+1)T}{N}\Big)(W_{nT/N})-u'_x(T-s,W_s)\Big|^2ds\bigg]\\
&\quad\le\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|P_{T/N}u'_x\Big(T-\frac{(n+1)T}{N}\Big)(W_{nT/N})\\
&\qquad\qquad-u'_x\Big(T-\frac{(n+1)T}{N},W_{nT/N}\Big)\Big|^2ds\bigg]\\
&\quad+\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|u'_x\Big(T-\frac{(n+1)T}{N},W_{nT/N}\Big)-u'_x(T-s,W_s)\Big|^2ds\bigg].
\end{aligned}
\tag{5.7}
$$

Since u''_{xx} is bounded, it is easy to see from the property of Gaussian kernel that $|P_{T/N}u'_x(T-(n+1)T/N,x)-u'_x(T-(n+1)T/N,x)|^2$ converge to 0 uniformly in the order of $1/N$ when $N\to\infty$. Therefore

$$
\begin{aligned}
&\frac{1}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|P_{T/N}u'_x\Big(T-\frac{(n+1)T}{N}\Big)(W_{nT/N})-u'_x(T-s,W_s)\Big|^2ds\bigg]\\
&\quad=O(N^{-3})+\frac{C}{N}E\bigg[\int_{nT/N}^{(n+1)T/N}\Big|u'_x\Big(T-\frac{(n+1)T}{N},W_{nT/N}\Big)-u'_x(T-s,W_s)\Big|^2ds\bigg]
\end{aligned}
$$

$$\begin{aligned}
&\le O(N^{-3}) + \frac{C}{N}E\left[\int_{nT/N}^{(n+1)T/N} |W_{nT/N} - W_s|^2\,ds\right] \\
&= O(N^{-3}) + \frac{C}{N}\int_{nT/N}^{(n+1)T/N} \left|\frac{nT}{N} - s\right| ds \\
&\le O(N^{-3}),
\end{aligned} \tag{5.8}$$

where we used the fact that u'_x is Hölder continuous in t with exponential greater than $1/2$ and Lipschitz in x (see [5, page 46] for the definition of $H_{2+\alpha}$).

Finally, by (5.5), (5.6), and (5.8), we get

$$E^{1/2}\left[|X_{nT/N}^{(n,N)} - X_{nT/N}^{(n+1,N)}|^2\right] = O(N^{-3/2}). \tag{5.9}$$

Applying Theorem 4.1, we deduce that

$$\sup_i E^{1/2}\left[|X_{iT/N}^{(n,N)} - X_{iT/N}^{(n+1,N)}|^2\right] = O(N^{-3/2}). \tag{5.10}$$

It is easy to check that the last term has a bound which is independent on n. Summing up the above inequalities over n, we deduce that

$$\sup_i E^{1/2}\left[|X_{iT/N}^{(N)} - Y_{iT/N}|^2\right] \le \sum_n \sup_i E^{1/2}\left[|X_{iT/N}^{(n,N)} - X_{iT/N}^{(n+1,N)}|^2\right] = O(N^{-1/2}). \tag{5.11}$$

□

6. Convergence of $Q^{(N)}$ to Z. We can also prove the convergence of $Q^{(N)}$ to Z as follows: according to the discussion in Section 2, we may assume that $Z_t = u_x(T - t, W_t)$, where $u \in H_{2+\alpha}$, for simplicity. Since u_x is Hölder continuous with exponent greater than $1/2$ in t and with bounded derivative in x, it is easy to see that Z_t is Hölder continuous with exponent greater than $1/2$ in $L_2(\Omega)$.

For any pair $0 \le j/N < k/N \le T$,

$$\begin{aligned}
&\sum_{j\le i<k}\int_{iT/N}^{(i+1)T/N} E|Q_s^{(N)} - Z_s|^2\,ds \\
&\quad = E\left[\left(\int_{jT/N}^{kT/N} (Q_s^{(N)} - Z_s)\,dW_s\right)^2\right] \\
&\quad \le 2E|Y_{kT/N} - Y_{jT/N} - X_{kT/N}^{(N)} + X_{jT/N}^{(N)}|^2 \\
&\qquad + 2E\left|\sum_{j\le i<k}\int_{iT/N}^{(i+1)T/N} (f(Y_s, Z_s, s) - f(X_{iT/N}^{(N)}, Q_{iT/N}^{(N)}, s))\,ds\right|^2
\end{aligned}$$

$$\begin{aligned}
&\le 2E|Y_{kT/N} - Y_{jT/N} - X^{(N)}_{kT/N} + X^{(N)}_{jT/N}|^2 \\
&\quad + \frac{2(k-j)T}{N} E\left[\sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} (f(Y_s, Z_s, s) - f(X^{(N)}_{iT/N}, Q^{(N)}_{iT/N}, s))^2\, ds\right] \\
&\le 2E|Y_{kT/N} - Y_{jT/N} - X^{(N)}_{kT/N} + X^{(N)}_{jT/N}|^2 \\
&\quad + \frac{C(k-j)T}{N} E\left[\sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} (Y_s - X^{(N)}_{iT/N})^2\, ds\right] \\
&\quad + \frac{C(k-j)T}{N} E\left[\sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} (Z_s - Q^{(N)}_{iT/N})^2\, ds\right].
\end{aligned} \tag{6.1}$$

Therefore

$$\begin{aligned}
&\sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Q^{(N)}_s - Z_s|^2\, ds - \frac{C(k-j)T}{N} \sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Q^{(N)}_{iT/N} - Z_s|^2\, ds \\
&\quad \le 2E|Y_{kT/N} - Y_{jT/N} - X^{(N)}_{kT/N} + X^{(N)}_{jT/N}|^2 \\
&\qquad + \frac{C(k-j)T}{N} E\left[\sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} (Y_s - X^{(N)}_{iT/N})^2\, ds\right] \\
&\quad = O(N^{-1/2}).
\end{aligned} \tag{6.2}$$

However,

$$\begin{aligned}
&\sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Q^{(N)}_{iT/N} - Z_s|^2\, ds \\
&= \sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Q^{(N)}_{iT/N} - E[Z_s | W_{iT/N}]|^2\, ds + \sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Z_s - E[Z_s | W_{iT/N}]|^2\, ds \\
&= \sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|E[Q^{(N)}_s - Z_s | W_{iT/N}]|^2\, ds \\
&+ \sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Z_s - E[Z_s | W_{iT/N}]|^2\, ds \\
&\le \sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Q^{(N)}_s - Z_s|^2\, ds + 2\sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Z_s - Z_{iT/N}|^2\, ds \\
&+ \sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Z_{iT/N} - E[Z_s | W_{iT/N}]|^2\, ds \\
&= \sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E|Q^{(N)}_s - Z_s|^2\, ds + O(N^{-1/2}),
\end{aligned} \tag{6.3}$$

where we used the fact that $Q^{(N)}_{iT/N} = E[Q^{(N)}_s | W_{iT/N}]$ when s is greater than iT/N and the Hölder L_2-continuity of Z_s.

Thus when $(k-j)T/N$ is less than $1/2C$, we deduce from (6.2) and (6.3) that

$$\sum_{j\le i<k} \int_{iT/N}^{(i+1)T/N} E\left|Q^{(N)}_{iT/N} - Z_s\right|^2 ds = O\left(N^{-1/2}\right). \tag{6.4}$$

If we decompose $[0,T]$ into sub-intervals with length less than $1/2C$, quoting (6.4) and summing up, we get the following theorem.

Theorem 6.1. *There is a constant C such that*

$$\sum_{(i;\, 0\le iT/N\le T)} \int_{iT/N}^{(i+1)T/N} E\left|Q^{(N)}_{iT/N} - Z_s\right|^2 ds \le CN^{-1/2}. \tag{6.5}$$

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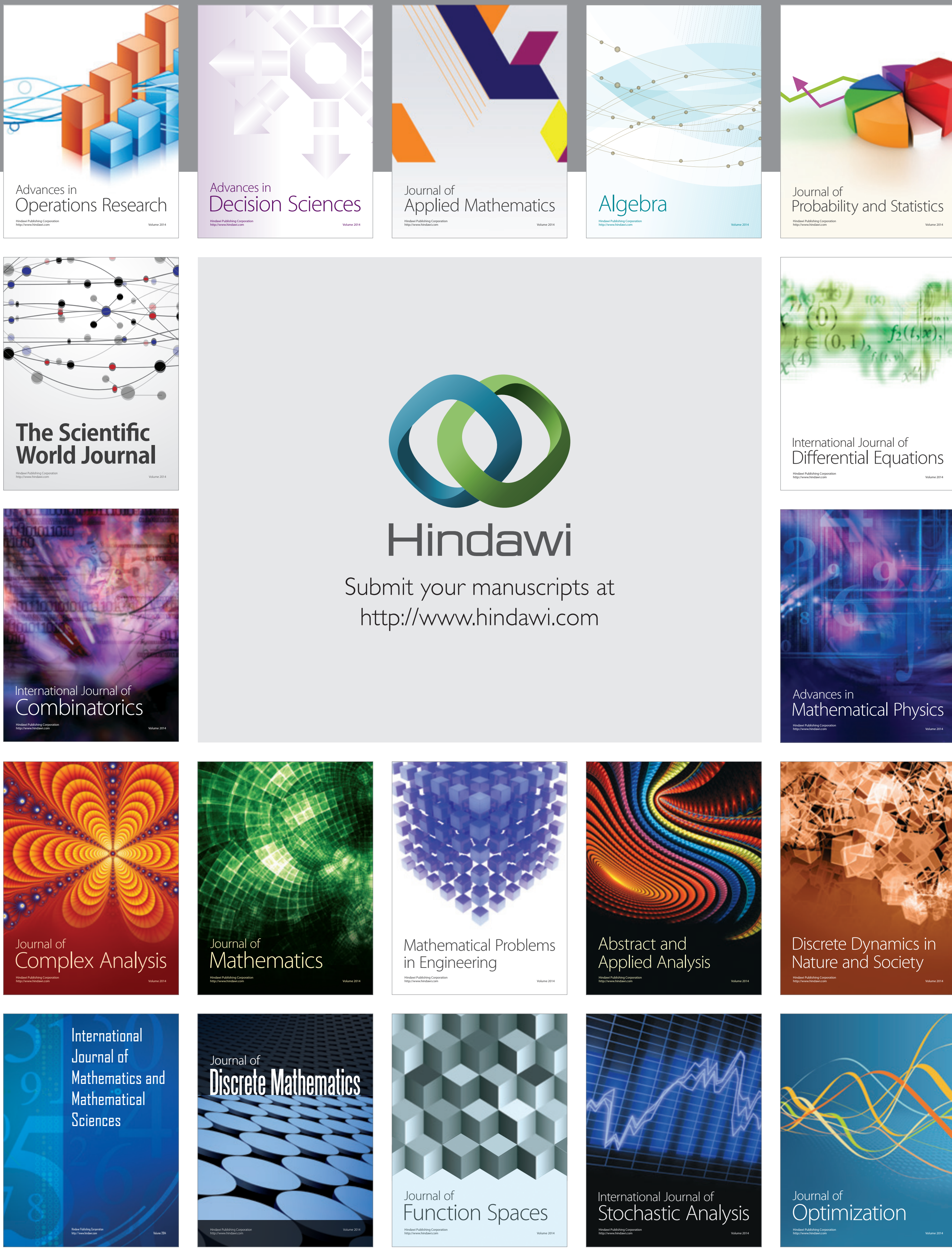
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