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SUFFICIENT CONDITIONS FOR UNIVALENCE IN $\mathbb{C}^n$

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The method of subordination chains is used to establish new univalence criteria for holomorphic mappings in the unit ball of $\mathbb{C}^n$. Various criteria involving the first and the second derivative of a holomorphic mapping in the unit ball of $\mathbb{C}^n$ are developed.

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1. Introduction. Let $\mathbb{C}^n$ be the space of n-complex variables $z = (z_1,\dots,z_n)$ with the usual inner product $\langle z,w\rangle = \sum_{j=1}^{n} z_j\overline{w}_j$ and Euclidean norm $\|z\| = \langle z,z\rangle^{1/2}$.

Let $H(B^n)$ denote the class of mappings $f(z) = (f_1(z),\dots,f_n(z))$, $z = (z_1,\dots,z_n)$, that are holomorphic in the unit ball $B^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ with values in $\mathbb{C}^n$. A mapping $f \in H(B^n)$ is said to be *locally biholomorphic in* B^n if f has a local inverse at each point in B^n or, equivalently, if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j}\right)_{1\le j,k\le n} \tag{1.1}$$

is nonsingular at each point $z \in B^n$.

The second derivative of a mapping $f \in H(B^n)$ is a symmetric bilinear operator $D^2f(z)(\cdot,\cdot)$ on $\mathbb{C}^n\times\mathbb{C}^n$, and $D^2f(z)(z,\cdot)$ is the linear operator obtained by restricting $D^2f(z)$ to $\{z\}\times\mathbb{C}^n$. The matrix representation for $D^2f(z)(z,\cdot)$ is

$$D^2f(z)(z,\cdot) = \left(\sum_{m=1}^{n}\frac{\partial^2 f_k(z)}{\partial z_j\partial z_m}z_m\right)_{1\le j,k\le n}. \tag{1.2}$$

We denote by $\mathcal{L}(\mathbb{C}^n)$ the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$, that is, the $n\times n$ complex matrices $A = (A_{jk})$ with the usual operator norm

$$\|A\| = \sup\{\|Az\| : \|z\| \le 1\}, \quad A \in \mathcal{L}(\mathbb{C}^n). \tag{1.3}$$

Let $f,g \in H(B^n)$. We say that f is *subordinate* to g ($f \prec g$) in B^n if there exists a mapping $v \in H(B^n)$ with $\|v(z)\| \le \|z\|$, for all $z \in B^n$ such that $f(z) = g(v(z))$, $z \in B^n$.

A function $L : B^n\times[0,\infty) \to \mathbb{C}^n$ is a *univalent subordination chain* if for each $t \in [0,\infty)$ and $L(\cdot,t) \in H(B^n)$, $L(\cdot,t)$ is univalent in B^n and $L(\cdot,s) \prec L(\cdot,t)$ whenever $0 \le s \le t < \infty$.

We will use the following theorem to prove our results.

Theorem 1.1 [1]. *Let* $L(z,t) = a_1(t)z+\cdots$, $a_1(t) \ne 0$, *be a function from* $B^n\times[0,\infty)$ *into* $\mathbb{C}^n$ *such that*

(i) *for each* $t \geq 0$, $L(\cdot,t) \in H(B^n)$*;*
(ii) $L(z,t)$ *is a locally absolutely continuous function of* $t \in [0,\infty)$*, locally uniformly with respect to* $z \in B^n$*;*
(iii) $a_1(t) \in C^1_{[0,\infty)}$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$*.*

Let $h(z,t)$ *be a function from* $B^n \times [0,\infty)$ *into* $\mathbb{C}^n$ *which satisfies the following conditions:*
(iv) *for each* $t \geq 0$, $h(\cdot,t) \in H(B^n)$*;*
(v) *for each* $z \in B^n$, $h(z,\cdot)$ *is a measurable function on* $[0,\infty)$*;*
(vi) $h(0,t) = 0$ *and* $\operatorname{Re}\langle h(z,t), z\rangle \geq 0$*, for each* $t \geq 0$ *and for all* $z \in B^n$*;*
(vii) *for each* $T > 0$ *and* $r \in (0,1)$*, there exists a number* $K = K(r,T)$ *such that* $\|h(z,t)\| \leq K(r,T)$*, when* $\|z\| \leq r$ *and* $t \in [0,T]$*.*

Suppose that $L(z,t)$ *satisfies*

$$\frac{\partial L(z,t)}{\partial t} = DL(z,t)h(z,t), \quad a.e.\ t \geq 0,\ \forall z \in B^n. \tag{1.4}$$

Further, suppose that there is a sequence $(t_m)_{m\geq 0}$*,* $t_m > 0$*, with* $\lim_{m\to\infty} t_m = \infty$ *such that*

$$\lim_{m\to\infty} \frac{L(z,t_m)}{a_1(t_m)} = F(z), \tag{1.5}$$

locally uniformly in B^n*. Then for each* $t \in [0,\infty)$*,* $L(\cdot,t)$ *is univalent in* B^n*.*

2. Univalence criteria. We obtain various univalence criteria involving the first and the second derivative of a holomorphic mapping in the unit ball B^n. Some of them represent the n-dimensional versions of univalence criteria for holomorphic functions in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem 2.1. *Let* $f \in H(B^n)$*,* $f(0) = 0$*, and* $Df(0) = I$*. Let* α, c *be complex numbers such that* $c \neq -1$ *and* $(\alpha-1)/(c+1) \notin [0,\infty)$*. If*

$$\left\|[Df(z)-\alpha I]^{-1}[cDf(z)+\alpha I]\right\| \leq 1, \tag{2.1}$$

$$\begin{aligned}\Big\|\|z\|^2[Df(z)-\alpha I]^{-1}&[cDf(z)+\alpha I]\\ &+(1-\|z\|^2)[Df(z)-\alpha I]^{-1}D^2f(z)(z,\cdot)\Big\| \leq 1,\end{aligned} \tag{2.2}$$

for all $z \in B^n$*, then* f *is a univalent mapping in* B^n*.*

Proof. We define

$$L(z,t) = f(e^{-t}z)+\frac{1}{1+c}(e^t-e^{-t})[Df(e^{-t}z)-\alpha I](z), \quad (z,t) \in B^n\times[0,\infty). \tag{2.3}$$

We will prove that $L(z,t)$ satisfies the conditions of Theorem 1.1. We have

$$a_1(t) = e^{-t}+\frac{1-\alpha}{1+c}(e^t-e^{-t}), \quad t \in [0,\infty), \tag{2.4}$$

and hence $a_1(t) \neq 0$, for all $t \geq 0$, $\lim_{t\to\infty}|a_1(t)| = \infty$ and $a_1(t) \in C^1[0,\infty)$.

It is easy to check that $L(z,t) = a_1(t)z + O(1)$ as $t \to \infty$ locally uniformly in B^n. Hence (1.5) holds with $F(z) = z$.

The function $L(z,t)$ satisfies the absolute continuity requirements of Theorem 1.1. Using (2.3), we obtain

$$DL(z,t) = \frac{e^t}{1+c}[Df(e^{-t}z) - \alpha I][I - E(z,t)], \quad (z,t) \in B^n \times [0,\infty), \tag{2.5}$$

where $E(z,t)$ is the linear operator defined by

$$\begin{aligned} E(z,t) = &-e^{-2t}[Df(e^{-t}z) - \alpha I]^{-1}[cDf(e^{-t}z) + \alpha I] \\ &-(1-e^{-2t})[Df(e^{-t}z) - \alpha I]^{-1}D^2 f(e^{-t}z)(e^{-t}z,\cdot). \end{aligned} \tag{2.6}$$

We define

$$\begin{aligned} A(z,t) &= [Df(e^{-t}z) - \alpha I]^{-1}[cDf(e^{-t}z) + \alpha I], \\ B(z,t) &= [Df(e^{-t}z) - \alpha I]^{-1}D^2 f(e^{-t}z)(e^{-t}z,\cdot), \\ W(z,t,\lambda) &= \lambda A(z,t) + (1-\lambda)B(z,t), \quad \lambda \in [0,1]. \end{aligned} \tag{2.7}$$

Using (2.1) and (2.2), we obtain $\|A(z,t)\| \le 1$ and $||W(z,t,\lambda_z)|| \le 1$, $z \in B^n$, $t \ge 0$, where $\lambda_z = e^{-2t}\|z\|^2$. Since $\lambda_z < e^{-2t} \le 1$, $z \in B^n$, $t \ge 0$, there exists $u \in (0,1]$ such that $e^{-2t} = u + (1-u)\lambda_z$. Then

$$||E(z,t)|| = ||uA(z,t) + (1-u)W(z,t,\lambda_z)|| \le u||A(z,t)|| + (1-u)||W(z,t,\lambda_z)|| \le 1, \tag{2.8}$$

for all $z \in B^n$ and $t \ge 0$. Using the principle of the maximum [2], we obtain $\|E(z,t)\| < 1$.

Since $\|E(z,t)\| < 1$, for all $(z,t) \in B^n \times [0,\infty)$, it results that $I - E(z,t)$ is an invertible operator.

Using (2.3), we obtain

$$\begin{aligned} \frac{\partial L(z,t)}{\partial t} &= \frac{e^t}{1+c}[Df(e^{-t}z) - \alpha I][I + E(z,t)](z) \\ &= DL(z,t)[I - E(z,t)]^{-1}[I + E(z,t)](z). \end{aligned} \tag{2.9}$$

Hence $L(z,t)$ satisfies the differential equation (1.4) for all $t \ge 0$ and $z \in B^n$, where

$$h(z,t) = [I - E(z,t)]^{-1} \cdot [I + E(z,t)](z). \tag{2.10}$$

It remains to prove that $h(z,t)$ satisfies conditions (iv), (v), (vi), and (vii) of Theorem 1.1. Obviously, $h(z,t)$ satisfies the holomorphy and measurability requirements and $h(0,t) = 0$. Using the inequality

$$\begin{aligned} ||h(z,t) - z|| &\le ||E(z,t)(h(z,t) + z)|| \\ &\le ||E(z,t)|| \cdot ||h(z,t) + z|| \\ &< ||h(z,t) + z||, \end{aligned} \tag{2.11}$$

we obtain $\operatorname{Re}\langle h(z,t), z\rangle \ge 0$, for all $z \in B^n$ and $t \ge 0$.

The inequality $\|[I-E(z,t)]^{-1}\| \le [I-\|E(z,t)\|]^{-1}$ implies that

$$\|h(z,t)\| \le \frac{1+\|E(z,t)\|}{1-\|E(z,t)\|}\|z\|. \tag{2.12}$$

Since all the conditions of Theorem 1.1 are satisfied, it results that the functions $L(z,t)$, $t \ge 0$, are univalent in B^n. Obviously, $f(z) = L(z,0)$ is also a univalent mapping on B^n. □

Corollary 2.2. *Let $f \in H(B^n)$ be locally univalent in B^n, $f(0) = 0$, and $Df(0) = I$. Let c be a complex number such that $c \ne -1$ and $|c| \le 1$. If*

$$\left\|c\|z\|^2 I + (1-\|z\|^2)[Df(z)]^{-1}D^2f(z)(z,\cdot)\right\| \le 1, \quad z \in B^n, \tag{2.13}$$

then the mapping f is univalent on B^n.

Proof. For $\alpha = 0$ and $c \in \mathbb{C}\setminus\{-1\}$, $|c| \le 1$ the conditions of Theorem 2.1 are satisfied and hence the mapping f is univalent in B^n. □

Remark 2.3. Corollary 2.2 represents the n-dimensional version of Ahlfors and Becker's univalence criterion [1]. If $c = 0$, we have the n-dimensional version of Becker's univalence result [3].

Corollary 2.4. *Let $f \in H(B^n)$, $f(0) = 0$, and $Df(0) = I$, and let α be a complex number with $\alpha \notin [1,\infty)$. If*

$$\|[Df(z)-\alpha I](z)\| \ge |\alpha|\|z\|, \tag{2.14}$$

$$\left\|\alpha\|z\|^2[Df(z)-\alpha I]^{-1} + (1-\|z\|^2)[Df(z)-\alpha I]^{-1}D^2f(z)(z,\cdot)\right\| \le 1, \tag{2.15}$$

for all $z \in B^n$, then f is a univalent mapping on B^n.

Proof. Using (2.14), we have that $Df(z)-\alpha I$ is an invertible operator and

$$\|[Df(z)-\alpha I]^{-1}\| \le \frac{1}{|\alpha|}, \quad z \in B^n. \tag{2.16}$$

The conclusion of the corollary follows from Theorem 2.1 with $c = 0$. □

Theorem 2.5. *Let $f \in H(B^n)$ with $f(0) = 0$ and $Df(0) = I$. Let α and c be complex numbers such that $c \ne -1$ and $(\alpha-1)/(c+1) \notin [0,\infty)$. If*

$$\begin{gathered} \|[Df(z)-\alpha I]^{-1}[cDf(z)+\alpha I]\| \le 1, \\ \|[Df(z)-\alpha I]^{-1}D^2f(z)(z,\cdot)\| \le 1, \end{gathered} \tag{2.17}$$

for all $z \in B^n$, then the mapping f is univalent on B^n.

Proof. Using (2.17), we obtain

$$\big\|\|z\|^2[Df(z)-\alpha I]^{-1}[cDf(z)+\alpha I]+(1-\|z\|^2)[Df(z)-\alpha I]^{-1}D^2f(z)(z,\cdot)\big\| \\ \le \|z\|^2+1-\|z\|^2=1, \quad \forall z\in B^n. \tag{2.18}$$

Hence, the conditions of Theorem 2.1 are satisfied and then f is a univalent mapping on B^n. □

Corollary 2.6. *Let $f\in H(B^n)$, $f(0)=0$, and $Df(0)=I$. Let α be a complex number such that $\alpha\notin[1,\infty)$. If*

$$\big\|[Df(z)-\alpha I](z)\big\|\ge|\alpha|\|z\|, \tag{2.19}$$

$$\big\|D^2f(z)(z,\cdot)\big\|\le|\alpha|, \tag{2.20}$$

for all $z\in B^n$, then f is a univalent mapping on B^n.

Proof. Using (2.19) and (2.20), we have

$$\begin{aligned} \big\|[Df(z)-\alpha I]^{-1}\big\| &\le \frac{1}{|\alpha|},\\ \big\|[Df(z)-\alpha I]^{-1}D^2f(z)(z,\cdot)\big\| &\\ &\le \big\|[Df(z)-\alpha I]^{-1}\big\|\cdot\big\|D^2f(z)(z,\cdot)\big\|\\ &\le \frac{1}{|\alpha|}\cdot|\alpha|\le 1,\quad z\in B^n. \end{aligned} \tag{2.21}$$

Using Theorem 2.5 with $c=0$, we obtain that f is univalent on B^n. □

Corollary 2.7. *Let $f\in H(B^n)$ such that $f(0)=0$ and $Df(0)=I$. If*

$$\operatorname{Re}\langle Df(z)(z),z\rangle>0, \tag{2.22}$$

for all $z\in B^n$, then the mapping f is univalent on B^n.

Proof. Let α be a real number such that $\alpha<0$. Since

$$\big\|[Df(z)-\alpha I](z)\big\|^2=\big\|Df(z)(z)\big\|^2+|\alpha|^2\|z\|^2-2\alpha\operatorname{Re}\langle Df(z)(z),z\rangle\ge|\alpha|^2\|z\|^2 \tag{2.23}$$

it results that (2.19) holds true, for all $z\in B^n$ and $\alpha<0$.

If $\alpha\to-\infty$, then (2.20) also holds true. Using Corollary 2.6, we obtain that f is univalent on B^n. □

Remark 2.8. When $n=1$, (2.22) becomes $\operatorname{Re} f'(z)>0$ and hence Corollary 2.7 represents the n-dimensional version of Alexander-Noshiro's univalence criterion.

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