EQUIVALENCE RESULTS FOR DISCRETE ABEL MEANS

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We present theorems showing when the discrete Abel mean and the Abel summability method are equivalent for bounded sequences and when two discrete Abel means are equivalent for bounded sequences.

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1. Introduction and notation. The well-known Abel summability method is a sequence-to-function transformation which is defined as follows: for a sequence $s := \{s_n\}$ of complex numbers, define

$$f(x) := (1-x) \sum_{k=0}^{\infty} s_k x^k, \quad (1.1)$$

for all x for which the series converges. If $f(x)$ exists for each $x \in (0,1)$ and $\lim_{x \rightarrow 1^-} f(x) = L$, then the sequence s is Abel summable to L . The discrete Abel mean is a sequence-to-sequence transformation given by the summability matrix A_λ whose nk th entry is

$$A_\lambda[n, k] := \frac{1}{\lambda(n)} \left(1 - \frac{1}{\lambda(n)}\right)^k, \quad n, k = 0, 1, 2, 3, \dots, \quad (1.2)$$

where $\lambda := \{\lambda(n)\}$ is a strictly increasing sequence of real numbers such that $\lambda(0) \geq 1$ and $\lambda(n) \rightarrow \infty$. Then the sequence s is A_λ -summable to L provided that

$$\lim_{n \rightarrow \infty} (A_\lambda s)_n = \lim_{n \rightarrow \infty} \frac{1}{\lambda(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{\lambda(n)}\right)^k = L. \quad (1.3)$$

In [1], Armitage and Maddox proved inclusion and Tauberian theorems for the discrete Abel mean. In this paper, we expand upon the work of these authors by examining equivalence properties of the A_λ method for bounded sequences.

For a given sequence s , define a sequence a by $a_0 := s_0$ and $a_n := s_n - s_{n-1}$ for $n \geq 1$. Then, $s_n = \sum_{k=0}^n a_k$ and for every n ,

$$(A_\lambda s)_n = \frac{1}{\lambda(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{\lambda(n)}\right)^k = \sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{\lambda(n)}\right)^k. \quad (1.4)$$

Also, define the sequence t by

$$t_n := \sum_{k=1}^n k a_k. \quad (1.5)$$

A straightforward induction argument yields

$$t_n = \sum_{k=0}^n (s_n - s_k). \quad (1.6)$$

If B and C are two summability methods, then C includes B , denoted $B \subset C$, provided that every sequence which is B -summable is also C -summable to the same limit. If $B \subset C$ and $C \subset B$, then B and C are equivalent, denoted $B \sim C$.

2. Equivalence results. For any sequence λ , A_λ is clearly a regular (i.e., limit preserving) method. In [1], Armitage and Maddox proved the following inclusion results for the A_λ method.

THEOREM 2.1 (see [1]). *Let $E(\lambda) := \{\lambda(n) : n = 0, 1, 2, \dots\}$ and $E(\mu) := \{\mu(n) : n = 0, 1, 2, \dots\}$. Then*

- (1) $A_\lambda \subset A_\mu$ *if and only if* $E(\mu) \setminus E(\lambda)$ *is a finite set;*
- (2) $A_\mu \sim A_\lambda$ *if and only if the symmetric difference* $E(\lambda) \triangle E(\mu)$ *is a finite set.*

COROLLARY 2.2 (see [1]). *For every λ , A_λ strictly includes the Abel method.*

The main result of this section is that A_λ is equivalent to the Abel method for bounded sequences provided that $\lambda(n+1)/\lambda(n) \rightarrow 1$. To show this we need the following two lemmas.

LEMMA 2.3 (see [1]). *If $\sum_{k=0}^{\infty} a_k x^k$ converges for all $x \in (0, 1)$, then*

$$\sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} t_k \Delta\left(\frac{x^k}{k}\right), \quad 0 < x < 1, \quad (2.1)$$

where $\Delta(x^k/k) = x^k/k - x^{k+1}/(k+1)$.

LEMMA 2.4. *If s is a bounded sequence, then $t_n = O(n)$.*

PROOF. Let s be a bounded sequence. By (1.6),

$$\begin{aligned} |t_n| &= \left| \sum_{k=0}^n (s_n - s_k) \right| = \left| (n+1)s_n - \sum_{k=0}^n s_k \right| \\ &\leq (n+1)\|s\|_{\infty} + \sum_{k=0}^n |s_k| \\ &\leq (n+1)\|s\|_{\infty} + (n+1)\|s\|_{\infty} \\ &= O(n). \end{aligned} \quad (2.2)$$

□

THEOREM 2.5. *If $\lim_{n \rightarrow \infty} (\lambda(n+1)/\lambda(n)) = 1$, then A_λ is equivalent to the Abel method for bounded sequences.*

PROOF. By Corollary 2.2, A_λ includes the Abel method. So assume that

$$\lim_{n \rightarrow \infty} \left(\frac{\lambda(n+1)}{\lambda(n)} \right) = 1, \quad (2.3)$$

let s be a bounded sequence, that is, A_λ -summable to L , and let a be the sequence such that $s_n = \sum_{k=0}^n a_k$. Let $x_n := 1 - 1/\lambda(n)$. Then, for a given $x \in (x_0, 1)$, there exists an n such that $x_n < x \leq x_{n+1}$. By (1.1) and (1.4),

$$\begin{aligned} |f(x) - (A_\lambda s)_n| &= \left| (1-x) \sum_{k=0}^{\infty} s_k x^k - \frac{1}{\lambda(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{\lambda(n)}\right)^k \right| \\ &= \left| \sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{\infty} a_k x_n^k \right|. \end{aligned} \quad (2.4)$$

By Lemma 2.3, this becomes

$$\begin{aligned} |f(x) - (A_\lambda s)_n| &= \left| \sum_{k=1}^{\infty} t_k \Delta \left(\frac{x^k}{k} \right) - \sum_{k=1}^{\infty} t_k \Delta \left(\frac{x_n^k}{k} \right) \right| \\ &= \left| \sum_{k=1}^{\infty} t_k \int_{x_n}^x t^{k-1} (1-t) dt \right| \\ &\leq \sum_{k=1}^{\infty} |t_k| \int_{x_n}^{x_{n+1}} t^{k-1} (1-t) dt. \end{aligned} \quad (2.5)$$

By Lemma 2.4, there exists an $M > 0$ such that $|t_k| \leq kM$. Hence,

$$\begin{aligned} |f(x) - (A_\lambda s)_n| &\leq M \sum_{k=1}^{\infty} k \int_{x_n}^{x_{n+1}} t^{k-1} (1-t) dt \\ &= M \int_{x_n}^{x_{n+1}} (1-t) \sum_{k=1}^{\infty} k t^{k-1} dt \\ &= M \int_{x_n}^{x_{n+1}} \frac{1}{1-t} dt \\ &= -M (\log(1-x_{n+1}) - \log(1-x_n)) \\ &= -M \left(\log \left(\frac{1}{\lambda(n+1)} \right) - \log \left(\frac{1}{\lambda(n)} \right) \right) \\ &= M \log \left(\frac{\lambda(n+1)}{\lambda(n)} \right) \\ &= o(1). \end{aligned} \quad (2.6)$$

Since s is A_λ -summable to L , we see that $\lim_{x \rightarrow 1^-} f(x) = L$. That is, s is Abel summable to L , and hence, A_λ is equivalent to the Abel method for bounded sequences. $\square$

The next theorem presents an equivalence relationship between the discrete Abel means when λ and μ are asymptotic.

THEOREM 2.6. *Let λ and μ be strictly increasing sequences of real numbers such that $\lambda(0) \geq 1$, $\mu(0) \geq 1$, $\lambda(n) \rightarrow \infty$, $\mu(n) \rightarrow \infty$, and $\lim_{n \rightarrow \infty} (\mu(n)/\lambda(n)) = 1$. Then A_λ is equivalent to A_μ for bounded sequences.*

PROOF. We proceed as in the proof of [Theorem 2.5](#). Let s be a bounded sequence and let a be the sequence such that $s_n = \sum_{k=0}^n a_k$. Let $M(n) := \max\{\lambda(n), \mu(n)\}$, $m(n) := \min\{\lambda(n), \mu(n)\}$, $x_n := 1 - 1/m(n)$, and $y_n := 1 - 1/M(n)$. Then $0 \leq x_n \leq y_n < 1$ and for a given n ,

$$\begin{aligned} |(A_\mu s)_n - (A_\lambda s)_n| &= \left| \frac{1}{\mu(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{\mu(n)}\right)^k - \frac{1}{\lambda(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{\lambda(n)}\right)^k \right| \\ &= \left| \frac{1}{M(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{M(n)}\right)^k - \frac{1}{m(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{m(n)}\right)^k \right| \quad (2.7) \\ &= \left| \sum_{k=0}^{\infty} a_k y_n^k - \sum_{k=0}^{\infty} a_k x_n^k \right|. \end{aligned}$$

By [Lemma 2.3](#),

$$\begin{aligned} |(A_\mu s)_n - (A_\lambda s)_n| &= \left| \sum_{k=1}^{\infty} t_k \Delta\left(\frac{y_n^k}{k}\right) - \sum_{k=1}^{\infty} t_k \Delta\left(\frac{x_n^k}{k}\right) \right| \\ &= \left| \sum_{k=1}^{\infty} t_k \int_{x_n}^{y_n} t^{k-1} (1-t) dt \right| \quad (2.8) \\ &\leq \sum_{k=1}^{\infty} |t_k| \int_{x_n}^{y_n} t^{k-1} (1-t) dt. \end{aligned}$$

By [Lemma 2.4](#), there exists an $M > 0$ such that $|t_k| \leq kM$. Hence,

$$\begin{aligned} |(A_\mu s)_n - (A_\lambda s)_n| &\leq M \sum_{k=1}^{\infty} k \int_{x_n}^{y_n} t^{k-1} (1-t) dt \\ &= M \int_{x_n}^{y_n} (1-t) \sum_{k=1}^{\infty} k t^{k-1} dt \\ &= M \int_{x_n}^{y_n} \frac{1}{1-t} dt \quad (2.9) \\ &= -M(\log(1-y_n) - \log(1-x_n)) \\ &= -M\left(\log\left(\frac{1}{M(n)}\right) - \log\left(\frac{1}{m(n)}\right)\right) \\ &= M \log\left(\frac{M(n)}{m(n)}\right) \\ &= o(1), \end{aligned}$$

since $\lim_{n \rightarrow \infty} (M(n)/m(n)) = \lim_{n \rightarrow \infty} (\mu(n)/\lambda(n)) = 1$. Hence, if s is A_λ -summable to L , then

$$0 \leq |(A_\mu s)_n - L| \leq |(A_\mu s)_n - (A_\lambda s)_n| + |(A_\lambda s)_n - L| = o(1) + o(1) = o(1). \quad (2.10)$$

Similarly, if s is A_μ -summable to L , then

$$0 \leq |(A_\lambda s)_n - L| \leq |(A_\lambda s)_n - (A_\mu s)_n| + |(A_\mu s)_n - L| = o(1) + o(1) = o(1). \quad (2.11)$$

Thus, A_λ and A_μ are equivalent for bounded sequences. $\square$

To see that $\lim_{n \rightarrow \infty} (\mu(n)/\lambda(n)) = 1$ is not a necessary condition in [Theorem 2.6](#), simply consider the sequences $\lambda(n) := n^2$ and $\mu(n) := n^3$. Then

$$\lim_{n \rightarrow \infty} \frac{\lambda(n+1)}{\lambda(n)} = \lim_{n \rightarrow \infty} \frac{\mu(n+1)}{\mu(n)} = 1, \quad (2.12)$$

and hence, by [Theorem 2.5](#), A_λ , A_μ , and the Abel method are all equivalent for bounded sequences. However, λ and μ are not asymptotic.

REFERENCES

- [1] D. H. Armitage and I. J. Maddox, *Discrete Abel means*, *Analysis* **10** (1990), no. 2-3, 177-186.

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