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AN EQUIVALENCE THEOREM CONCERNING POPULATION GROWTH IN A VARIABLE ENVIRONMENT

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We give conditions under which two solutions x and y of the Kolmogorov equation $\dot{x} = xf(t,x)$ satisfy $\lim y(t)/x(t) = 1$ as $t \to \infty$. This conclusion is important for two reasons: it shows that the long-time behavior of the population is independent of the initial condition and it applies to ecological systems in which the coefficients are time dependent. Our first application is to an equation of Weissing and Huisman for growth and competition in a light gradient. Our second application is to a nonautonomous generalization of the Turner-Bradley-Kirk-Pruitt equation, which even before generalization, includes several problems of ecological interest.

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1. Introduction. The Kolmogorov equation [4] of population growth in a temporally variable environment is

$$\dot{x} = xf(t,x), \quad x(0) > 0, \tag{1.1}$$

where x is a real-valued function of the real variable t. We assume that $f(t,x)$ is continuous for $t \geq 0$ and $x > 0$ and that solutions of (1.1) exist and are positive for $t > 0$. If $f(t,x)$ is continuous for $x \geq 0$, then $\dot{x} = x\varphi(t)$, where $\varphi(t) = f(t,x(t))$ is continuous, and the solution is automatically positive.

Two solutions x and y are said to be *asymptotically equivalent*, and we write $x \sim y$, if

$$\lim_{t\to\infty} \frac{y(t)}{x(t)} = 1. \tag{1.2}$$

The relation $x \sim y$ is an *equivalence* in that

$$x \sim x, \qquad x \sim y \Longleftrightarrow y \sim x, \qquad x \sim y, \qquad y \sim z \Longrightarrow x \sim z. \tag{1.3}$$

If x is bounded, $x \sim y$ implies

$$\lim_{t\to\infty} \left| x(t) - y(t) \right| = 0. \tag{1.4}$$

When $x \sim y$ or $y - x \to 0$ for all solutions in a suitable class K, these solutions have essentially the same long-time behavior, and the effect of initial conditions is transient. That is why the relation is important.

Our main result gives conditions under which $x \sim y$. Although we have called it an equivalence theorem, it is also a stability theorem. We recall that a stationary solution $y = a$ is *asymptotically stable* relative to a given class K if every other solution $y \in K$ satisfies $\lim_{t\to\infty} y(t) = a$. When $a \neq 0$, this is the same as $\lim_{t\to\infty}(y/a) = 1$. In our theory, the role of a is taken by any particular solution $x \in K$, and instead of $\lim y(t) = a$ we require $y \sim x$. Either solution, x or y, can be viewed as a perturbation of the other obtained by a change in the initial condition. Hence, our main theorem asserts that any given solution is asymptotically stable with respect to changes of initial conditions, provided that we stay within the class K. An earlier investigation with a similar objective is given by Cohen [1]. However, Cohen's model is stochastic and its analysis involves a study of certain matrix products (Hajnal [2]). The present study, though thematically related to Cohen's, employs completely different methods. Further details, including clarification of the class K, are given after the main theorem.

We will apply our theory to equations

$$\dot{x} = g(t,x) \tag{1.5}$$

in which the factor x on the right is missing. For example, the remarkable investigations of Huisman and Weissing [3, 11] lead to an equation of that kind. However, (1.5) can be put in the form (1.1) provided that $g(t,x)$ is continuous, $g(t,0) = 0$, and

$$g_x = \frac{\partial g}{\partial x} \tag{1.6}$$

exists as a right-hand derivative at $(t,0)$. In that case we can replace (1.5) by $\dot{x} = xf(t,x)$, where $f(t,x) = g(t,x)/x$ for $x \neq 0$ and $f(t,0) = g_x(t,0)$. This remark is used below.

2. The equivalence theorem. The following hypothesis is due to Vance and Coddington [9]:

$$f_x(t,x) \leq -\gamma(x)\lambda(t), \qquad \int_0^\infty \lambda(t)\,dt = \infty. \tag{2.1}$$

Here λ is continuous and nonnegative for $t \geq 0$ and $\gamma(x)$ is continuous and positive for $x > 0$.

This hypothesis in isolation leads nowhere. The reason is that without some other condition there is no way of knowing that

$$\int_0^\infty \gamma(x(t))\lambda(t)\,dt = \infty, \tag{2.2}$$

and this is what is actually needed in the analysis. In [9, Theorems 3 and 5], it is explicitly required that

$$M_1 \le x(t) \le M_2, \quad 0 \le t < \infty, \tag{2.3}$$

where M_i are positive constants depending on the solution x. In Theorem 7.1, a condition of this kind is obtained by denying the conclusion. On the interval $M_1 \le x \le M_2$ the continuous function $y(x)$ has a positive lower bound δ, so the Vance-Coddington hypothesis in its entirety implies

$$f_x(x,t) \le -\delta\lambda(t), \qquad \int_0^\infty \lambda(t)dt = \infty. \tag{2.4}$$

If two solutions x and y are being considered, with constants δ_1 and δ_2, respectively, we have (2.4) with

$$\delta = \min(\delta_1, \delta_2). \tag{2.5}$$

Upon replacing λ by $\delta\lambda$, we are led to the following new hypothesis.

Condition E*I*. Let I be an open interval, finite or infinite. The function $f(t,x)$ satisfies condition EI if there exists a continuous function $\lambda \ge 0$, depending on I, such that

$$\int_0^\infty \lambda(t)dt = \infty, \qquad \xi \in I \Longrightarrow f_x(t,\xi) \le -\lambda(t). \tag{2.6}$$

The letters EI are intended to suggest the phrase "equivalence relative to I," and Theorem 2.1 will be referred to as the *equivalence theorem*. Condition EI will be applied with I a value interval for the solutions $x(t)$ under consideration, that is, an interval such that $x(t) \in I$ for $t \ge 0$. It is not required that I be the smallest interval with this property.

The value interval I plays the same role that in the Vance-Coddington theory is taken by their hypothesis involving y together with the assumption that the solutions being considered are bounded away from 0 and ∞. The new formulation is especially useful for equations containing a parameter k, since it may happen that all solutions exceeding k belong to one equivalence class while those less than k belong to another. An example is given in connection with the Turner-Bradley-Kirk-Pruitt equation below.

Here is our main theorem.

Theorem 2.1. *Suppose x and y are two solutions of (1.1) with a common value interval $I = I(x) = I(y)$ relative to which condition* EI *holds. Then*

$$\inf y(t) > 0 \Longrightarrow \lim_{t\to\infty} \frac{y(t)}{x(t)} = 1. \tag{2.7}$$

In the light of this theorem, solutions x and y are considered to be "in a suitable class" if $I(x) = I(y) = I$. The equivalence theorem implies that if one

solution with $I(x) = I$ is bounded away from zero, then all solutions are, and if in addition one is bounded above, then all solutions are. It generalizes and sharpens [9, Theorem 5]. The latter does not introduce the value interval I and assumes a priori that both solutions are bounded away from 0 and ∞.

Proof. Let $v = y/x$. Then

$$\dot{v} = \frac{x\dot{y} - y\dot{x}}{x^2} = \frac{y}{x}(f(t,y) - f(t,x)) = \frac{y}{x}(y-x)f_x(t,\xi), \tag{2.8}$$

where ξ is between $x(t)$ and $y(t)$. Hence $\dot{v} = (v-1)\theta(t)$ with

$$\theta(t) = y(t)f_x(t,\xi) \le -y(t)\lambda(t) \le -(\inf y)\lambda(t). \tag{2.9}$$

The separable equation $\dot{v} = (v-1)\theta(t)$ is easily solved and gives $v(t) \to 1$, since

$$\int_0^\infty \theta(t)dt = -\infty. \tag{2.10}$$

The above discussion ignores a minor technical difficulty that is now explained. Since $\theta(t)$ involves the function $f_x(t,\xi)$, where the dependence of ξ on t is difficult to control, it is not immediately obvious that $\theta(t)$ is continuous or even integrable. The resolution of this difficulty involves two steps, the first of which leads to a uniqueness theorem of independent interest.

First step. Here we assume only that $f_x(t,\xi) \le 0$ for $\xi \in I$. This is implied by condition EI but is much weaker. With $w = v - 1$ the proof of Theorem 2.1 gives $\dot{w} = \theta(t)w$ where $\theta(t) \le 0$. Given $t_0 \ge 0$, we will show that

$$w(t_0) = 0 \Longrightarrow w(t) = 0 \quad \text{for } t \ge t_0. \tag{2.11}$$

If not, assume without loss of generality that $w(t_2) > 0$ at some value $t_2 > t_0$. Go back toward t_0 until you first reach a value t_1 at which $w(t_1) = 0$. On the interval (t_1, t_2) the mean-value theorem gives a contradiction, $w(t_2) \le 0$.

In terms of x and y this is a uniqueness theorem; namely, it asserts that if $x(t_0) = y(t_0)$ at some $t_0 \ge 0$, then $x(t) = y(t)$ for $t > t_0$. However, the hypothesis $f_x \le 0$ does not require that f_x be bounded, and hence it does not yield the local Lipschitz condition on which uniqueness is usually based.

Second step. In view of the above result we can assume that $w(t) \ne 0$ for large t. The differential equation satisfied by x and y shows that x and y are of class C^1 and hence $\dot{w}$ is continuous. The equation $\theta = \dot{w}/w$ now shows that θ is continuous for large t, which is what we need. □

3. Persistence and extinction. Theorem 2.1 could also be worded as follows: suppose I is a value interval relative to which condition EI holds. Then

either (i) the ratio of any two solutions x and y with value intervals $I(x) = I(y) = I$ tends to 1 as $t \to \infty$, or (ii) every solution with value interval $I(x) = I$ satisfies $\inf x(t) = 0$. It is easily shown that if

$$\liminf_{t\to\infty} \int_0^t f(\tau,0)d\tau = -\infty, \tag{3.1}$$

then (ii) holds, and if

$$0 \le s \le t \Longrightarrow \int_s^t f(\tau,0)d\tau \le B \tag{3.2}$$

for some constant B, then (i) holds. However, these remarks do not distinguish between extinction and persistence; that is, between $\lim x(t) = 0$ and $\inf x(t) > 0$. This matter is discussed next, assuming continuity of $f(x,t)$ for $x \ge 0$ and $t \ge 0$.

Let T and c be large and small positive constants, respectively. It is said that the *long-time average* of $f(t,0)$ is bounded away from 0 positively or negatively if, for all $t > 0$,

$$\frac{1}{T}\int_t^{t+T} f(\tau,0)d\tau \ge c \qquad \text{or} \qquad \frac{1}{T}\int_t^{t+T} f(\tau,0)d\tau \le -c, \tag{3.3}$$

respectively. Suppose that $f(t,0)$ is bounded below, that $f_x \le 0$, and that $f_x(t,x)$ is bounded below for small x. Then (i) persistence holds if the long-time average of $f(t,0)$ is bounded away from 0 positively, and (ii) extinction holds if it is bounded away from 0 negatively.

Part (i) of this result is essentially the same as [9, Theorem 2], if we take into account the accompanying remarks regarding uniform continuity at $x = 0$. Hence we give the proof only for (ii).

Proof of (ii). The above hypothesis for (ii) is worded so as to show a parallelism with (i), but in fact this hypothesis is far stronger than necessary. Instead of assuming that the long-time average is bounded away from 0 negatively, we assume only that

$$\int_0^\infty f(t,0)dt = -\infty. \tag{3.4}$$

Instead of the condition $f_x \le 0$, we assume only that $f(t,x) \le f(t,0)$ for $x > 0$. The differential equation now gives (ii) as follows:

$$x(t) = x(0)e^{\int_0^t f(\tau,x(\tau))d\tau} \le x(0)e^{\int_0^t f(\tau,0)d\tau} \longrightarrow 0. \tag{3.5}$$

These results are in several respects sharp. Confining attention to (i), we note that if $\dot{x} = -x^2$, then $f(t,0) = 0$ and $f_x(t,x) = -1$, but all solutions tend to 0. Thus persistence does not follow if the long-time average of $f(t,0)$ is

only greater than or equal to 0. In the equation

$$\dot{x} = x\frac{1-xt}{1+xt}, \tag{3.6}$$

we have $f(t,0) = 1$, $|f(t,x)| \le 1$, and $f_x \le 0$. Nevertheless all solutions tend to 0. The trouble is that f_x is not bounded below. In the equation

$$\dot{x} = x(2+t\sin t-\cos t-x), \tag{3.7}$$

it is easily checked that

$$\frac{1}{2\pi}\int_t^{t+2\pi} f(\tau,0)d\tau = 2-\cos t \ge 1, \tag{3.8}$$

and clearly $f_x = -1$. Nevertheless $\inf x(t) = 0$, though the proof of this is too long for inclusion here. Part (i) fails because $f(t,0)$ is not bounded below.

As a final example, we consider the familiar logistic equation

$$\dot{x} = x(k-x), \tag{3.9}$$

where $k \ge 0$ is constant. Here $\lim_{t\to\infty} x(t) = k$. Hence if $k > 0$, then all solutions are bounded away from 0 below, and if $k = 0$, then all solutions tend to 0. Our results give a similar dichotomy for equations that are not quite so easy to solve explicitly. □

4. An equation of Weissing and Huisman. As a first application, we consider the equation

$$\dot{W} = \frac{p_{\max}}{k}\ln\left(\frac{H+I_{\text{in}}}{H+I_{\text{in}}e^{-kW}}\right) - lW, \qquad W(0) > 0, \tag{4.1}$$

which was introduced in Huisman and Weissing [3] to describe light-limited growth and competition among phytoplankton species in a mixed water column. The function W represents the total biomass in the water column and study of its behavior is the object of the theory. The symbol I_{in} represents the light intensity at the top of the water column, k the light extinction coefficient, H the half saturation constant of specific carbon uptake rate, $p_{\max}$ the maximum specific carbon uptake rate, and l the specific rate of carbon loss. In the theory of Huisman and Weissing, these coefficients are assumed to be positive constants. We will prove the following theorem.

Theorem 4.1. *Suppose that the coefficients $p_{\max}$, k, H, I_{in}, l in (4.1) are positive continuous functions of t which are bounded above and bounded away from 0. Then all solutions W are positive and bounded. Moreover, either*

(i) *the difference of any two solutions tends to 0 as $t \to \infty$, or*

(ii) *every solution satisfies $\inf W(t) = 0$.*

The conclusions (i) and (ii) are not mutually exclusive. We take all coefficients as time-dependent because it is easy to do. More realistically, we could assume that all coefficients are positive constants except I_{in}. Allowing time-dependence of the incident intensity was mentioned as desirable in Weissing and Huisman [11].

In the course of the proof, we will show that $\dot{W} = Wf(t,W)$ with

$$f(t,0) = p_{\max}\frac{R}{1+R} - l, \quad R = \frac{I_{\text{in}}}{H}. \tag{4.2}$$

It will be seen also that f and f_W are bounded and that $f_W \le 0$. Hence persistence or extinction follows if the long-time average of

$$p_{\max}\frac{I_{\text{in}}}{I_{\text{in}}+H} - l \tag{4.3}$$

is bounded away from zero positively or negatively, respectively.

4.1. Reformulation. Using the ratios $R = I_{\text{in}}/H$ and $r = p_{\max}/k$, we write (4.1) in the form

$$\dot{W} = r\ln\left(\frac{1+R}{1+Re^{-kW}}\right) - lW. \tag{4.4}$$

By hypothesis, the coefficients r, R, l, k are positive continuous functions of t, bounded both above and away from 0. This assumption, weaker than that in Theorem 4.1, is sufficient for our purposes. Since (4.4) implies

$$\dot{W} \le r\ln(1+R) - lW \le c_1 - c_2W, \tag{4.5}$$

where c_1 and c_2 are positive constants, W exists for all $t > 0$ and is bounded above.

Equation (4.4) is not in Kolmogorov form. However, the procedure suggested in the introduction gives $\dot{W} = Wf(t,W)$, where

$$f(t,W) = \frac{r}{W}\ln\left(\frac{1+R}{1+Re^{-kW}}\right) - l \tag{4.6a}$$

for $W \neq 0$ and

$$f(t,0) = \frac{rkR}{1+R} - l = p_{\max}\frac{R}{1+R} - l. \tag{4.6b}$$

To compute the sign of f_W, we use the formula

$$f(t,W) = p_{\max}\int_0^1 \frac{R}{R+e^{kWs}}\,ds - l \tag{4.7}$$

which is valid for $W \ge 0$. Equation (4.7) can be deduced by following the derivation of (13) in Huisman and Weissing [3], using their equations (6), (2), (1), and

the equation $W = wz$ given in the second column, line 13, of page 508. Once (4.7) is available, it is easily checked by evaluating the integral. The passage from (4.6) to (4.7) is not obvious, but that from (4.7) to (4.6) is routine.

Proof. If alternative (ii) does not hold, at least one solution, y, satisfies $\inf y(t) > 0$. Let x be any other solution. Since (4.6) has the Kolmogorov form with continuous f, all solutions are positive and bounded as seen above. Hence, a common value interval I for x and y has the form $(0, m)$ where $m < \infty$. For $\xi \in I$,

$$1 \le e^{k\xi s} \le e^{kms} \le e^{km} \le b, \quad 0 \le s \le 1, \tag{4.8}$$

where b is an upper bound for e^{km}. By (4.7),

$$\frac{\partial f}{\partial W} = -p_{\max} \int_0^1 \frac{Rks}{\left(R + e^{kWs}\right)^2} e^{kWs} ds, \tag{4.9}$$

which gives

$$f_W(t,\xi) \le -p_{\max} \int_0^1 \frac{Rks}{(R+b)^2} ds = -p_{\max} \frac{kR}{2(R+b)^2}. \tag{4.10}$$

Condition EI holds with λ equal to the (constant) minimum of the expression on the right, and the conclusion (i) follows from Theorem 2.1. □

5. Supplementary remarks

5.1. The autonomous case. If the coefficients R, r, l, k are positive constants, we write $f(W)$ instead of $f(t,W)$ and $f'(W)$ instead of f_W. Since $f'(W) < 0$ for $W \ge 0$ and $f(W)$ is negative for W large, the solution exists and is positive for $t \ge 0$. Its detailed behavior depends on

$$f(0) = p_{\max} \frac{R}{1+R} - l. \tag{5.1}$$

The population persists if and only if $f(0) > 0$ and in that case there is a unique value W^* such that $\lim W(t) = W^*$. The value W^* is the positive root of $f(W) = 0$, defined implicitly by

$$p_{\max} \int_0^1 \frac{R}{R + e^{kW^* s}} ds = l. \tag{5.2}$$

In particular, if a single solution is bounded away from 0 then all solutions are, and all tend to W^*. Hence their ratio tends to 1, in agreement with Theorem 4.1.

5.2. A generalization. In [11], the authors introduce a generalization of their theory. Under mild assumptions, they show that

$$\dot{W} = F(G), \tag{5.3}$$

where F is increasing, $F(0) = 0$, and for a positive function g,

$$G(W) = \int_0^W g(I(s))ds. \tag{5.4}$$

Clearly $G(0) = 0$. If we set $H(W) = F(G(W))$, then $H(0) = 0$ and $\dot{W} = H(W)$. Assuming that the needed derivatives exist,

$$H'(W) = F'(G(W))G'(W) = F'(G(W))g(I(W)) \tag{5.5}$$

and $H'(0) = F'(0)g(I(0))$. Since $I(0) = I_{\text{in}}$ in [11], all the conditions are available for a theorem analogous to Theorem 4.1. The details will not be given here.

6. The generalized TBKP equation. With $p+1$, n, k, c positive constants and $x(0) > 0$, the equation

$$\dot{x} = cx^{1-np}(k^n - x^n)^{1+p} \tag{6.1}$$

was introduced in Turner et al. [8] and is here called the TBKP equation. An objection to (6.1) is that it requires initial values $x_0 < k$. Indeed, if $x > k$, the expression

$$(k^n - x^n)^{1+p} \tag{6.2}$$

becomes imaginary unless p has certain special values, and even then the behavior as $x \to \infty$ may be inappropriate.

To deal with this problem, we introduce the *odd power function*

$$y^{\{m\}} = (\operatorname{sgn} y)|y|^m \tag{6.3}$$

and replace (6.1) by

$$\dot{x} = cx^{1-np}(k^n - x^n)^{\{1+p\}}, \qquad x(0) = x_0 > 0. \tag{6.4}$$

The assumption that n, k, and $p+1$ are positive constants is retained, but instead of the constant c we introduce a positive continuous function $c(t)$. We also introduce an additional growth term of the form $d(t)x$, where $d(t)$ is continuous but can change sign. Finally, we allow a quadratic self-limiting term $-e(t)x^2$, where $e(t)$ is continuous and nonnegative. The generalized TBKP equation replacing (6.1) is therefore

$$\dot{x} = c(t)x^{1-np}(k^n - x^n)^{\{1+p\}} + d(t)x - e(t)x^2, \qquad x(0) = x_0 > 0. \tag{6.5}$$

This has the Kolmogorov form $\dot{x} = xf(t,x)$ with

$$f(t,x) = c(t)x^{-np}(k^n - x^n)^{\{1+p\}} + d(t) - e(t)x. \tag{6.6}$$

Equation (6.5) includes equations of Turner et al. [8], von Bertalanffy [10], Richards [6], and Thornley and Johnson [7] as well as the standard logistic, Gompertz, and Malthus population growth equations. Besides time-dependent coefficients, (6.5) allows a self-limiting term that does not appear in any of the special cases cited. It is seen in Section 7 that all solutions exist for $0 \le t < \infty$ and are positive.

We will prove the following theorem.

Theorem 6.1. *In (6.5), suppose that $p \ge 0$ and that*

$$\int_0^\infty e(t)dt = \infty. \tag{6.7}$$

If a single solution is bounded away from 0, then all solutions are, and $\lim y(t)/x(t) = 1$ holds for every pair of solutions. If, in addition, at least one solution is bounded above, then all solutions are, and every pair satisfies $\lim_{t\to\infty} |x(t) - y(t)| = 0$.

Proof. When $y \ne 0$, the equation $(d/dy)y^{\{\alpha\}} = \alpha|y|^{\alpha-1}$ holds for any constant α and yields

$$f_x(t,x) = -nc(t)(x^n + pk^n)x^{-np-1}\left|x^n - k^n\right|^p - e(t). \tag{6.8}$$

The result now follows from Theorems 2.1 and 7.1. □

6.1. Further discussion. Theorems 4.1 and 6.1 hardly use the full force of Theorem 2.1, in that the value interval I plays only a minor role. A more complete development would distinguish the cases $\sup x(t) < k$ and $\inf x(t) > k$. For example if $p > 0$ as in Theorem 6.1 and

$$\sup x(t) < k, \qquad \sup y(t) < k, \qquad \inf y(t) > 0, \tag{6.9}$$

then the conclusion follows from the condition

$$\int_0^\infty (c(t) + e(t))dt = \infty, \tag{6.10}$$

which can hold even when $e(t) = 0$. More subtle results of this kind are valid when $p < 0$. To be of practical use, however, the needed information about $I(x)$ must be deduced from the differential equation and the initial conditions. A full development of these ideas would take us too far afield, and the interested reader is referred to Redheffer [5].

7. A remark on continuity. Continuity of $f(t,x)$ for $x > 0$ does not imply $x(t) > 0$ for solutions of the Kolmogorov equation, as seen by the example $f(t,x) = -1/x$ for which $\dot{x} = -1$. Nevertheless (except in results that actually involve $f(t,0)$) continuity for $x > 0$ suffices for the problems considered here. Distinguishing continuity for $x > 0$ from that for $x \ge 0$ may seem like mere hair splitting, but in fact the following theorem increases the scope of Theorem 2.1.

Theorem 7.1. *Let $\dot{x} = xf(t,x)$, where $f(t,x)$ is continuous for $x > 0$. Suppose further that there are continuous functions g and h, with g positive, such that*

$$0 < x < g(t) \Longrightarrow f(t,x) > -h(t). \tag{7.1}$$

Then $x(0) > 0$ implies $x(t) > 0$ on the interval of existence of x.

Proof. Since $g(t)$ can be reduced at will, we can, and do, assume $g(0) < x(0)$. Let the interval of existence be $(0,\gamma)$ with $\gamma \le \infty$ and let $(0,\beta)$ be the longest interval on which $x(t) > 0$. If $\beta < \gamma$, then

$$\liminf_{t\to\beta} x(t) = 0. \tag{7.2}$$

Go back towards 0 from β until you first encounter a point α at which $x(\alpha) = g(\alpha)$. There must be such a point since $x(0) > g(0)$ and $x(t) < g(t)$ near β. Then $0 < x(t) < g(t)$ on (α,β), so

$$\dot{x} \ge -xh(t), \quad \alpha \le t < \beta. \tag{7.3}$$

For $\alpha \le t < \beta$, this gives

$$x(t) \ge x(\alpha)e^{-\int_\alpha^t h(\tau)d\tau}. \tag{7.4}$$

Hence $\liminf_{t\to\beta} x(t) > 0$, which contradicts (7.2). □

For example in the generalized TBKP equation, $x \le k \Rightarrow f(t,x) \ge d(t) - ke(t)$. Theorem 7.1 applies with $g(t) = k$ and $h(t) = d(t) - ke(t)$. For large x we have $f(t,x) \le d(t)$, which together with the previous result shows that $x(t)$ exists for all $t > 0$ and is positive. Nevertheless $f(t,x)$ is discontinuous at $x = 0$ whenever $p > 0$.

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