SUPER AND SUBSOLUTIONS FOR ELLIPTIC EQUATIONS ON ALL OF $\mathbb{R}^n$

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Received 26 October 2001

By construction sub and supersolutions for the following semilinear elliptic equation $-\Delta u(x) = \lambda g(x)f(u(x))$, $x \in \mathbb{R}^n$, which arises in population genetics, we derive some results about the theory of existence of solutions as well as asymptotic properties of the solutions for every n and for the function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that g is smooth and is negative at infinity.

2000 Mathematics Subject Classification: 35J60.

1. Introduction. In this paper, we discuss the existence and nonexistence of solutions as well as asymptotic properties of the solutions of the equation

$$-\Delta u(x) = \lambda g(x)f(u(x)), \quad x \in \mathbb{R}^n, \quad 0 \leq u(x) \leq 1 \quad (1.1)$$

which arises in population genetics (see [7, 11]). The unknown function u corresponds to the relative frequency of an allele and is hence constrained to have values between 0 and 1. The real parameter $\lambda > 0$ corresponds to the reciprocal of a diffusion coefficient.

We assume throughout that $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth which changes sign on $\mathbb{R}^n$. Also we will assume throughout that f satisfies the condition $f : [0, 1] \rightarrow \mathbb{R}$ is a smooth function such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and $f(u) > 0$ for all $0 < u < 1$.

By the definition of f , it is clear that (1.1) has the trivial solutions $u \equiv 0$ and $u \equiv 1$.

The existence of solutions for (1.1) in the bounded region case with Dirichlet or Neumann boundary conditions is discussed in [7, 11], but in this case all of $\mathbb{R}^n$ is much more complicated (see [3, 6, 7, 8, 9, 12, 13]). The results obtained in [7] with the assumption that g is negative at infinity show that the existence theory for solutions of (1.1) is very different for the two cases $n = 1, 2$ and $n \geq 3$.

Some of the nontrivial solutions were bifurcating off the trivial solution $u \equiv 0$. In order to investigate these bifurcation phenomena, it was necessary to understand the eigenvalues and eigenfunctions of the corresponding linearized problem

$$-\Delta u(x) = \lambda g(x)f'(0)u(x), \quad x \in \mathbb{R}^n. \quad (1.2)$$

The existence of positive principal eigenfunctions of (1.2) with the following conditions on g was considered in [6]:

(i) g is negative and bounded away from zero at infinity; or

(ii) $|g(x)| \leq k/(1 + |x|^2)^\alpha$, $n \geq 3$,

for some constants $k > 0$ and $\alpha > 1$, and these results for the case $g^+ \in L^{n/2}(\mathbb{R}^n)$, $n \geq 3$ where $g^+(x) = \max\{g(x), 0\}$ are extended in [3].

In this paper, we investigate the existence of solutions of (1.1) with the assumption that g or g^+ are small at infinity.

Our analysis is based on the construction of sub and supersolutions.

It is proved in [2] that the positive principal eigenvalue of the Dirichlet boundary value problem

$$\begin{aligned} -\Delta u(x) &= \lambda g(x)u(x), \quad x \in D, \\ u(x) &= 0, \quad x \in \partial D, \end{aligned} \quad (1.3)$$

where D is a bounded domain with smooth boundary has the variational characterization

$$\lambda_1^+(D) = \inf \left\{ \int_D |\nabla u(x)|^2 dx : u \in H_0^1(D), \int_D g u^2 dx = 1 \right\}. \quad (1.4)$$

Also, it is well known that the above infimum is attained and a minimizer $\phi_1 > 0$ is smooth, that is, $c^2(\overline{D})$. Hence ϕ_1 satisfies the Dirichlet boundary value problem (1.3), so ϕ_1 is a principal eigenfunction corresponding to principal eigenvalue $\lambda_1^+(D)$.

Suppose, however, that $g = g^+ - g^-$ where $g^+(x) = \max\{g(x), 0\}$ and $g^-(x) = \min\{g(x), 0\}$.

If $n \geq 3$ and $g^+ \in L^{n/2}(\mathbb{R}^n)$, then for all $u \in H_0^1(D)$ such that $\int_D g u^2 dx = 1$ we have

$$\begin{aligned} 1 &= \int_D g u^2 dx \leq \int_D g^+ u^2 dx \\ &\leq \|g^+\|_{L^{n/2}(D)} \|u\|_{L^{2n/(n-2)}(D)}^2 \\ &\leq c(n) \|g^+\|_{L^{n/2}(D)} \|\nabla u\|_{L^2(D)}^2, \end{aligned} \quad (1.5)$$

where $c(n)$ is the embedding constant of $H_0^1(D)$ into $L^{2n/(n-2)}(D)$ and is independent of D (see Brézis and Nirenberg [5, page 443]). Thus

$$\lambda_1^+(D) \geq \|\nabla u\|_{L^2(D)}^2 \geq \{c(n) \|g^+\|_{L^{n/2}(D)}\}^{-1} > 0. \quad (1.6)$$

Also, it is well known (see [1]) that if $g^+ \in L^{n/2}(\mathbb{R}^n)$, then $\lambda^* = \lim_{R \rightarrow \infty} \lambda_1^+(B_R(0))$ exists and λ^* is the principal eigenvalue of the equation

$$-\Delta u(x) = \lambda g(x)u(x), \quad x \in \mathbb{R}^n \quad (1.7)$$

and there exists a corresponding principal eigenfunction ϕ such that $\phi(x) \rightarrow 0$ as $|x| \rightarrow \infty$. In addition, λ^* can be characterized as follows (see [1, Lemma 2.7])

$$\lambda^* = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx : u \in C_0^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} g u^2 dx = 1 \right\}. \quad (1.8)$$

THEOREM 1.1 (see [10]). *If $\lambda > \lambda^*$, then there exists $\underline{u} \geq 0$ ($\underline{u} \neq 0$) with compact support such that $\underline{u}$ is a subsolution of*

$$\begin{aligned} -\Delta u(x) &= \lambda g(x)f(u(x)), \quad x \in B_R(0), \\ u(x) &= 0, \quad x \in \partial B_R(0) \end{aligned} \quad (1.9)$$

for all R sufficiently large, also we can choose $\underline{u}$ sufficiently small.

2. Sub and supersolutions for $n \geq 3$. We assume $D \subset \mathbb{R}^n$ is a bounded region with smooth boundary. We consider the following boundary value problem:

$$\begin{aligned} -\Delta u(x) &= \lambda g(x)f(u(x)), \quad x \in D, \\ u(x) &= 0, \quad x \in \partial D. \end{aligned} \quad (2.1)$$

If $\lambda > 0$ be fixed, we can choose $c > 0$ such that for u , $0 \leq u \leq 1$, the function $u \rightarrow \lambda g(x)f(u) + cu$, for every $x \in D$, is an increasing function.

Let $h(x, u) = \lambda g(x)f(u) + cu$, then we have $h(x, 0) \equiv 0$ and $h(x, 1) \equiv c$. We can write (2.1) as

$$\begin{aligned} -\Delta u(x) + cu(x) &= h(x, u(x)), \quad x \in D, \\ u(x) &= 0, \quad x \in \partial D. \end{aligned} \quad (2.2)$$

It is well known that (2.2) has a unique solution $u = Kf$ (see Amann [4]), where K is given by an integral operator whose kernel is the Green's function for the problem, that is,

$$(Kf)(x) = \int_D G(x, y)h(y, u(y))dy. \quad (2.3)$$

In (2.3), $G(x, y)$ is the Green's function of the operator $-\Delta + c$ with Dirichlet boundary condition, also we can write (2.3) as $u = KN(u)$ in where $K : c(\overline{D}) \rightarrow c^\alpha(\overline{D})$ is a compact linear integral operator with kernel G (see [4]) and $N : c(\overline{D}) \rightarrow c(\overline{D})$ is the Nemytskii operator corresponding to h . Since $h(x, \cdot)$ is increasing, it is easy to see that N is an increasing operator, that is, if $u_1 \geq u_2$, then $Nu_1 \geq Nu_2$.

We call $u \in c^2(D)$ is a subsolution of (2.2) or equivalently (2.1) if we have

$$\begin{aligned} -\Delta u(x) + cu(x) &\leq h(x, u(x)), \quad x \in D, \\ u(x) &\leq 0, \quad x \in \partial D, \end{aligned} \quad (2.4)$$

and $u \in c(\overline{D})$ is a subsolution of (2.3) if

$$u(x) \leq \int_D G(x, y)h(y, u(y))dy, \quad x \in D, \quad (2.5)$$

that is, $u \leq KN(u)$. The definition of supersolution is quite similar.

It is well known that if v, w are sub and supersolutions of (2.2) (or for (2.3)), respectively, and $v \leq w$, then there exists a solution u of (2.2) (of (2.3)) such that $v \leq u \leq w$.

3. The case when $n = 1, 2$. In this section, we consider the problem

$$\begin{aligned} -\Delta u(x) &= \lambda g(x)f(u(x)), \quad x \in \mathbb{R}^n, \\ 0 &\leq u(x) \leq 1, \quad x \in \mathbb{R}^n, \end{aligned} \quad (3.1)$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function which changes sign on $\mathbb{R}^n$ and it has the following condition: (G) there exists $R_0 > 0$ such that $g(x) < 0$ for all of $x \in \mathbb{R}^n$, whenever $|x| > R_0$.

Also $f \in C^1([0, 1])$ with the conditions

$$f(0) = 0 = f(1), \quad f'(0) > 0, \quad f'(1) < 0, \quad f(u) > 0, \quad 0 < u < 1. \quad (3.2)$$

THEOREM 3.1 (see [7]). *Let u be a nontrivial solution of (4.1). Then there exists a real constant k such that $0 < u(x) < k < 1$ for all of x in $\mathbb{R}^n$.*

Now by using Theorem 3.1 and condition (G) on g , we conclude that

$$\Delta u(x) > 0 \quad (3.3)$$

for all of $x \in \mathbb{R}^n$ with $|x| > R_0$.

THEOREM 3.2. *Let u be a nontrivial solution of (4.1). Then u is nonconstant in out of the ball $B_{R_0}(0)$.*

PROOF. Using assumption on g , we have $\Delta u(x) > 0$ for all of $x \in \mathbb{R}^n$ with $|x| > R_0$, so $|\nabla u(x)| > 0$ whenever $|x| > R_0$. Hence u is a nonconstant function in out of the ball $B_{R_0}(0)$. $\square$

THEOREM 3.3. *Let $n = 1$ and u be a nontrivial solution of (4.1). Then u is a strictly decreasing function on (R_0, ∞) and increasing function on $(-\infty, -R_0)$.*

PROOF. By using assumption on g , we have $u''(x) > 0$ for all of $x \in \mathbb{R}^n$ with $|x| > R_0$. So, u can have only one of the possibilities (a) and (b) in Figure 3.1.

Figure 3.1(a) is impossible because we must have $0 \leq u(x) \leq 1$ for all $x \in \mathbb{R}^n$. So u satisfy in Figure 3.1(b), thus u is strictly decreasing in out of ball $B_{R_0}(0)$. $\square$

THEOREM 3.4. *Let $n = 2$ and u be a solution of (4.1) which is radially symmetric, then u is a strictly monotone function in out of the ball $B_{R_1}(0)$, where $R_1 > R_0$.*

PROOF. It is obvious by using maximum principle. $\square$

4. The case when $n \geq 3$. Let g satisfy condition (G). It is easy to see that

$$\bar{u}(x) = \begin{cases} 1, & |x| \leq R_0, \\ \left(\frac{R_0}{|x|}\right)^{(n-2)}, & |x| > R_0, \end{cases} \quad (4.1)$$

is a supersolution of (4.1), so we are ready to prove the following theorem.

THEOREM 4.1. *If $\lambda > \lambda^*$, then there exists a nonconstant solution u of (4.1) such that*

$$\lim_{|x| \rightarrow \infty} u(x) = 0. \quad (4.2)$$

FIGURE 3.1

PROOF. We consider $\bar{u}$ as a supersolution of (4.1). Also there exists a subsolution $\underline{u}$ of (4.1) with compact support and sufficiently small (see [10]). So we can choose $\underline{u}$ such that $\underline{u} \leq \bar{u}$, so there exists a solution u of (4.1) such that $\underline{u} \leq u \leq \bar{u}$. Also by using the definition of $\bar{u}$, we have $\lim_{|x| \rightarrow \infty} u(x) = 0$. $\square$

THEOREM 4.2. Let $\alpha > 1$ and $\lambda > 0$ be arbitrary. Then there exists a supersolution $\bar{u}$ of (4.1) such that $|\bar{u}(x)| \leq c|x|^{-\beta}$ for a constant $c > 0$, and

$$\beta = \begin{cases} n-2, & n < 2\alpha, \\ 2\alpha-2, & n > 2\alpha. \end{cases} \quad (4.3)$$

PROOF. Using condition (G) of the function g , we have

$$|g^+(x)| \leq \frac{k}{(1+|x|^2)^\alpha}, \quad (4.4)$$

where $k \geq M(1+R_0^2)^\alpha$, $M = \max g^+(x)$. So using [10, Lemma 4.3], the proof is complete. $\square$

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