GLOBAL ATTRACTIVITY IN A GENOTYPE SELECTION MODEL

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Received 20 July 2001

We obtain a sufficient condition for the global attractivity of the genotype selection model $y_{n+1} = y_n e^{\beta_n(1-2y_{n-k})} / (1 - y_n + y_n e^{\beta_n(1-2y_{n-k})})$, $n \in \mathbb{N}$. Our results improve the results established by Grove et al. (1994) and Kocić and Ladas (1993).

2000 Mathematics Subject Classification: 39A10, 39A12.

1. Introduction. Let $\mathbb{Z}$ denote the set of all integers. For $a, b \in \mathbb{Z}$, define $\mathbb{N}(a) = \{a, a+1, \dots\}$, $\mathbb{N} = \mathbb{N}(0)$, and $\mathbb{N}(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$.

Consider the following nonlinear delay difference equation:

$$y_{n+1} = \frac{y_n e^{\beta_n(1-2y_{n-k})}}{1 - y_n + y_n e^{\beta_n(1-2y_{n-k})}}, \quad n \in \mathbb{N}, \quad (1.1)$$

where $k \in \mathbb{N}$ and $\{\beta_n\}$ is a sequence of positive real numbers.

When $k = 0$ and $\beta_n \equiv \beta$ for all $n \in \mathbb{N}$, (1.1) was introduced by May [2, pages 513-560] as an example of a map generated by a simple model for frequency-dependent natural selection. The local stability of the equilibrium $\bar{y} = 1/2$ of (1.1) was investigated by May [2]. In [1] (see also [3]), Grove further investigated the stability of the equilibrium $\bar{y} = 1/2$ of (1.1) and proved that when $\beta_n \equiv \beta$, the equilibrium $\bar{y} = 1/2$ of (1.1) is locally asymptotically stable if $0 < \beta < 4 \cos(k\pi/(2k+1))$ and is unstable if $0 < \beta < 4 \cos(k\pi/(2k+1))$. Furthermore, if

$$0 < \beta \leq \frac{2}{k}, \quad k \in \mathbb{N}(1). \quad (1.2)$$

Then this equilibrium is a global attractor of all solution $\{y_n\}$ of (1.1) with initial conditions $y_{-k}, y_{-k+1}, \dots, y_0 \in (0, 1)$.

On the basis of computer observations, the authors of [1] also observe that condition (1.2) is probably far from sharp when $k \in \mathbb{N}(2)$. Therefore, it is highly desirable to improve condition (1.2).

The purpose of this paper is to obtain new sufficient conditions for the global attractivity of the equilibrium $\bar{y} = 1/2$ of (1.1). Our main result is the following theorem.

THEOREM 1.1. *Assume that $\{\beta_n\}$ is a positive sequence which satisfies*

$$\sum_{i=n-k}^n \beta_i \leq 3 + \frac{1}{k+1}, \quad (1.3)$$

for all large n , and

$$\sum_{i=0}^{\infty} \beta_i = \infty. \quad (1.4)$$

Then every solution $\{y_n\}$ of (1.1) with initial conditions $y_{-k}, y_{-k+1}, \dots, y_0 \in (0, 1)$ will tend to $\bar{y} = 1/2$.

COROLLARY 1.2. Assume that $\beta_n \equiv \beta$ for all $n \in \mathbb{N}$ and

$$\beta \leq \frac{3}{k+1} + \frac{1}{(k+1)^2}. \quad (1.5)$$

Then every solution $\{y_n\}$ of (1.1) with initial conditions $y_{-k}, y_{-k+1}, \dots, y_0 \in (0, 1)$ will tend to $\bar{y} = 1/2$.

It is easy to see that when $k \in \mathbb{N}(2)$, (1.5) is an improvement on (1.2).

By a solution of (1.1), we mean a sequence $\{y_n\}$ that is defined for $n \in \mathbb{N}(-k)$ and that satisfies (1.1) for $n \in \mathbb{N}$. If $a_{-k}, a_{-k+1}, \dots, a_0$ are $k+1$ given constants, then (1.1) has a unique solution satisfying the initial conditions

$$x_i = a_i \quad \text{for } i \in \mathbb{N}(-k, 0). \quad (1.6)$$

For the sake of convenience, throughout, we use the convention

$$\sum_{n=i}^j r_n \equiv 0, \quad \text{whenever } j \leq i-1. \quad (1.7)$$

2. Proof of Theorem 1.1. Let $\{y_n\}$ be a solution of (1.1) with initial conditions $y_{-k}, y_{-k+1}, \dots, y_0 \in (0, 1)$. Then clearly, $y_n \in (0, 1)$ for all $n \in \mathbb{N}(-k)$. By introducing the substitution

$$x_n = \ln \frac{y_n}{1-y_n}, \quad n \in \mathbb{N}(-k), \quad (2.1)$$

we obtain

$$\Delta x_n + r_n f(x_{n-k}) = 0, \quad n \in \mathbb{N}, \quad (2.2)$$

$$x_{-k}, x_{-k+1}, \dots, x_0 \in (-\infty, \infty), \quad (2.3)$$

where

$$\Delta x_n = x_{n+1} - x_n, \quad r_n = \frac{1}{2} \beta_n, \quad f(x) = 2 - \frac{4}{e^x + 1}. \quad (2.4)$$

It is easy to see that

$$f(0) = 0, \quad x f(x) > 0 \quad \forall x \in \mathbb{R}, \quad (2.5)$$

$$f'(x) = \frac{4e^x}{(e^x + 1)^2} \quad \forall x \in \mathbb{R}. \quad (2.6)$$

Thus, f is increasing, we also have

$$f'(x) < \frac{4e^x}{(2\sqrt{e^x})^2} = 1 \quad \text{for } x \neq 0, \quad (2.7)$$

which implies that

$$|f(x)| < |x| \quad \text{for } x \neq 0. \quad (2.8)$$

Define h as follows

$$h(x) = \max \{f(x), -f(-x)\} \quad \text{for } x > 0. \quad (2.9)$$

We have from (2.5), (2.8), and the increasing property of f that $h(x)$ is increasing in $[0, \infty)$, and

$$|f(x)| \leq h(|x|) < |x| \quad \text{for } x \neq 0. \quad (2.10)$$

We will now prove that

$$\lim_{n \rightarrow \infty} x_n = 0. \quad (2.11)$$

There are two cases to consider.

CASE 1. The sequence $\{x_n\}$ is eventually nonnegative or eventually nonpositive. We assume that $\{x_n\}$ is eventually nonnegative, then there exists an integer $n_0 \in \mathbb{N}(k)$ such that $x_{n-k} \geq 0$ for all $n \in \mathbb{N}(n_0)$. By (2.2), we have $\Delta x_n \leq 0$ for all $n \in \mathbb{N}(n_0)$ and there exists $a \geq 0$ such that

$$\lim_{n \rightarrow \infty} x_n = a. \quad (2.12)$$

If $a > 0$, by the increasing property of f , it follows that

$$\Delta x_n \leq -r_n f(a) \quad \forall n \in \mathbb{N}(n_0 + k). \quad (2.13)$$

Summing (2.13) from $n_0 + k$ to $n - 1$ and using (1.4), we have

$$x_n - x_{n_0+k} \leq -f(a) \sum_{i=n_0+k}^{n-1} r_i \rightarrow -\infty \quad \text{as } n \rightarrow \infty, \quad (2.14)$$

which contradicts (2.12). The case when $\{x_n\}$ is eventually nonpositive can be dealt with similarly.

CASE 2. The sequence $\{x_n\}$ is oscillatory. By (1.3) and (2.4), then there exists an integer $n^* \in \mathbb{N}(2k)$ such that

$$\sum_{i=n-k}^n r_i \leq \alpha = \frac{3}{2} + \frac{1}{2(k+1)}, \quad n \in \mathbb{N}(n^* - 2k), \quad (2.15)$$

$$x_{n^*-1} x_{n^*} \leq 0, \quad x_{n^*} \neq 0. \quad (2.16)$$

By virtue of the choice of n^* , there exists a real number $\lambda \in [0, 1)$ such that

$$x_{n^*-1} + \lambda(x_{n^*} - x_{n^*-1}) = 0. \quad (2.17)$$

Let l be a positive constant such that

$$\max_{n \in \mathbb{N}(n^*-2k-1, n^*-1)} |x_n| \leq l. \quad (2.18)$$

By (2.2), (2.10), (2.18), and the increasing property of h , we have

$$|\Delta x_n| \leq r_n h(l), \quad n \in \mathbb{N}(n^*-1, n^*+k-1). \quad (2.19)$$

Which, together with (2.17), implies that

$$\begin{aligned} |x_{n-k}| &= |x_{n-k} - x_{n^*-1} - \lambda(x_{n^*} - x_{n^*-1})| \\ &= \left| - \sum_{j=n-k}^{n^*-2} \Delta x_j - \lambda \Delta x_{n^*-1} \right| \\ &\leq h(l) \left(\sum_{j=n-k}^{n^*-2} r_j + \lambda r_{n^*-1} \right), \quad n \in \mathbb{N}(n^*-1, n^*+k-1). \end{aligned} \quad (2.20)$$

In view of (2.2), (2.10), and (2.20), we obtain

$$|\Delta x_n| \leq r_n h(l) \left(\sum_{j=n-k}^{n^*-2} r_j + \lambda r_{n^*-1} \right), \quad n \in \mathbb{N}(n^*-1, n^*+k-1). \quad (2.21)$$

Now we show that

$$|x_n| \leq h(l) \quad \forall n \in \mathbb{N}(n^*, n^*+k). \quad (2.22)$$

There are two possible cases to consider.

CASE 1. Suppose that $d = \sum_{i=n^*}^{n^*+k-1} r_i + (1-\lambda)r_{n^*-1} \leq 1$. By (2.15), (2.17), and (2.21) we have for $n \in \mathbb{N}(n^*, n^*+k)$

$$\begin{aligned} |x_n| &= |x_n - x_{n^*-1} - \lambda(x_{n^*} - x_{n^*-1})| \\ &= \left| \sum_{i=n^*}^{n-1} \Delta x_i + (1-\lambda)\Delta x_{n^*-1} \right| \\ &\leq \sum_{i=n^*}^{n^*+k-1} r_i h(l) \left(\sum_{j=i-k}^{n^*-2} r_j + \lambda r_{n^*-1} \right) + (1-\lambda)r_{n^*-1} h(l) \left(\sum_{j=n^*-k-1}^{n^*-2} r_j + \lambda r_{n^*-1} \right) \\ &= h(l) \sum_{i=n^*}^{n^*+k-1} r_i \left[\sum_{j=i-k}^i r_j - \sum_{j=n^*}^i r_j - (1-\lambda)r_{n^*-1} \right] \\ &\quad + h(l)(1-\lambda)r_{n^*-1} \left[\sum_{j=n^*-k-1}^{n^*-1} r_j - (1-\lambda)r_{n^*-1} \right] \end{aligned}$$

$$\begin{aligned}
&\leq h(l) \left[\alpha d - \sum_{i=n^*}^{n^*+k-1} r_i \sum_{j=n^*}^i r_j - (1-\lambda)r_{n^*-1}d \right] \\
&= h(l) \left[\alpha d - \frac{1}{2} \left(\sum_{i=n^*}^{n^*+k-1} r_i \right)^2 - \frac{1}{2} \sum_{i=n^*}^{n^*+k-1} r_i^2 - (1-\lambda)r_{n^*-1}d \right] \\
&= h(l) \left[\alpha d - \frac{1}{2}d^2 - \frac{1}{2} \left(\sum_{i=n^*}^{n^*+k-1} r_i^2 + (1-\lambda)^2 r_{n^*-1}^2 \right) \right].
\end{aligned} \tag{2.23}$$

Since

$$\sum_{i=n^*}^{n^*+k-1} r_i^2 + (1-\lambda)^2 r_{n^*-1}^2 \geq \frac{1}{k+1} \left(\sum_{i=n^*}^{n^*+k-1} r_i + (1-\lambda)r_{n^*-1} \right)^2 = \frac{d^2}{k+1}. \tag{2.24}$$

We obtain

$$\begin{aligned}
|x_n| &\leq h(l) \left[\alpha d - \left(\frac{1}{2} + \frac{1}{2(k+1)} \right) d^2 \right] \\
&\leq h(l) \left[\alpha - \left(\frac{1}{2} + \frac{1}{2(k+1)} \right) \right] \\
&= h(l).
\end{aligned} \tag{2.25}$$

CASE 2. Suppose that $d = \sum_{i=n^*}^{n^*+k-1} r_i + (1-\lambda)r_{n^*-1} > 1$. In this case, there exists an integer $m \in \mathbb{N}(n^*, n^*+k)$ such that

$$\sum_{i=m}^{n^*+k-1} r_i \leq 1, \quad \sum_{i=m-1}^{n^*+k-1} r_i > 1. \tag{2.26}$$

Therefore, there is an $\eta \in (0, 1]$ such that

$$\sum_{i=m}^{n^*+k-1} r_i + (1-\eta)r_{m-1} = 1. \tag{2.27}$$

By (2.15), (2.17), (2.19), and (2.21), we have for $n \in \mathbb{N}(n^*, n^*+k)$

$$\begin{aligned}
|x_n| &= |x_n - x_{n^*-1} - \lambda \Delta x_{n^*-1}| \\
&= \left| \sum_{i=n^*}^{n-1} \Delta x_i + (1-\lambda) \Delta x_{n^*-1} \right| \\
&= \sum_{j=n^*}^{n^*+k-1} |\Delta x_j| + (1-\lambda) |\Delta x_{n^*-1}|
\end{aligned}$$

$$\begin{aligned}
&= (1-\lambda) |\Delta x_{n^*-1}| + \sum_{j=n^*}^{m-2} |\Delta x_j| + \eta |\Delta x_{m-1}| + (1-\eta) |\Delta x_{m-1}| + \sum_{j=m}^{n^*+k-1} |\Delta x_j| \\
&\leq h(l) \left((1-\lambda) r_{n^*-1} + \sum_{j=n^*}^{m-2} r_j + \eta r_{m-1} \right) + h(l) (1-\eta) r_{m-1} \left(\sum_{j=m-1-k}^{n^*-2} r_j + \lambda r_{n^*-1} \right) \\
&\quad + h(l) \sum_{j=m}^{n^*+k-1} r_j \left(\sum_{i=j-k}^{n^*-2} r_i + \lambda r_{n^*-1} \right) \\
&= h(l) \left[(1-\lambda) r_{n^*-1} + \sum_{j=n^*}^{m-1} r_j - (1-\eta) r_{m-1} \right] \\
&\quad + h(l) (1-\eta) r_{m-1} \left[\sum_{j=m-1-k}^{m-1} r_j - \sum_{j=n^*}^{m-1} r_j - (1-\lambda) r_{n^*-1} \right] \\
&\quad + h(l) \sum_{j=m}^{n^*+k-1} r_j \left[\sum_{i=j-k}^j r_i - \sum_{i=m}^j r_i - \sum_{i=n^*}^{m-1} r_i - (1-\lambda) r_{n^*-1} \right] \\
&\leq h(l) \left[\alpha - (1-\eta) r_{m-1} - \sum_{j=m}^{n^*+k-1} r_j \sum_{i=m}^j r_i \right] \\
&= h(l) \left[\alpha - (1-\eta) r_{m-1} - \frac{1}{2} \left(\sum_{j=m}^{n^*+k-1} r_j \right)^2 - \frac{1}{2} \sum_{j=m}^{n^*+k-1} r_j^2 \right] \\
&= h(l) \left[\alpha - (1-\eta) r_{m-1} - \frac{1}{2} (1 - (1-\eta) r_{m-1})^2 - \frac{1}{2} \sum_{j=m}^{n^*+k-1} r_j^2 \right] \\
&= h(l) \left[\alpha - \frac{1}{2} - \frac{1}{2} \left(\sum_{j=m}^{n^*+k-1} r_j^2 + (1-\eta)^2 r_{m-1}^2 \right) \right].
\end{aligned} \tag{2.28}$$

Since

$$\sum_{j=m}^{n^*+k-1} r_j^2 + (1-\eta)^2 r_{m-1}^2 \geq \frac{1}{n^* - m + k + 1} \left(\sum_{j=m}^{n^*+k-1} r_j + (1-\eta) r_{m-1} \right)^2 \geq \frac{1}{k+1}. \tag{2.29}$$

We obtain

$$|x_n| \leq h(l) \left(\alpha - \frac{1}{2} - \frac{1}{2(k+1)} \right) = h(l). \tag{2.30}$$

Furthermore, we can prove that

$$|x_n| \leq h(l) \quad \forall n \in \mathbb{N}(n^*). \tag{2.31}$$

Assume, for the sake of contradiction, that (2.31) is not true. Then there exists $m_1 \in \mathbb{N}(n^* + k + 1)$ such that $|x_{m_1}| > h(l)$ and $|x_n| \leq h(l)$ for $n \in \mathbb{N}(n^*, m_1 - 1)$. Set

$$m_2 = \max \{n \in \mathbb{N}(n^*, m_1) : x_{n-1}x_n \leq 0, x_n \neq 0\}. \quad (2.32)$$

In case $m_1 \leq m_2 + k$. From (2.10), we have

$$\max_{n \in \mathbb{N}(m_2 - 2k - 1, m_2 - 1)} |x_n| \leq h(l) < l. \quad (2.33)$$

By a similar method to the proof of (2.22), we obtain

$$|x_n| \leq h(l) \quad \forall n \in \mathbb{N}(m_2, m_2 + k) \quad (2.34)$$

which contradicts the definition of m_1 . In case $m_1 - 1 \geq m_2 + k$, it follows from the choice of m_1 and m_2 that

$$x_n > 0 \quad \text{or} \quad x_n < 0 \quad \forall n \in \mathbb{N}(m_2, m_1). \quad (2.35)$$

Assume that $x_n > 0$ for all $n \in \mathbb{N}(m_2, m_1)$. (In case $x_n < 0$, the proof is similar.) From (2.2) we have

$$\Delta x_n \leq 0 \quad \text{for } n \in \mathbb{N}(m_1 - 1, m_1 + k) \quad (2.36)$$

which implies that

$$x_{m_1} \leq x_{m_1 - 1} \leq h(l). \quad (2.37)$$

This contradicts the definition of m_1 . Thus (2.31) holds.

From the argument above, we can establish a sequence $\{n_i\}$ of positive integers with $n_1 = n^*$, $n_{i+1} - n_i > 2k$ such that

$$x_{n_i - 1}x_{n_i} \leq 0, \quad x_{n_i} \neq 0, \quad (2.38)$$

and a sequence $\{z_i\}$ with $z_1 = l$, $z_{i+1} = h(z_i)$ such that

$$\max_{n \in \mathbb{N}(n_i - 2k - 1, n_i - 1)} |x_n| \leq z_i, \quad |x_n| \leq z_{i+1} \quad \forall n \in \mathbb{N}(n_i). \quad (2.39)$$

By (2.10), we obtain

$$\lim_{i \rightarrow \infty} z_i = 0 \quad (2.40)$$

which, together with (2.39), implies that $\lim_{n \rightarrow \infty} x_n = 0$. The proof is complete. $\square$

ACKNOWLEDGMENT. This work was supported by the Science Foundation of Hunan Educational Committee of China.

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