

A CHARACTERIZATION OF HARMONIC FOLIATIONS BY THE VOLUME PRESERVING PROPERTY OF THE NORMAL GEODESIC FLOW

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We prove that a Riemannian foliation with the flat normal connection on a Riemannian manifold is harmonic if and only if the geodesic flow on the normal bundle preserves the Riemannian volume form of the canonical metric defined by the adapted connection.

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1. Introduction. Let (M, g_M) be a Riemannian manifold. A foliation \mathcal{F} on M is *Riemannian* and g_M *bundle-like* if all the leaves are locally equi-distant to each other. Such a foliation is characterized by the property that a geodesic orthogonal to the foliation at one point is orthogonal everywhere. For a Riemannian foliation, considerable efforts have been made to give global characterizations of the property that it is harmonic, that is, all of its leaves are minimal submanifolds. For examples, a Riemannian foliation is harmonic if and only if either one of the following conditions holds: (1) it is an extremal of the energy functional for special variations (see [2]); (2) it is an extremal of the energy of the foliation under certain variations of the Riemannian metric of the manifold (see [1]). In this paper, we give a dynamical characterization of the harmonicity of a Riemannian foliation which has the flat normal connection in the sense of Oshikiri [4].

Let \mathcal{F} be a Riemannian foliation of dimension p and codimension q on a Riemannian manifold M of dimension n ($p + q = n$) with bundle-like metric g_M . Throughout, we work in the smooth category and the following notations are used:

- TM is the tangent bundle of M .
- L and L^\perp are the tangent bundle and the normal bundle of \mathcal{F} , respectively.
- ΓTM , ΓL , and ΓL^\perp are the spaces of sections of TM , L , and L^\perp , respectively.
- $\pi : TM \rightarrow L^\perp$, $\pi^\perp : TM \rightarrow L$, and $P_{\mathcal{F}} : L^\perp \rightarrow M$ are the canonical projections.
- ∇^M is the Levi-Civita connection associated with g_M .

Since \mathcal{F} is Riemannian, there exists a unique torsion-free metric connection ∇ on L^\perp which is called *adapted* and given as follows (see [2]): for $Z \in \Gamma L^\perp$,

$$\nabla_X Z = \begin{cases} \pi[X, Z] & \text{for } X \in \Gamma L, \\ \pi(\nabla_X^M Z) & \text{for } X \in \Gamma L^\perp. \end{cases} \quad (1.1)$$

Associated with the above connection there is a bundle map $C_{\mathcal{F}} : TL^\perp \rightarrow L^\perp$ called the

connection map associated with \mathcal{F} given as follows. For $\xi \in T_Z L^\perp$ with $(dP_{\mathcal{F}})(\xi) \neq 0$,

$$C_{\mathcal{F}}(\xi) = \nabla_{\sigma(0)} Z, \tag{1.2}$$

where Z is a curve in L^\perp such that $d/dt|_{t=0} Z = \xi$ and $\sigma(t) = P_{\mathcal{F}}(Z(t))$. This map gives a metric \tilde{g} on L^\perp defined by

$$\tilde{g}(\xi, \eta) = g_M((dP_{\mathcal{F}})_Z(\xi), (dP_{\mathcal{F}})_Z(\eta)) + g_M(C_{\mathcal{F}}(\xi), C_{\mathcal{F}}(\eta)) \tag{1.3}$$

for $\xi, \eta \in T_Z L^\perp$. We denote the Riemannian volume form on L^\perp associated with \tilde{g} by $\tilde{\mu}$.

We define a local flow ϕ_t on L^\perp , called the *normal geodesic flow* of \mathcal{F} as follows. For $z \in L^\perp$, let γ be a geodesic with initial velocity z . Since \mathcal{F} is Riemannian, $\dot{\gamma}(t) \in L^\perp$ for each t in the domain of γ . We put $\phi_t(z) = \dot{\gamma}(t)$ for $z \in L^\perp$ and t in the domain of γ .

A foliation \mathcal{F} is said to *have the flat normal connection* if the normal bundle L^\perp of \mathcal{F} admits an orthonormal frame field $\{E_{p+1}, \dots, E_n\}$ such that $g_M(\nabla_Z^M E_\alpha, E_\beta) = 0$ for all $\alpha, \beta = p + 1, \dots, n$ and all $Z \in \Gamma L^\perp$.

The purpose of this paper is to prove the following theorem.

THEOREM 1.1. *Let \mathcal{F} be a Riemannian foliation on a Riemannian manifold which has a flat normal connection and $\tilde{\mu}$ the Riemannian volume form on L^\perp corresponding to \tilde{g} . Then \mathcal{F} is harmonic if and only if (ϕ_t) preserves $\tilde{\mu}$.*

2. The proof. Let ζ be a vector field on L^\perp generated by the geodesic flow. It suffices to show that \mathcal{F} is harmonic if and only if $(\Theta_\zeta \tilde{\mu})(z) = 0$ at any given point $z \in L^\perp$, where Θ_ζ denotes the Lie derivative. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space of M at the point $m = P_{\mathcal{F}}(z)$ such that $e_i \in L_m$ for $i = 1, \dots, p$ and $e_\alpha \in L_m^\perp$ for $\alpha = p + 1, \dots, n$. In a neighborhood of m , we may choose a frame $\{E_\alpha : \alpha = p + 1, \dots, n\}$ of L^\perp , called an *adapted frame*, satisfying the following properties: $E_\alpha(m) = e_\alpha$, $\alpha = p + 1, \dots, n$, $\nabla_{e_\alpha} E_\beta = \pi(\nabla_{e_\alpha}^M E_\beta) = 0$ and $\nabla_X E_\alpha = \pi([X, E_\alpha]) = 0$ for any smooth section X of L on U (see [3]). Since \mathcal{F} has the flat normal connection, we may choose E_α so that $\nabla_{E_\alpha} E_\beta = 0$ for $\alpha, \beta = p + 1, \dots, n$. Completing this frame by an orthonormal frame $\{E_i : i = 1, \dots, p\}$ of L with $E_i(m) = e_i$, we get a local orthonormal frame $\{E_1, \dots, E_n\}$ of TM on a neighborhood U of m with $E_A(m) = e_A$ for $A = 1, \dots, n$. Let E_A^H for $A = 1, \dots, n$ be the *horizontal lift* of E_A to TL^\perp , that is, the unique vector field on a neighborhood of z in L^\perp such that $dP_{\mathcal{F}}(E_A^H) = E_A$ and $C_{\mathcal{F}}(E_A^H) = 0$, and E_α^V for $\alpha = p + 1, \dots, n$ the *vertical lift* of E_α on TL^\perp , that is, the vector field on a neighborhood of z such that $dP(E_\alpha^V) = 0$ and $C_{\mathcal{F}}(E_\alpha^V) = E_\alpha$. We put $E_A^H(z) = e_A^H$ and $E_\alpha^V(z) = e_\alpha^V$. Now we compute

$$\begin{aligned} & [(\Theta_\zeta \tilde{\mu})(z)](e_1^H, \dots, e_n^H, e_{p+1}^V, \dots, e_n^V) \\ &= - \sum_{i=1}^p \tilde{\mu}(e_1^H, \dots, [\zeta, E_i^H](z), \dots, e_p^H, e_{p+1}^H, \dots, e_n^H, e_{p+1}^V, \dots, e_n^V) \\ &\quad - \sum_{\alpha=p+1}^n \tilde{\mu}(e_1^H, \dots, e_p^H, e_{p+1}^H, \dots, [\zeta, E_\alpha^H](z), \dots, e_n^H, e_{p+1}^V, \dots, e_n^V) \\ &\quad - \sum_{\alpha=p+1}^n \tilde{\mu}(e_1^H, \dots, e_n^H, e_{p+1}^V, \dots, [\zeta, E_\alpha^V](z), \dots, e_n^V). \end{aligned} \tag{2.1}$$

But,

$$\begin{aligned}
 \tilde{\mu}(e_1^H, \dots, [\zeta, E_i^H](z), \dots, e_p^H, e_{p+1}^H, \dots, e_n^H, e_{p+1}^V, \dots, e_n^V) \\
 = \tilde{g}([\zeta, E_i^H](z), e_i^H) = g_M((dP_{\mathcal{F}})[\zeta, E_i^H](m), e_i), \\
 \tilde{\mu}(e_1^H, \dots, e_p^H, e_{p+1}^H, \dots, [\zeta, E_\alpha^H](z), \dots, e_n^H, e_{p+1}^V, \dots, e_n^V) \\
 = g_M((dP_{\mathcal{F}})([\zeta, E_\alpha^H](z)), e_\alpha),
 \end{aligned}
 \tag{2.2}$$

where $m = P_{\mathcal{F}}(z)$ and α is the second fundamental form of \mathcal{F} (see [2]).

Let W_i be any vector field on M satisfying $W_i(\varphi_t^i m) = \tilde{\varphi}_t^i z$ for the local flows (φ_t^i) of E_i and $(\tilde{\varphi}_t^i)$ of E_i^H . From $dP_{\mathcal{F}} \circ E_i^H = E_i \circ P_{\mathcal{F}}$, we have $P_{\mathcal{F}} \circ \tilde{\varphi}_t^i = \varphi_t^i \circ P_{\mathcal{F}}$ for any t . Therefore,

$$\begin{aligned}
 dP_{\mathcal{F}}([\zeta, E_i^H](z)) &= \frac{d}{dt} \Big|_{t=0} (dP_{\mathcal{F}} \circ d\tilde{\varphi}_{-t}^i)(\zeta(\tilde{\varphi}_t^i(z))) \\
 &= \frac{d}{dt} \Big|_{t=0} (d\varphi_{-t}^i \circ dP_{\mathcal{F}})(\zeta(\tilde{\varphi}_t^i(z))) \\
 &= \frac{d}{dt} \Big|_{t=0} (d\varphi_{-t}^i \circ \tilde{\varphi}_t^i)(z) \\
 &= \frac{d}{dt} \Big|_{t=0} (d\varphi_{-t}^i)(W_i(\varphi_t^i(m))) \\
 &= [W_i, E_i](m).
 \end{aligned}
 \tag{2.3}$$

Hence we have

$$\begin{aligned}
 g_M(dP_{\mathcal{F}}([\zeta, E_i^H](z)), E_i(z)) &= g_M([W_i, E_i], E_i)(m) \\
 &= g_M(W_i, \nabla_{E_i}^M E_i)(m) \\
 &= g_M(W_i(m), \alpha(E_i, E_i)(m)) \\
 &= g_M(z, \alpha(E_i(m), E_i(m))).
 \end{aligned}
 \tag{2.4}$$

Thus, we have

$$\begin{aligned}
 - \sum_{i=1}^p \tilde{\mu}(e_1^H, \dots, [\zeta, E_i^H](z), \dots, e_p^H, e_{p+1}^H, \dots, e_n^H, e_{p+1}^V, \dots, e_n^V) \\
 = -g_M\left(z, \sum_{i=1}^p \alpha(E_i(m), E_i(m))\right) \\
 = -g_M(z, \tau(m)),
 \end{aligned}
 \tag{2.5}$$

where $\tau(m)$ is the mean curvature vector of \mathcal{F} at m (see [2]).

On the other hand, we have

$$g_M((dP_{\mathcal{F}}[\zeta, E_\alpha^H])(m), e_\alpha) = g_M([W_\alpha, E_\alpha](m), e_\alpha),
 \tag{2.6}$$

where W_α is any vector field on M satisfying $W_\alpha(\varphi_t^\alpha m) = \tilde{\varphi}_t^\alpha z$ for the local flows φ_t^α of E_α and $\tilde{\varphi}_t^\alpha$ of E_α^H , $\alpha = p + 1, \dots, n$. Since $W_\alpha(\varphi_t^\alpha m)$ is an integral curve of E_α^H , we have $\pi(\nabla_{E_\alpha^H}^M W_\alpha) = C_{\mathcal{F}}(E_\alpha^H) = 0$. Moreover, by the choice of $\{E_\alpha\}$, we have $\pi(\nabla_{W_\alpha}^M E_\alpha)(m) = 0$. Therefore,

$$g_M((dP_{\mathcal{F}}[\zeta, E_\alpha^H])(m), e_\alpha) = g_M((\nabla_{W_\alpha}^M E_\alpha)(m) - (\nabla_{E_\alpha^H}^M W_\alpha)(m), e_\alpha) = 0. \tag{2.7}$$

Thus, to complete the proof, it suffices to show that

$$\tilde{\mu}(e_1^H, \dots, e_n^H, e_{p+1}^V, \dots, [\zeta, E_\alpha^V](z), \dots, e_n^V) = 0, \tag{2.8}$$

that is,

$$g_M(C_{\mathcal{F}}([\zeta, E_\alpha^V](z)), e_\alpha) = 0. \tag{2.9}$$

For this purpose, we introduce a local coordinate system around a point $z \in L^\perp$ as follows: let $(x^A)_{A=1, \dots, n} : U \rightarrow \mathbb{R}^n$ be a distinguished chart on a neighborhood U of $m \in M$. To $z \in P_{\mathcal{F}}^{-1}(U)$ with $P_{\mathcal{F}}(z) = m$, we assign $(x^1(m), \dots, x^n(m), z^{p+1}(m), \dots, z^n(m))$ as its coordinates, where $z = \sum_{\alpha=p+1}^n z^\alpha(m) E_\alpha(m)$. Let y be a geodesic orthogonal to the leaves of \mathcal{F} and $(x^A(t) : A = 1, \dots, n)$ its local coordinates.

Write

$$\dot{y}(t) = \sum_{\alpha=p+1}^n z^\alpha(t) E_\alpha(y(t)). \tag{2.10}$$

By the choice of $\{E_\alpha\}$, we get

$$\frac{d}{dt} z^\alpha = 0 \tag{2.11}$$

for $\alpha = p + 1, \dots, n$. Moreover, if we express E_α as $E_\alpha = \sum_{A=1}^n f_\alpha^A (\partial / \partial x^A)$, where f_α^A is a smooth function on U , we have

$$\sum_{A=1}^n \left(\frac{d}{dt} x^A \right) \frac{\partial}{\partial x^A} = \dot{y} = \sum_{\alpha=p+1}^n z^\alpha E_\alpha = \sum_{\alpha=p+1}^n \sum_{A=1}^n z^\alpha f_\alpha^A \frac{\partial}{\partial x^A}. \tag{2.12}$$

Equations (2.10) and (2.11) imply that $(x^A(t), z^\alpha(t))$ satisfy

$$\frac{d}{dt} x^A = \sum_{\alpha=p+1}^n z^\alpha f_\alpha^A, \quad \frac{d}{dt} z^\alpha = 0. \tag{2.13}$$

It follows that ζ can be locally expressed as

$$\zeta = \sum_{\alpha, A} z^\alpha f_\alpha^A \frac{\partial}{\partial x^A}. \tag{2.14}$$

A simple computation using the above expression of ζ gives

$$[\zeta, E_\alpha^V] = - \sum_A \left(f_\alpha^A + \sum_\beta z^\beta E_\alpha(f_\beta^A) \right) \frac{\partial}{\partial x^A}. \tag{2.15}$$

It is easy to show that for a vector field $\xi = \sum_A \xi^A (\partial/\partial x^A) + \sum_\alpha \tilde{\xi}^\alpha (\partial/\partial z^\alpha)$, $C_{\mathcal{F}}(\xi)$ is given by

$$C_{\mathcal{F}}(\xi) = \sum_\alpha \left(\tilde{\xi}^\alpha + \sum_{\beta,A} \Gamma_{\beta A}^\alpha z^\beta \xi^A \right) E_\alpha, \tag{2.16}$$

where $z = \sum_\alpha z^\alpha E_\alpha$ and $\nabla_{\partial_A} E_\alpha = \sum_{y=p+1}^n \Gamma_{\alpha A}^y E_y$. Therefore,

$$C_{\mathcal{F}}([\zeta, E_\alpha^V]) = - \sum_{\delta, \sigma, A} \left\{ f_\alpha^A + \sum_\beta z^\beta E_\alpha(f_\beta^A) \right\} \Gamma_{\sigma A}^\delta z^\sigma E_\delta. \tag{2.17}$$

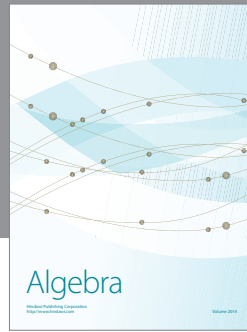
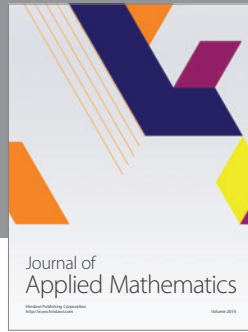
But $\Gamma_{\sigma A}^\delta = 0$ on U for $A = 1, \dots, n$ and $\delta, \sigma = p + 1, \dots, n$ by the choice of the frame $\{E_A\}$. Hence $C_{\mathcal{F}}([\zeta, E_\alpha^V]) = 0$ and the proof is complete. \square

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REFERENCES

[1] M. D. Hvidsten and P. Tondeur, *A characterization of harmonic foliations by variations of the metric*, Proc. Amer. Math. Soc. **98** (1986), no. 2, 359-362.
 [2] F. W. Kamber and P. Tondeur, *Harmonic foliations*, Harmonic Maps (New Orleans, La, 1980), Lecture Notes in Math., vol. 949, Springer-Verlag, Berlin, 1982, pp. 87-121.
 [3] F. W. Kamber, P. Tondeur, and G. Tóth, *Transversal Jacobi fields for harmonic foliations*, Michigan Math. J. **34** (1987), no. 2, 261-266.
 [4] G. Oshikiri, *Some remarks on minimal foliations*, Tôhoku Math. J. (2) **39** (1987), no. 2, 223-229.

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