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ON THE DIOPHANTINE EQUATION $x^2+p^{2k+1}=4y^n$

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It has been proved that if p is an odd prime, $y>1$, $k\geq 0$, n is an integer greater than or equal to 4, $(n,3h)=1$ where h is the class number of the field $Q(\sqrt{-p})$, then the equation $x^2+p^{2k+1}=4y^n$ has exactly five families of solution in the positive integers x, y. It is further proved that when $n=3$ and $p=3a^2\pm 4$, then it has a unique solution $k=0$, $y=a^2\pm 1$.

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1. Introduction. The purpose of this note is to compute positive integral solutions of the equation $x^2+p^{2k+1}=4y^n$, where p is an odd prime and n is any integer greater than or equal to 3. The special case when $p=3$ and $k=0$ was treated by Nagell [7] and Ljunggren [3] who proved that this equation has the only solutions $y=1$ and $y=7$ with $n=3$. Later on, Ljunggren [4, 5], Persson [8], and Stolt [9] studied the general equation $x^2+D=4y^n$ and proved that it has a solution under certain necessary conditions on D. Le [2] and Mignotte [6] proved that the equation $D_1x^2+D_2^m=4y^n$ has a finite number of solutions under certain conditions on m and n but did not compute these solutions. We will prove the following theorem.

Theorem 1.1. *The Diophantine equation*

$$x^2+p^{2k+1}=4y^n,\quad y>1, \tag{1.1}$$

where p is an odd prime, $k\geq 0$, n is an integer greater than or equal to 4, $(n,3h)=1$, where h is the class number of the field $Q(\sqrt{-p})$ has exactly five families of solutions given in Table 1.1.

Table 1.1

p	n	k	x	y
7	5	$5M$	$11\cdot 7^{5M}$	$2\cdot 7^{2M}$
7	13	$13M$	$181\cdot 7^{13M}$	$2\cdot 7^{2M}$
7	7	$7M+1$	$13\cdot 7^{7M}$	$2\cdot 7^{2M}$
11	5	$5M$	$31\cdot 11^{5M}$	$3\cdot 11^{2M}$
19	7	$7M$	$559\cdot 19^{7M}$	$5\cdot 19^{2M}$

We start by the usual method of factorizing in the field $Q(\sqrt{-p})$, then we use a recent result of Bilu et al. [1], about primitive divisors of a Lucas number.

We start by giving some important definitions.

Definition 1.2. A Lucas pair is a pair (α,β) of algebraic integers, such that $\alpha+\beta$ and $\alpha\beta$ are nonzero coprime rational integers and α/β is not a root of unity. Given a Lucas pair (α,β), we define the corresponding sequence of Lucas numbers by $u_n(\alpha,\beta)=(\alpha^n-\beta^n)/(\alpha-\beta)$ (where $n=0,1,2,\dots$).

A prime number p is a primitive divisor of $u_n(\alpha,\beta)$ if p divides u_n, but does not divide $(\alpha-\beta)^2u_1u_2\cdots u_{n-1}$.

The following result has been proved in [1].

Lemma 1.3. *For $n>30$, the nth term of any Lucas sequence has a primitive divisor.*

Also in [1], for $5\le n\le 30$, all values of the pairs (α,β) have been listed for which the nth term of the Lucas sequence $u_n(\alpha,\beta)$ has no primitive divisors.

We first consider the case when $(p,x)=1$ and prove the following theorem.

Theorem 1.4. *Equation (1.1), where n and p satisfy the conditions of Theorem 1.1, has no solution in the positive integers x when $(p,x)=1$ except when $p=7,11$, or 19.*

Proof. First suppose that n is an odd integer. Without loss of generality, we can suppose that n is an odd prime. Factorizing (1.1), we obtain

$$\left(\frac{x+p^k\sqrt{-p}}{2}\right)\cdot\left(\frac{x-p^k\sqrt{-p}}{2}\right)=y^n. \tag{1.2}$$

We can easily verify that the two numbers on the left-hand side are relatively prime integers in $Q(\sqrt{-p})$. So that

$$\frac{x+p^k\sqrt{-p}}{2}=\left(\frac{a+b\sqrt{-p}}{2}\right)^n, \tag{1.3}$$

where a and b are rational integers such that $a\equiv b(\bmod 2)$ and $4y=a^2+pb^2$, where $(a,pb)=1$.

Let

$$\alpha=\frac{a+b\sqrt{-p}}{2},\qquad \bar{\alpha}=\frac{a-b\sqrt{-p}}{2}. \tag{1.4}$$

Then from (1.3), we get

$$\frac{\alpha^n-\bar{\alpha}^n}{\alpha-\bar{\alpha}}=\frac{p^k}{b}. \tag{1.5}$$

By equating imaginary parts in (1.3), we can easily conclude from (1.5) that

$$\frac{\alpha^n-\bar{\alpha}^n}{\alpha-\bar{\alpha}}=\begin{cases}\pm 1 & \text{if } (p,n)=1,\\ \pm p & \text{if } n\mid p.\end{cases} \tag{1.6}$$

It can be verified that $(\alpha,\bar{\alpha})$ is a Lucas pair as defined earlier and the only positive prime divisor of the corresponding nth Lucas number

$$u_n=\frac{\alpha^n-\bar{\alpha}^n}{\alpha-\bar{\alpha}} \tag{1.7}$$

is p which is not a primitive divisor because it divides $(\alpha-\bar{\alpha})^2=pb^2$. So the Lucas number defined in (1.7) has no primitive divisors. Using Lemma 1.3 and [1, Table 2], we deduce that (1.1) has no solutions when $n>13$. When $5\le n\le 13$, again using [1, Table 2], we find all values of α for which the Lucas number $u_n(\alpha,\beta)$ has no primitive divisors. We consider each value of n separately.

When $n=13$, then $\alpha=(1+\sqrt{-7})/2$ which correspondingly gives $k=0$, $a=1$, $b=1$, $p=7$ and consequently, $y=(a^2+pb^2)/4=2$, $x=181$ is the only solution of the equation $x^2+p^{2k+1}=4y^{13}$.

When $n=11$, there is no α for which $u_{11}(\alpha,\bar{\alpha})$ has no primitive divisors and so no solution of (1.1).

When $n=7$, the values of α for which $u_7(\alpha,\bar{\alpha})$ has no primitive divisors, are $\alpha=(1+\sqrt{-7})/2,(1+\sqrt{-19})/2$ which give $y=2$ as a solution of $x^2+7^3=4y^7$ $(x=13)$ and $y=5$ as a solution of $x^2+19=4y^7$ $(x=559)$. Similarly, for $n=5$, we get $y=2$ as a solution of $x^2+7=4y^5$ $(x=11)$ and $y=3$ as a solution of $x^2+11=4y^5$ $(x=31)$.

Now we will prove that there is no solution for (1.1) when n is even. It suffices to consider that $n=4$.

Factorizing $x^2+p^{2k+1}=4y^4$, we get

$$(2y^2+x)\cdot(2y^2-x)=p^{2k+1}. \tag{1.8}$$

Since $(p,x)=(p,y)=1$, then

$$2y^2+x=p^{2k+1}, \qquad 2y^2-x=1 \tag{1.9}$$

which gives $4y^2=p^{2k+1}+1$. This can easily be checked to have no solution with $y>1$. □

Proof of Theorem 1.1. Suppose that $p\mid x$. Let $x=p^{\lambda}x_1$, $y=p^{\mu}y_1$, where $(x_1,p)=(y_1,p)=1$ and $\lambda,\mu\ge 1$. Substituting in (1.1), we get

$$p^{2\lambda}\cdot x_1^2+p^{2k+1}=4p^{n\mu}\cdot y_1^n. \tag{1.10}$$

We have the following three cases.

Case 1. If $2\lambda=\min(2\lambda,2k+1,n\mu)$, then

$$x_1^2+p^{2k-2\lambda+1}=4p^{n\mu-2\lambda}\cdot y_1^n. \tag{1.11}$$

This equation is impossible modulo p unless $n\mu-2\lambda=0$, and then we get $x_1^2+p^{2(k-\lambda)+1}=4y_1^n$, where $(x_1,p)=(y_1,p)=1$. According to Theorem 1.4, this equation has no solution for all $n\ge 4$ except when $n=13,7,5$, $k=\lambda$, and $n=7$, $k=\lambda+1$.

Accordingly, when $n=13$, we have $13\mu=2\lambda$, then $\lambda=13M$, $\mu=2M$ and so the solutions of (1.1) are $p=7$, $x=181\cdot 7^{13M}$, $y=2\cdot 7^{2M}$. Similarly, considering $n=5,7$, we get exactly the families of solutions given in the statement of Theorem 1.1.

Case 2. If $2k+1=\min(2\lambda,2k+1,n\mu)$, then

$$p^{2\lambda-2k-1}\cdot x_1^2+1=4p^{n\mu-2k-1}\cdot y_1^n. \tag{1.12}$$

This equation is known to have no solution [7].

Case 3. If $n\mu = \min(2\lambda, 2k+1, n\mu)$, then

$$p^{2\lambda-n\mu}\cdot x_1^2 + p^{2k+1-n\mu} = 4y_1^n. \tag{1.13}$$

This equation is possible only if $2\lambda - n\mu = 0$ or $2k+1-n\mu = 0$. If $2\lambda - n\mu = 0$, we get $x_1^2 + p^{2(k-\lambda)+1} = 4y_1^n$, which is an equation of the same form as considered in Case 1.

If $2k+1-n\mu = 0$, we get $p(p^{\lambda-k-1}\cdot x_1)^2 + 1 = 4y_1^n$, which is known to have no solution [6]. This completes the proof of Theorem 1.1. □

Note 1.5. When $n = 3$, factorizing (1.1), we get

$$\frac{x + 3^k\sqrt{-3}}{2} = \varepsilon\left(\frac{a + b\sqrt{-3}}{2}\right)^3, \tag{1.14}$$

$$\frac{x + p^k\sqrt{-p}}{2} = \left(\frac{a + b\sqrt{-p}}{2}\right)^3, \quad p \neq 3, \tag{1.15}$$

where $\varepsilon = \omega$ or ω^2 and ω is a cube root of unity. From (1.14), we easily deduce that $k = 0$ and $y = 1$ and 7 are the only solutions as proved in [3]. We treat (1.15) by the same way as before by taking $\alpha = (a + b\sqrt{-p})/2$ and $\bar{\alpha} = (a - b\sqrt{-p})/2$, so we get $(\alpha^3 - \bar{\alpha}^3)/(\alpha - \bar{\alpha}) = \pm 1$. It can be easily proved that $(\alpha, \bar{\alpha})$ is a Lucas pair as defined above. Using [1, Table 2], we find the following two values of α for which the Lucas number $u_3(\alpha, \bar{\alpha})$ has no primitive divisors:

$$\alpha = \begin{cases} \dfrac{m + \sqrt{\pm 4 - 3m^2}}{2}, & m > 1, \\ \dfrac{m + \sqrt{\pm 4\cdot 3^k - 3m^2}}{2}, & m \not\equiv 0 \pmod 3), \end{cases} \tag{1.16}$$

where $(k, m) \neq (1, 2)$.

The first value of α gives $b = 1$, $k = 0$ and consequently, $p = 3a^2 \pm 4$, $y = a^2 \pm 1$, and $x = a(2a^2 \pm 3)$ is the solution of (1.1) with $n = 3$. No solution is found for the second value of α since $p \neq 3$. Hence, we have the following theorem.

Theorem 1.6. *The Diophantine equation*

$$x^2 + p^{2k+1} = 4y^3, \quad (p, x) = 1 \tag{1.17}$$

has the only solutions $k = 0$ and $y = 1$ and 7 when $p = 3$. When p is a prime greater than 3, such that $(3, h) = 1$, where h is the class number of the field $Q(\sqrt{-p})$, then it has solutions only if $p = 3a^2 \pm 4$, and then the solution is $k = 0$, $y = a^2 \pm 1$, and $x = a(2a^2 \pm 3)$.

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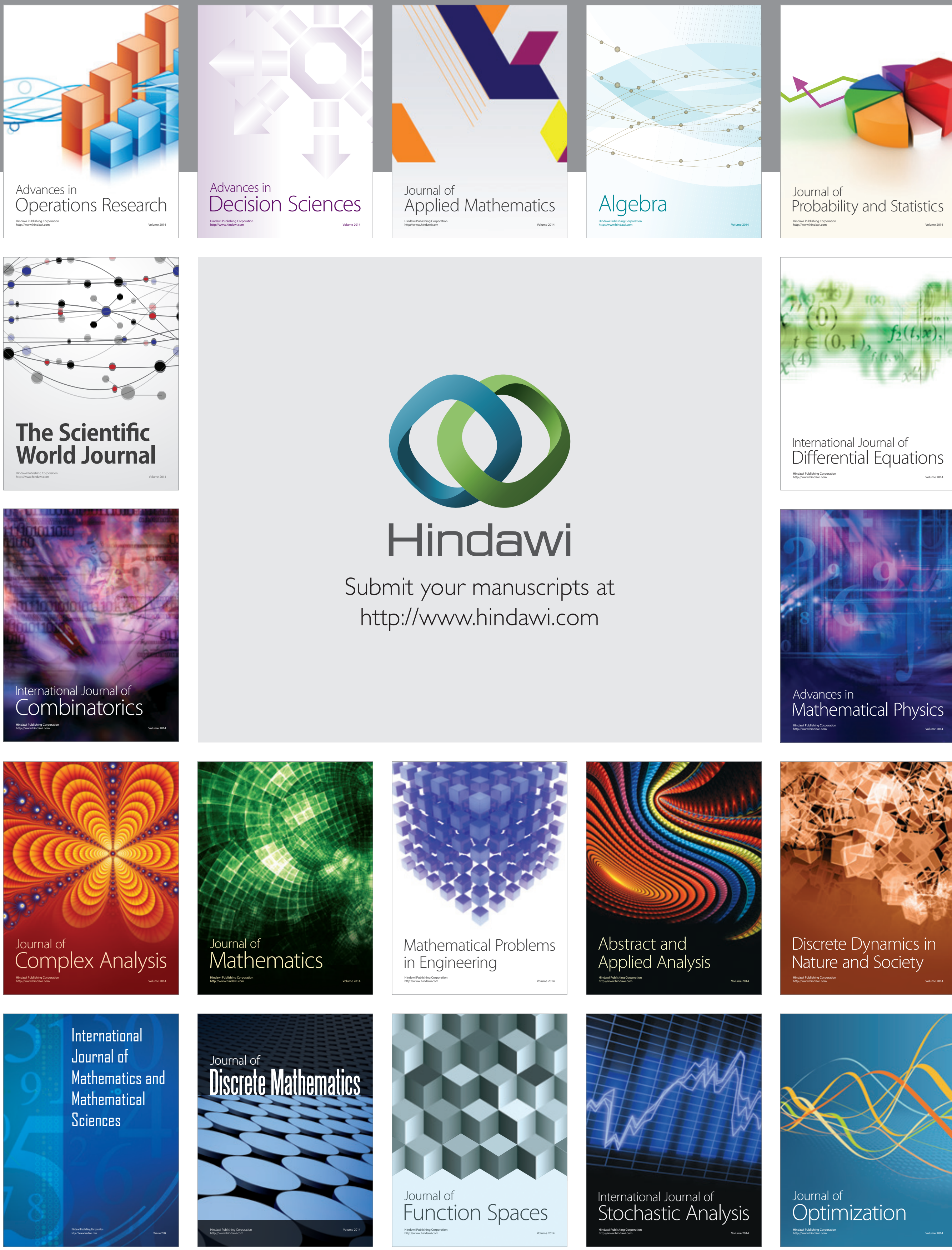
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