CR-SUBMANIFOLDS OF A NEARLY TRANS-SASAKIAN MANIFOLD

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This paper considers the study of CR-submanifolds of a nearly trans-Sasakian manifold, generalizing the results of trans-Sasakian manifolds and thus those of Sasakian manifolds.

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1. Introduction. In 1978, Bejancu introduced the notion of CR-submanifold of a Kaehler manifold [1]. Since then several papers on CR-submanifolds of Kaehler manifold have been published. On the other hand, CR-submanifolds of a Sasakian manifold have been studied by Kobayashi [6], Shahid et al. [10], Yano and Kon [11], and others. Bejancu and Papaghuic [2] studied CR-submanifolds of a Kenmotsu manifold. In 1985, Oubina introduced a new class of almost contact metric manifold known as trans-Sasakian manifold [7]. This class contains α -Sasakian and β -Kenmotsu manifold [5]. Geometry of CR-submanifold of trans-Sasakian manifold was studied by Shahid [8, 9]. A nearly trans-Sasakian manifold [4] is a more general concept.

The results of this paper are the generalization of the results obtained earlier by several authors, namely [6, 8, 9] and others.

2. Preliminaries. Let $\overline{M}$ be an n -dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) satisfying [3]

$$\begin{aligned}\phi^2 X &= -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi \circ \xi = 0, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),\end{aligned}\tag{2.1}$$

where X and Y are vector fields tangent to $\overline{M}$.

An almost contact metric structure (ϕ, ξ, η, g) on $\overline{M}$ is called trans-Sasakian if [7]

$$(\overline{\nabla}_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\},\tag{2.2}$$

where α and β are nonzero constants, $\overline{\nabla}$ denotes the Riemannian connection of g on $\overline{M}$, and we say that the trans-Sasakian structure is of type (α, β) .

Further, an almost contact metric manifold $\overline{M}(\phi, \xi, \eta, g)$ is called nearly trans-Sasakian if [4]

$$\begin{aligned}(\overline{\nabla}_X \phi)(Y) + (\overline{\nabla}_Y \phi)(X) &= \alpha\{2g(X, Y)\xi - \eta(Y)X - \eta(X)Y\} \\ &\quad - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\}.\end{aligned}\tag{2.3}$$

It is clear that any trans-Sasakian manifold, and thus any Sasakian manifold, satisfies the above relation.

Let M be an m -dimensional isometrically immersed submanifold of a nearly trans-Sasakian manifold $\overline{M}$ and denote by the same g the Riemannian metric tensor field induced on M from that of $\overline{M}$.

DEFINITION 2.1. An m -dimensional Riemannian submanifold M of a nearly trans-Sasakian manifold $\overline{M}$ is called a CR-submanifold if ξ is tangent to M and there exists a differentiable distribution $D : x \in M \rightarrow D_x \subset T_x M$ such that

- (1) the distribution D_x is invariant under ϕ , that is, $\phi D_x \subset D_x$ for each $x \in M$;
- (2) the complementary orthogonal distribution $D^\perp : x \in M \rightarrow D_x^\perp \subset T_x M$ of D is anti-invariant under ϕ , that is, $\phi D_x^\perp \subset T_x^\perp M$ for all $x \in M$, where $T_x M$ and $T_x^\perp M$ are the tangent space and the normal space of M at x , respectively.

If $\dim D_x^\perp = 0$ (resp., $\dim D_x = 0$), then the CR-submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution D (resp., $D^\perp$) is called the horizontal (resp., vertical) distribution. Also, the pair $(D, D^\perp)$ is called ξ -horizontal (resp., vertical) if $\xi_x \in D_x$ (resp., $\xi_x \in D_x^\perp$) [6].

For any vector field X tangent to M , we put [6]

$$X = PX + QX, \quad (2.4)$$

where PX and QX belong to the distribution D and $D^\perp$.

For any vector field N normal to M , we put [6]

$$\phi N = BN + CN, \quad (2.5)$$

where BN (resp., CN) denotes the tangential (resp., normal) component of ϕN .

Let $\overline{\nabla}$ (resp., ∇) be the covariant differentiation with respect to the Levi-Civita connection on $\overline{M}$ (resp., M). The Gauss and Weingarten formulas for M are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y); \quad \overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.6)$$

for $X, Y \in TM$ and $N \in T^\perp M$, where h (resp., A) is the second fundamental form (resp., tensor) of M in $\overline{M}$, and $\nabla^\perp$ denotes the normal connection. Moreover, we have

$$g(h(X, Y), N) = g(A_N X, Y). \quad (2.7)$$

3. Some basic lemmas. First we prove the following lemma.

LEMMA 3.1. Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then

$$\begin{aligned} P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QX} Y) - P(A_{\phi QY} X) \\ = \phi P \nabla_X Y + \phi P \nabla_Y X + 2\alpha g(X, Y) P\xi - \alpha \eta(Y) PX \\ - \alpha \eta(X) PY - \beta \eta(Y) \phi PX + \beta \eta(X) \phi PY, \end{aligned} \quad (3.1)$$

$$\begin{aligned} Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QX} Y) - Q(A_{\phi QY} X) \\ = 2Bh(X, Y) - \alpha \eta(Y) QX - \alpha \eta(X) QY + 2\alpha g(X, Y) Q\xi, \end{aligned} \quad (3.2)$$

$$\begin{aligned} h(X, \phi PY) + h(Y, \phi PX) + \nabla_X^\perp \phi QY + \nabla_Y^\perp \phi QX \\ = \phi Q \nabla_Y X + \phi Q \nabla_X Y + 2Ch(X, Y) - \beta \eta(Y) \phi QX - \beta \eta(X) \phi QY \end{aligned} \quad (3.3)$$

for $X, Y \in TM$.

PROOF. From the definition of the nearly trans-Sasakian manifold and using (2.4), (2.5), and (2.6), we get

$$\begin{aligned} & \nabla_X \phi PY + h(X, \phi PY) - A_{\phi QY} X + \nabla_X^\perp \phi QY - \phi(\nabla_X Y + h(X, Y)) \\ & \quad + \nabla_Y \phi PX + h(Y, \phi PX) - A_{\phi QX} Y + \nabla_Y^\perp \phi QX - \phi(\nabla_Y X + h(X, Y)) \quad (3.4) \\ & = \alpha\{2g(X, Y)\xi - \eta(Y)X - \eta(X)Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} \end{aligned}$$

for any $X, Y \in TM$.

Now using (2.4) and equaling horizontal, vertical, and normal components in (3.4), we get the result. $\square$

LEMMA 3.2. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then*

$$\begin{aligned} 2(\overline{\nabla}_X \phi)(Y) &= \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] \\ & \quad + \alpha\{2g(X, Y)\xi - \eta(Y)X - \eta(X)Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} \quad (3.5) \end{aligned}$$

for any $X, Y \in D$.

PROOF. By Gauss formula (2.6), we get

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = \nabla_X \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X). \quad (3.6)$$

Also, we have

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) + \phi[X, Y]. \quad (3.7)$$

From (3.6) and (3.7), we get

$$(\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) = \nabla_X \phi Y + h(X, \phi Y) - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]. \quad (3.8)$$

Also for nearly trans-Sasakian manifolds, we have

$$(\overline{\nabla}_X \phi)(Y) + (\overline{\nabla}_Y \phi)(X) = \alpha\{2g(X, Y)\xi - \eta(Y)X - \eta(X)Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\}. \quad (3.9)$$

Combining (3.8) and (3.9), the lemma follows. $\square$

In particular, we have the following corollary.

COROLLARY 3.3. *Let M be a ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold, then*

$$2(\overline{\nabla}_X \phi)(Y) = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) - \phi[X, Y] + 2\alpha g(X, Y)\xi \quad (3.10)$$

for any $X, Y \in D$.

Similarly, by Weingarten formula (2.6), we get the following lemma.

LEMMA 3.4. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then*

$$2(\overline{\nabla}_Y \phi)(Z) = A_{\phi Y} Z - A_{\phi Z} Y + \nabla_Y^\perp \phi Z - \nabla_Z^\perp \phi Y - \phi[Y, Z] + \alpha\{2g(Y, Z)\xi - \eta(Y)Z - \eta(Z)Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} \quad (3.11)$$

for any $Y, Z \in D^\perp$.

COROLLARY 3.5. *Let M be a ξ -horizontal CR-submanifold of a nearly trans-Sasakian manifold, then*

$$2(\overline{\nabla}_Y \phi)(Z) = A_{\phi Y} Z - A_{\phi Z} Y + \nabla_Y^\perp \phi Z - \nabla_Z^\perp \phi Y - \phi[Y, Z] + 2\alpha g(Y, Z)\xi \quad (3.12)$$

for any $Y, Z \in D^\perp$.

LEMMA 3.6. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$, then*

$$2(\overline{\nabla}_X \phi)(Y) = \alpha\{2g(X, Y)\xi - \eta(Y)X - \eta(X)Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} - A_{\phi Y} X + \nabla_X^\perp \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] \quad (3.13)$$

for any $X \in D$ and $Y \in D^\perp$.

4. Parallel distributions

DEFINITION 4.1. The horizontal (resp., vertical) distribution D (resp., $D^\perp$) is said to be parallel [1] with respect to the connection ∇ on M if $\nabla_X Y \in D$ (resp., $\nabla_Z W \in D^\perp$) for any vector field $X, Y \in D$ (resp., $W, Z \in D^\perp$).

Now we prove the following proposition.

PROPOSITION 4.2. *Let M be a ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. If the horizontal distribution D is parallel, then*

$$h(X, \phi Y) = h(Y, \phi X) \quad (4.1)$$

for all $X, Y \in D$.

PROOF. Using parallelism of horizontal distribution D , we have

$$\nabla_X \phi Y \in D, \quad \nabla_Y \phi X \in D \quad \text{for any } X, Y \in D. \quad (4.2)$$

Thus using the fact that $QX = QY = 0$ for $Y \in D$, (3.2) gives

$$Bh(X, Y) = g(X, Y)Q\xi \quad \text{for any } X, Y \in D. \quad (4.3)$$

Also, since

$$\phi h(X, Y) = Bh(X, Y) + Ch(X, Y), \quad (4.4)$$

then

$$\phi h(X, Y) = g(X, Y)Q\xi + Ch(X, Y) \quad \text{for any } X, Y \in D. \quad (4.5)$$

Next from (3.3), we have

$$h(X, \phi Y) + h(Y, \phi X) = 2Ch(X, Y) = 2\phi h(X, Y) - 2g(X, Y)Q\xi, \quad (4.6)$$

for any $X, Y \in D$. Putting $X = \phi X \in D$ in (4.6), we get

$$h(\phi X, \phi Y) + h(Y, \phi^2 X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y)Q\xi \quad (4.7)$$

or

$$h(\phi X, \phi Y) - h(Y, X) = 2\phi h(\phi X, Y) - 2g(\phi X, Y)Q\xi. \quad (4.8)$$

Similarly, putting $Y = \phi Y \in D$ in (4.6), we get

$$h(\phi Y, \phi X) - h(X, Y) = 2\phi h(X, \phi Y) - 2g(X, \phi Y)Q\xi. \quad (4.9)$$

Hence from (4.8) and (4.9), we have

$$\phi h(X, \phi Y) - \phi h(Y, \phi X) = g(X, \phi Y)Q\xi - g(\phi X, Y)Q\xi. \quad (4.10)$$

Operating ϕ on both sides of (4.10) and using $\phi\xi = 0$, we get

$$h(X, \phi Y) = h(Y, \phi X) \quad (4.11)$$

for all $X, Y \in D$. □

Now, for the distribution $D^\perp$, we prove the following proposition.

PROPOSITION 4.3. *Let M be a ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. If the distribution $D^\perp$ is parallel with respect to the connection on M , then*

$$(A_{\phi Y}Z + A_{\phi Z}Y) \in D^\perp \quad \text{for any } Y, Z \in D^\perp. \quad (4.12)$$

PROOF. Let $Y, Z \in D^\perp$, then using Gauss and Weingarten formula (2.6), we obtain

$$\begin{aligned} & -A_{\phi Z}Y + \nabla_Y^\perp \phi Z - A_{\phi Y}Z + \nabla_Z^\perp \phi Y \\ & = \phi \nabla_Y Z + \phi \nabla_Z Y + 2\phi h(Y, Z) \\ & \quad + \alpha \{2g(X, Y)\xi - \eta(Y)Z - \eta(Z)Y\} - \beta \{\eta(Y)\phi Z + \eta(Z)\phi Y\} \end{aligned} \quad (4.13)$$

for any $Y, Z \in D^\perp$. Taking inner product with $X \in D$ in (4.13), we get

$$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = g(\nabla_Y Z, \phi X) + g(\nabla_Z Y, \phi X). \quad (4.14)$$

If the distribution $D^\perp$ is parallel, then $\nabla_Y Z \in D^\perp$ and $\nabla_Z Y \in D^\perp$ for any $Y, Z \in D^\perp$. So from (4.14) we get

$$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = 0 \quad \text{or} \quad g(A_{\phi Y}Z + A_{\phi Z}Y, X) = 0 \quad (4.15)$$

which is equivalent to

$$(A_{\phi Y}Z + A_{\phi Z}Y) \in D^\perp \quad \text{for any } Y, Z \in D^\perp, \quad (4.16)$$

and this completes the proof. □

DEFINITION 4.4 [6]. A CR-submanifold is said to be mixed totally geodesic if $h(X, Z) = 0$ for all $X \in D$ and $Z \in D^\perp$.

The following lemma is an easy consequence of (2.7).

LEMMA 4.5. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.*

DEFINITION 4.6 [6]. A normal vector field $N \neq 0$ is called D -parallel normal section if $\nabla_X^\perp N = 0$ for all $X \in D$.

Now we have the following proposition.

PROPOSITION 4.7. *Let M be a mixed totally geodesic ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then the normal section $N \in \phi D^\perp$ is D -parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.*

PROOF. Let $N \in \phi D^\perp$. Then from (3.2) we have

$$Q(\nabla_Y \phi X) = 0 \quad \text{for any } X \in D, Y \in D^\perp. \quad (4.17)$$

In particular, we have $Q(\nabla_Y X) = 0$. By using it in (3.3), we get

$$\nabla_X^\perp \phi QY = \phi Q \nabla_X Y \quad \text{or} \quad \nabla_X^\perp N = -\phi Q \nabla_X \phi N. \quad (4.18)$$

Thus, if the normal section $N \neq 0$ is D -parallel, then using Definition 4.6 and (4.18), we get

$$\phi Q(\nabla_X \phi N) = 0 \quad (4.19)$$

which is equivalent to $\nabla_X \phi N \in D$ for all $X \in D$. The converse part easily follows from (4.18). $\square$

5. Integrability conditions of distributions. First we calculate the Nijenhuis tensor $N_\phi(X, Y)$ on a nearly trans-Sasakian manifold $\overline{M}$. For this, first we prove the following lemma.

LEMMA 5.1. *Let $\overline{M}$ be a nearly trans-Sasakian manifold, then*

$$\begin{aligned} (\overline{\nabla}_{\phi X} \phi)(Y) &= 2\alpha g(\phi X, Y)\xi - \eta(Y)\phi X + \beta\eta(Y)X - \beta\eta(X)\eta(Y)\xi \\ &\quad - \eta(X)\overline{\nabla}_Y \xi + \phi(\overline{\nabla}_Y \phi)(X) + \eta(\overline{\nabla}_Y X)\xi \end{aligned} \quad (5.1)$$

for any $X, Y \in T\overline{M}$.

PROOF. From the definition of nearly trans-Sasakian manifold $\overline{M}$, we have

$$\begin{aligned} (\overline{\nabla}_{\phi X} \phi)(Y) &= 2\alpha g(\phi X, Y)\xi - \eta(Y)\phi X + \beta\eta(Y)X \\ &\quad - \beta\eta(Y)\eta(X)\xi - (\overline{\nabla}_Y \phi)(\phi X). \end{aligned} \quad (5.2)$$

Also, we have

$$\begin{aligned} (\overline{\nabla}_Y \phi)(\phi X) &= \overline{\nabla}_Y \phi^2 X - \phi \overline{\nabla}_Y \phi X \\ &= \overline{\nabla}_Y \phi^2 X - \phi \overline{\nabla}_Y \phi X + \phi(\phi \overline{\nabla}_Y X) - \phi(\phi \overline{\nabla}_Y X) \\ &= -\overline{\nabla}_Y X + \eta(X)\overline{\nabla}_Y \xi - \phi(\overline{\nabla}_Y \phi X - \phi \overline{\nabla}_Y X) - \phi(\phi \overline{\nabla}_Y X) \\ &= -\overline{\nabla}_Y X + \eta(X)\overline{\nabla}_Y \xi - \phi(\overline{\nabla}_Y \phi)(X) - \overline{\nabla}_Y X - \eta(\overline{\nabla}_Y X)\xi. \end{aligned} \quad (5.3)$$

Using (5.3) in (5.2), we get

$$\begin{aligned} (\bar{\nabla}_{\phi X} \phi)(Y) &= 2\alpha g(\phi X, Y)\xi - \eta(Y)\phi X + \beta\eta(Y)X - \beta\eta(X)\eta(Y)\xi \\ &\quad - \eta(X)\bar{\nabla}_Y \xi + \phi(\bar{\nabla}_Y \phi)(X) + \eta(\bar{\nabla}_Y X)\xi \end{aligned} \quad (5.4)$$

for any $X, Y \in T\bar{M}$, which completes the proof of the lemma. $\square$

On a nearly trans-Sasakian manifold $\bar{M}$, Nijenhuis tensor is given by

$$N_\phi(X, Y) = (\bar{\nabla}_{\phi X} \phi)(Y) - (\bar{\nabla}_{\phi Y} \phi)(X) - \phi(\bar{\nabla}_X \phi)(Y) + \phi(\bar{\nabla}_Y \phi)(X) \quad (5.5)$$

for any $X, Y \in T\bar{M}$.

From (5.1) and (5.5), we get

$$\begin{aligned} N_\phi(X, Y) &= 4\alpha g(\phi X, Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y + \beta\eta(Y)X \\ &\quad - \beta\eta(X)Y - \eta(X)\bar{\nabla}_Y \xi + \eta(Y)\bar{\nabla}_X \xi + \eta(\bar{\nabla}_Y X)\xi \\ &\quad - \eta(\bar{\nabla}_X Y)\xi + 2\phi(\bar{\nabla}_Y \phi)(X) - 2\phi(\bar{\nabla}_X \phi)(Y). \end{aligned} \quad (5.6)$$

Thus using (2.3) in the above equation and after some calculations, we obtain

$$\begin{aligned} N_\phi(X, Y) &= 4\alpha g(\phi X, Y)\xi + (2\alpha - 1)\eta(Y)\phi X + (2\alpha + 1)\eta(X)\phi Y - \beta\eta(Y)X \\ &\quad - 3\beta\eta(X)Y + 4\beta\eta(X)\eta(Y)\xi - \eta(X)\bar{\nabla}_Y \xi + \eta(Y)\bar{\nabla}_X \xi \\ &\quad + \eta(\bar{\nabla}_Y X)\xi - \eta(\bar{\nabla}_X Y)\xi + 4\phi(\bar{\nabla}_Y \phi)(X) \end{aligned} \quad (5.7)$$

for any $X, Y \in T\bar{M}$.

Now we prove the following proposition.

PROPOSITION 5.2. *Let M be a ξ -vertical CR-submanifold of a nearly trans-Sasakian manifold $\bar{M}$. Then, the distribution D is integrable if the following conditions are satisfied:*

$$S(X, Z) \in D, \quad h(X, \phi Z) = h(\phi X, Z) \quad (5.8)$$

for any $X, Z \in D$.

PROOF. The torsion tensor $S(X, Y)$ of the almost contact structure (ϕ, ξ, η, g) is given by

$$S(X, Y) = N_\phi(X, Y) + 2d\eta(X, Y)\xi = N_\phi(X, Y) + 2g(\phi X, Y)\xi. \quad (5.9)$$

Thus, we have

$$S(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2g(\phi X, Y)\xi \quad (5.10)$$

for any $X, Y \in TM$.

Suppose that the distribution D is integrable. So for $X, Y \in D$, $Q[X, Y] = 0$ and $\eta([X, Y]) = 0$ as $\xi \in D^\perp$.

If $S(X, Y) \in D$, then from (5.7) and (5.9) we have

$$\{2(2\alpha + 1)g(\phi X, Y)\xi + \eta([X, Y])\xi + 4(\phi \nabla_Y \phi X + \phi h(Y, \phi X) + Q \nabla_Y X + h(X, Y))\} \in D \quad (5.11)$$

or

$$2(2\alpha+1)g(\phi X, Y)Q\xi + \eta([X, Y])Q\xi + 4(\phi Q \nabla_Y \phi X + \phi h(Y, \phi X) + Q \nabla_Y X + h(X, Y)) = 0 \quad (5.12)$$

for any $X, Y \in D$.

Replacing Y by ϕZ for $Z \in D$ in the above equation, we get

$$2(2\alpha+1)g(\phi X, \phi Z)Q\xi + 4(\phi Q \nabla_{\phi Y} \phi X + \phi h(\phi Z, \phi X) + Q \nabla_{\phi Z} X + h(X, \phi Z)) = 0. \quad (5.13)$$

Interchanging X and Z for $X, Z \in D$ in (5.13) and subtracting these relations, we obtain

$$\phi Q[\phi X, \phi Z] + Q[X, \phi Z] + h(X, \phi Z) - h(Z, \phi X) = 0 \quad (5.14)$$

for any $X, Z \in D$ and the assertion follows. $\square$

Now, we prove the following proposition.

PROPOSITION 5.3. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then*

$$A_{\phi Y}Z - A_{\phi Z}Y = \alpha(\eta(Y)Z - \eta(Z)Y) + \frac{1}{3}\phi P[Y, Z] \quad (5.15)$$

for any $Y, Z \in D^\perp$.

PROOF. For $Y, Z \in D^\perp$ and $X \in T(M)$, we get

$$\begin{aligned} 2g(A_{\phi Z}Y, X) &= 2g(h(X, Y), \phi Z) \\ &= g(h(X, Y), \phi Z) + g(h(X, Y), \phi Z) \\ &= g(\overline{\nabla}_X Y, \phi Z) + g(\overline{\nabla}_Y X, \phi Z) \\ &= g(\overline{\nabla}_X Y + \overline{\nabla}_Y X, \phi Z) \\ &= -g(\phi(\overline{\nabla}_X Y + \overline{\nabla}_Y X), Z) \\ &= -g(\overline{\nabla}_Y \phi X + \overline{\nabla}_X \phi Y + \alpha\eta(X)Y + \alpha\eta(Y)X - 2\alpha g(X, Y)\xi, Z) \\ &= -g(\overline{\nabla}_Y \phi X, Z) - g(\overline{\nabla}_X \phi Y, Z) - \alpha\eta(X)g(Y, Z) \\ &\quad - \alpha\eta(Y)g(X, Z) + 2\alpha\eta(Z)g(X, Y) \\ &= g(\overline{\nabla}_Y Z, \phi X) + g(A_{\phi Y}Z, X) - \alpha\eta(X)g(Y, Z) \\ &\quad - \alpha\eta(Y)g(X, Z) + 2\alpha\eta(Z)g(X, Y). \end{aligned} \quad (5.16)$$

The above equation is true for all $X \in T(M)$, therefore, transvecting the vector field X both sides, we obtain

$$2A_{\phi Z}Y = A_{\phi Y}Z - \phi \overline{\nabla}_Y Z - \alpha g(Y, Z)\xi - \alpha\eta(Y)Z + 2\alpha\eta(Z)Y \quad (5.17)$$

for any $Y, Z \in D^\perp$. Interchanging the vector fields Y and Z , we get

$$2A_{\phi Y}Z = A_{\phi Z}Y - \phi \overline{\nabla}_Z Y - \alpha g(Y, Z)\xi - \alpha\eta(Z)Y + 2\alpha\eta(Y)Z. \quad (5.18)$$

Subtracting (5.17) and (5.18), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = \alpha(\eta(Y)Z - \eta(Z)Y) + \frac{1}{3}\phi P[Y, Z] \quad (5.19)$$

for any $Y, Z \in D^\perp$, which completes the proof. $\square$

THEOREM 5.4. *Let M be a CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then, the distribution $D^\perp$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = \alpha(\eta(Y)Z - \eta(Z)Y), \quad \text{for any } Y, Z \in D^\perp. \quad (5.20)$$

PROOF. First suppose that the distribution $D^\perp$ is integrable. Then $[Y, Z] \in D^\perp$ for any $Y, Z \in D^\perp$. Since P is a projection operator on D , so $P[Y, Z] = 0$. Thus from (5.15) we get (5.20). Conversely, we suppose that (5.20) holds. Then using (5.15), we have $\phi P[Y, Z] = 0$ for any $Y, Z \in D^\perp$. Since $\text{rank } \phi = 2n$. Therefore, either $P[Y, Z] = 0$ or $P[Y, Z] = k\xi$. But $P[Y, Z] = k\xi$ is not possible as P is a projection operator on D . Thus, $P[Y, Z] = 0$, which is equivalent to $[Y, Z] \in D^\perp$ for any $Y, Z \in D^\perp$ and hence $D^\perp$ is integrable. $\square$

COROLLARY 5.5. *Let M be a ξ -horizontal CR-submanifold of a nearly trans-Sasakian manifold $\overline{M}$. Then, the distribution $D^\perp$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = 0 \quad (5.21)$$

for any $Y, Z \in D^\perp$.

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