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LOCAL STABILITY OF THE ADDITIVE FUNCTIONAL EQUATION AND ITS APPLICATIONS

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The main purpose of this paper is to prove the Hyers-Ulam stability of the additive functional equation for a large class of unbounded domains. Furthermore, by using the theorem, we prove the stability of Jensen's functional equation for a large class of restricted domains.

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1. Introduction. The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [14] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [3] under the assumption that G_1 and G_2 are Banach spaces. Later, the result of Hyers was significantly generalized by Rassias [11]. It should be remarked that we can find in [4] a lot of references concerning the stability of functional equations (see also [2, 5, 6]).

In [12, 13], Skof investigated the Hyers-Ulam stability of the additive functional equation for many cases of restricted domains in $\mathbb{R}$. Later, Losonczi [9] proved the local stability of the additive equation for more general cases and applied the result to the proof of stability of the Hosszú's functional equation.

In Section 2, the Hyers-Ulam stability of the additive equation will be investigated for a large class of unbounded domains. Moreover, in Section 3, we will apply the previous result to the proof of the local stability of the Jensen's functional equation on unbounded domains.

Throughout this paper, let E_1 and E_2 be a real (or complex) normed space and a Banach space, respectively.

2. Stability of additive equation on restricted domains. Assume that $\varphi : (0,\infty) \to [0,\infty)$ is a decreasing mapping for which there exists a $d > 0$ such that

$$\varphi(s) \leq s, \tag{2.1}$$

for any $s \geq d$.

We now define

$$\begin{gathered} B_1 = \{(x,y) \in E_1 \setminus \{0\} \times E_1 : \|y\| < \varphi(\|x\|)\} \cup \{(0,y) \in E_1^2 : y \in E_1\}, \\ B_2 = \{(x,y) \in E_1^2 : \|x+y\| < d\}. \end{gathered} \tag{2.2}$$

In the following theorem, we generalize the theorems of Skof [12, 13] and of Losonczi [9] concerning the stability of the additive equation on restricted domains.

Theorem 2.1. *If a mapping $f : E_1 \to E_2$ with $\|f(0)\| \leq \varepsilon$ satisfies the inequality*

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon, \tag{2.3}$$

for some $\varepsilon \geq 0$ and all $(x,y) \in E_1^2 \setminus (B_1 \cup B_2)$, then there exists a unique additive mapping $A : E_1 \to E_2$ such that

$$||f(x) - A(x)|| \leq 39\varepsilon, \quad \forall x \in E_1. \tag{2.4}$$

Proof. First, we assume that $(x,y) \in B_2$ satisfies $x \neq 0$, $y \neq 0$, and $x+y \neq 0$. For this case, we can choose a $z_1 \in E_1$ with

$$\begin{gathered} ||z_1|| \geq \varphi(\|x+y\|), \qquad ||z_1|| \geq \varphi(\|x\|), \qquad ||x+z_1|| \geq \varphi(\|y\|), \\ ||x+y+z_1|| \geq d, \qquad ||x+z_1|| \geq d. \end{gathered} \tag{2.5}$$

Thus, the pairs $(x+y, z_1)$, (x, z_1), and $(y, x+z_1)$ do not belong to $B_1 \cup B_2$. Hence, it follows from (2.3) that

$$\begin{aligned} ||f(x+y) - f(x) - f(y)|| &\leq ||-f(x+y+z_1) + f(x+y) + f(z_1)|| \\ &\quad + ||f(x+z_1) - f(x) - f(z_1)|| \\ &\quad + ||f(x+y+z_1) - f(y) - f(x+z_1)|| \\ &\leq 3\varepsilon, \end{aligned} \tag{2.6}$$

for any $(x,y) \in B_2$ with $x \neq 0$, $y \neq 0$, and $x+y \neq 0$.

When $x = 0$ or $y = 0$, we have

$$||f(x+y) - f(x) - f(y)|| = ||f(0)|| \leq \varepsilon. \tag{2.7}$$

Taking this fact into account, we see that inequality (2.6) is valid for all $(x,y) \in B_2$ with $x+y \neq 0$.

We now assume that $(x,y) \in B_2$ satisfies $x+y=0$ and $\|x\| \geq d$. (In this case, $\|y\| = \|-x\| \geq d$.) In view of (2.1), both the pairs $(-x,-x)$ and $(x,-2x)$ do not belong to $B_1 \cup B_2$. Hence, it follows from (2.3) that

$$||f(-2x)-2f(-x)|| \leq \varepsilon, \qquad ||f(-x)-f(x)-f(-2x)|| \leq \varepsilon. \tag{2.8}$$

From the last two inequalities we get

$$\begin{aligned} ||f(x+y)-f(x)-f(y)|| &= ||f(0)-f(x)-f(-x)|| \\ &\leq ||f(0)||+||f(-2x)-2f(-x)|| \\ &\quad +||f(-x)-f(x)-f(-2x)|| \\ &\leq 3\varepsilon. \end{aligned} \tag{2.9}$$

Considering all the previous inequalities including (2.3), we conclude that f satisfies the inequality

$$||f(x+y)-f(x)-f(y)|| \leq 3\varepsilon, \tag{2.10}$$

for all $(x,y) \in E_1^2 \setminus (B_1 \cup B_2) \cup \{(u,v) \in B_2 : \|u\| \geq d\}$.

Now, let $(x,y) \in E_1^2$ be arbitrarily given with $\|x\| \geq d$ and $\|y\| \geq d$. Since φ is decreasing, we see by (2.1) that

$$\varphi(\|x\|) \leq \varphi(d) \leq d \leq \|y\|, \tag{2.11}$$

and this implies that $(x,y) \notin B_1$. If, moreover, the given pair (x,y) belongs to B_2, then $(x,y) \in \{(u,v) \in B_2 : \|u\| \geq d\}$. Otherwise, $(x,y) \in E_1^2 \setminus (B_1 \cup B_2)$. Hence, it follows from (2.10) that

$$||f(x+y)-f(x)-f(y)|| \leq 3\varepsilon, \tag{2.12}$$

for all $(x,y) \in E_1^2$ with $\|x\| \geq d$ and $\|y\| \geq d$.

Assume that $(x,y) \in E_1^2$ with $\|x\| < d$ and $\|y\| \geq 4d$. In this case, we may choose a $z_2 \in E_1$ with $2d \leq \|z_2\| < 3d$. Then, it holds that

$$\begin{array}{llll} ||x-z_2|| \geq d, & ||y+z_2|| \geq d, & ||x-z_2|| \geq d, & ||z_2|| \geq 2d, \\ ||-z_2|| \geq 2d, & ||y+z_2|| \geq d, & ||z_2|| \geq 2d, & ||-z_2|| \geq 2d. \end{array} \tag{2.13}$$

It then follows from (2.12) and (2.13) that

$$\begin{aligned}||f(x+y)-f(x)-f(y)|| &\le ||f(x+y)-f(x-z_2)-f(y+z_2)|| \\ &\quad +||-f(x)+f(x-z_2)+f(z_2)|| \\ &\quad +||-f(y)+f(-z_2)+f(y+z_2)|| \\ &\quad +||f(0)-f(z_2)-f(-z_2)||+||-f(0)|| \\ &\le 13\varepsilon,\end{aligned} \tag{2.14}$$

for $(x,y)\in E_1^2$ with $\|x\|<d$ and $\|y\|\ge 4d$.

Combining (2.12) and (2.14), we have

$$||f(x+y)-f(x)-f(y)|| \le 13\varepsilon, \tag{2.15}$$

for all $(x,y)\in E_1^2$ with $\|y\|\ge 4d$. Since the Cauchy difference $f(x+y)-f(x)-f(y)$ is symmetric with respect to x and y, we conclude that inequality (2.15) is true for all $(x,y)\in E_1^2$ with $\|x\|\ge 4d$ or $\|y\|\ge 4d$.

If $(x,y)\in E_1^2$ satisfies $\|x\|<4d$ and $\|y\|<4d$, then we can choose a $z_3\in E_1$ with $\|z_3\|\ge 8d$. Then, we have $\|x+z_3\|\ge 4d$. Since inequality (2.15) holds true for all $(x,y)\in E_1^2$ with $\|x\|\ge 4d$ or $\|y\|\ge 4d$, we get

$$\begin{aligned}||f(x+y)-f(x)-f(y)|| &\le ||-f(x+y+z_3)+f(x+y)+f(z_3)|| \\ &\quad +||f(x+z_3)-f(x)-f(z_3)|| \\ &\quad +||f(x+y+z_3)-f(y)-f(x+z_3)|| \\ &\le 39\varepsilon,\end{aligned} \tag{2.16}$$

for any $(x,y)\in E_1^2$ with $\|x\|<4d$ and $\|y\|<4d$.

Inequality (2.16) together with (2.15) yields

$$||f(x+y)-f(x)-f(y)|| \le 39\varepsilon, \quad \forall x,y\in E_1. \tag{2.17}$$

According to [1], there exists a unique additive mapping $A: E_1\to E_2$ that satisfies inequality (2.4) for each x in E_1. □

Corollary 2.2. *Let $d>0$ and $\varepsilon\ge 0$ be given. If a mapping $f: E_1\to E_2$ with $\|f(0)\|\le\varepsilon$ satisfies inequality (2.3) for all $x,y\in E_1$ with $\max\{\|x\|,\|y\|\}\ge d$ and $\|x+y\|\ge d$, then there exists a unique additive mapping $A: E_1\to E_2$ that satisfies inequality (2.4) for each $x\in E_1$.*

Proof. Because of the symmetry property of the Cauchy difference with respect to x and y, we can, without loss of generality, assume that f satisfies inequality (2.3) for all $x,y\in E_1$ with $\|y\|\ge d$ and $\|x+y\|\ge d$.

For a constant mapping $\varphi(s) = d$ $(s > 0)$, define

$$B_1 = \{(x,y) \in E_1 \setminus \{0\} \times E_1 : \|y\| < d\} \cup \{(0,y) \in E_1^2 : y \in E_1\},$$
$$B_2 = \{(x,y) \in E_1^2 : \|x+y\| < d\}. \tag{2.18}$$

Since

$$E_1^2 \setminus B_1 = \{(x,y) \in E_1 \setminus \{0\} \times E_1 : \|y\| \ge d\},$$
$$E_1^2 \setminus B_2 = \{(x,y) \in E_1^2 : \|x+y\| \ge d\}, \tag{2.19}$$

we have

$$E_1^2 \setminus (B_1 \cup B_2) = \{(x,y) \in E_1 \setminus \{0\} \times E_1 : \|y\| \ge d \text{ and } \|x+y\| \ge d\}. \tag{2.20}$$

Thus, it follows from our hypothesis that f satisfies inequality (2.3) for all $(x,y) \in E_1^2 \setminus (B_1 \cup B_2)$.

According to Theorem 2.1, there exists a unique additive mapping $A : E_1 \to E_2$ that satisfies inequality (2.4) for all $x \in E_1$. □

In 1983, Skof [12] presented an interesting asymptotic behavior of the additive mappings: a mapping $f : \mathbb{R} \to \mathbb{R}$ is additive if and only if $|f(x+y) - f(x) - f(y)| \to 0$ as $|x| + |y| \to \infty$.

Without difficulty, the above theorem of Skof can be extended to mappings from a real normed space to a Banach space. We now apply Corollary 2.2 to a generalization of Skof theorem.

Corollary 2.3. *A mapping $f : E_1 \to E_2$ is additive if and only if*

$$\|f(x+y) - f(x) - f(y)\| \longrightarrow 0 \tag{2.21}$$

as $\|x+y\| \to \infty$.

Proof. On account of the hypothesis, there exists a decreasing sequence (ε_n) with $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon_n, \tag{2.22}$$

for all $(x,y) \in E_1^2$ with $\|x+y\| \ge n$. With $y = 0$ and $\|x\| \to \infty$, our hypothesis implies that $f(0) = 0$.

By Corollary 2.2, there exists a unique additive mapping $A_n : E_1 \to E_2$ such that

$$\|f(x) - A_n(x)\| \le 39\varepsilon_n, \quad \forall x \in E_1. \tag{2.23}$$

Now, let l and m be integers with $m > l > 0$. Then, inequality (2.23) implies that

$$\|f(x) - A_m(x)\| \le 39\varepsilon_m \le 39\varepsilon_l, \tag{2.24}$$

for $x \in E_1$, and further, the uniqueness of A_n implies that $A_m = A_l$ for all integers $l, m > 0$, that is, $A_n = A_1$ for any $n \in \mathbb{N}$. By letting $m \to \infty$ in the last inequality, we get

$$||f(x) - A_1(x)|| = 0, \tag{2.25}$$

for any $x \in E_1$, which means that f is additive. The reverse assertion is trivial. □

3. Stability of Jensen's equation on restricted domains. Kominek investigated in [8] the Hyers-Ulam stability of the Jensen's functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \tag{3.1}$$

for the class of mappings defined on a bounded subset of $\mathbb{R}^N$. On the other hand, the author proved in [7] the Hyers-Ulam stability of that equation on unbounded domains.

In this section, we use Theorem 2.1 to generalize the theorems of the author and of Kominek.

Let $\varphi_1 : [0,\infty) \to [0,\infty)$ be a decreasing mapping that satisfies $\varphi_1(0) = d_0 > 0$. Define

$$\begin{aligned} B_1 &= \{(x,y) \in E_1 \setminus \{0\} \times E_1 : \|y\| < \varphi_1(\|x\|)\} \cup \{(0,y) \in E_1^2 : y \in E_1\}, \\ B_2 &= \{(x,y) \in E_1^2 : \|x+y\| < d_0\}, \\ D &= \{(0,y) \in E_1^2 : \|y\| \geq d_0\}. \end{aligned} \tag{3.2}$$

Theorem 3.1. *If a mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \varepsilon, \tag{3.3}$$

for some $\varepsilon \geq 0$ and all $(x,y) \in E_1^2 \setminus (B_1 \cup B_2) \cup D$, then there exists a unique additive mapping $A : E_1 \to E_2$ such that

$$||f(x) - A(x) - f(0)|| \leq 78\varepsilon, \tag{3.4}$$

for any $x \in E_1$.

Proof. If we substitute $g(x)$ for $f(x) - f(0)$ in (3.3), then

$$\left\| 2g\left(\frac{x+y}{2}\right) - g(x) - g(y) \right\| \leq \varepsilon, \tag{3.5}$$

for any $(x,y) \in E_1^2 \setminus (B_1 \cup B_2) \cup D$. With $x = 0$ and $\|y\| \geq d_0$, inequality (3.5) yields

$$\left\|2g\left(\frac{y}{2}\right) - g(y)\right\| \leq \varepsilon, \tag{3.6}$$

for each $y \in E_1$ with $\|y\| \geq d_0$. Replace y by $x+y$ ($\|x+y\| \geq d_0$) in inequality (3.6) to get

$$\left\|2g\left(\frac{x+y}{2}\right) - g(x+y)\right\| \leq \varepsilon, \tag{3.7}$$

for all $x,y \in E_1$ with $\|x+y\| \geq d_0$.

It follows from (3.5) and (3.7) that

$$\begin{aligned} &\|g(x+y) - g(x) - g(y)\| \\ &\qquad \leq \left\|g(x+y) - 2g\left(\frac{x+y}{2}\right)\right\| + \left\|2g\left(\frac{x+y}{2}\right) - g(x) - g(y)\right\| \\ &\qquad \leq 2\varepsilon, \end{aligned} \tag{3.8}$$

for every $(x,y) \in E_1^2 \setminus (B_1 \cup B_2) \cup D$ with $\|x+y\| \geq d_0$. Since $(x,y) \in E_1^2 \setminus (B_1 \cup B_2)$ implies that $\|x+y\| \geq d_0$, the mapping g surely satisfies

$$\|g(x+y) - g(x) - g(y)\| \leq 2\varepsilon, \tag{3.9}$$

for all $(x,y) \in E_1^2 \setminus (B_1 \cup B_2)$.

It trivially holds that $\varphi_1(s) \leq s$ for all $s \geq d_0$. On account of Theorem 2.1, there exists a unique additive mapping $A : E_1 \to E_2$ such that

$$\|g(x) - A(x)\| \leq 78\varepsilon, \tag{3.10}$$

for each x in E_1. □

Let $\varphi_2 : (0,\infty) \to [0,\infty)$ be a continuous and decreasing mapping that satisfies

$$0 < d = \inf\{s > 0 : \varphi_2(s) = 0\} < \infty. \tag{3.11}$$

Furthermore, assume that the restriction $\varphi_2|_{(0,d]}$ is strictly decreasing.

Now, we define

$$\begin{aligned} B_1 &= \{(x,y)\in E_1\setminus\{0\}\times E_1 : \|y\| < \varphi_2(\|x\|)\}\cup\{(0,y)\in E_1^2 : y\in E_1\},\\ B_2 &= \{(x,y)\in E_1^2 : \|x+y\| < d_0\},\\ D &= \{(0,y)\in E_1^2 : \|y\|\ge d_0\}, \end{aligned} \tag{3.12}$$

where we set $d_0 = \inf\{d, \lim_{s\to 0+}\varphi_2(s)\}$.

Corollary 3.2. *If a mapping $f: E_1 \to E_2$ satisfies inequality (3.3) for some $\varepsilon \ge 0$ and all $(x,y) \in E_1^2 \setminus (B_1 \cup B_2) \cup D$, then there exists a unique additive mapping $A: E_1 \to E_2$ satisfying inequality (3.4) for all $x \in E_1$.*

Proof. First, we define a mapping $\varphi_0 : [0,\infty) \to [0,\infty)$ by

$$\varphi_0(s) = \begin{cases} d_0, & \text{for } s = 0,\\ \inf\{\varphi_2(s), \inf\varphi_2^{-1}(s)\}, & \text{for } s > 0, \end{cases} \tag{3.13}$$

where we set $\varphi_2^{-1}(t) = \{s > 0 : \varphi_2(s) = t\}$ and $\inf\varnothing = \infty$. (We cannot exclude the case $\varphi_2^{-1}(s) = \varnothing$ from the above definition.) We define

$$\begin{aligned} \tilde{B}_1 &= \{(x,y)\in E_1\setminus\{0\}\times E_1 : \|y\| < \varphi_0(\|x\|)\}\cup\{(0,y)\in E_1^2 : y\in E_1\},\\ \tilde{B}_2 &= \{(x,y)\in E_1^2 : \|x+y\| < d_0\},\\ \tilde{D} &= \{(0,y)\in E_1^2 : \|y\|\ge d_0\}. \end{aligned} \tag{3.14}$$

The fact that $\varphi_0(s) \le \varphi_2(s)$ for all $s > 0$ implies that $\tilde{B}_1 \subset B_1$. Since $B_2 = \tilde{B}_2$ and $D = \tilde{D}$, we get

$$E_1^2 \setminus (B_1 \cup B_2) \cup D \subset E_1^2 \setminus (\tilde{B}_1 \cup \tilde{B}_2) \cup \tilde{D}. \tag{3.15}$$

Now, assume that $(x,y) \in E_1^2 \setminus (\tilde{B}_1 \cup \tilde{B}_2) \cup \tilde{D}$ but $(x,y) \notin E_1^2 \setminus (B_1 \cup B_2) \cup D$. Because $(x,y) \notin D$ and $(x,y) \notin B_2$, we have

$$x \ne 0, \qquad \|x+y\| \ge d_0. \tag{3.16}$$

Moreover, (x,y) should belong to $B_1 \setminus \tilde{B}_1$, that is,

$$0 < \inf\varphi_2^{-1}(\|x\|) \le \|y\| < \varphi_2(\|x\|). \tag{3.17}$$

(Since $\|x\| > 0$ and $\varphi_2|_{(0,d]}$ is strictly decreasing, we have $\inf \varphi_2^{-1}(\|x\|) > 0$.) If we assume that $(y,x) \in B_1$, then we get $\|x\| < \varphi_2(\|y\|)$. This fact implies that $\|y\| < \inf \varphi_2^{-1}(\|x\|)$, which is contrary to (3.17). Hence, by (3.16), we conclude that $(y,x) \notin B_1 \cup B_2$. This fact together with (3.3), yields

$$\left\| 2f\left(\frac{y+x}{2}\right) - f(y) - f(x) \right\| \le \varepsilon, \tag{3.18}$$

for all $(x,y) \in E_1^2 \setminus (\tilde{B}_1 \cup \tilde{B}_2) \cup \tilde{D}$.

We now define another mapping $\varphi : [0,\infty) \to [0,\infty)$ by

$$\varphi(s) = \begin{cases} d_0, & \text{for } s = 0, \\ \inf\{\varphi_2(s), \inf \varphi_2^{-1}(s)\}, & \text{for } 0 < s \le d_1, \\ \sup\{\varphi_2(s), \sup \varphi_2^{-1}(s)\}, & \text{for } s > d_1, \end{cases} \tag{3.19}$$

where $d_1 > 0$ is the unique fixed point of φ_2, that is, $d_1 = \varphi_2(d_1)$, and we set $\inf \varnothing = \infty$ and $\sup \varnothing = 0$.

Let $s_i > 0$ $(i = 1,2,3,4)$ be arbitrarily given with $0 < s_1 < s_2 \le d_1 < s_3 < s_4$. Since φ_2 is decreasing, we have

$$\begin{gathered} \lim_{s\to 0+} \varphi_2(s) \ge \varphi_2(s_1) \ge \varphi_2(s_2) \ge d_1 \ge \varphi_2(s_3) \ge \varphi_2(s_4), \\ d \ge \inf \varphi_2^{-1}(s_1) \ge \inf \varphi_2^{-1}(s_2) \ge d_1 \ge \sup \varphi_2^{-1}(s_3) \ge \sup \varphi_2^{-1}(s_4). \end{gathered} \tag{3.20}$$

Hence, we get

$$\varphi(0) \ge \varphi(s_1) \ge \varphi(s_2) \ge \varphi(s_3) \ge \varphi(s_4) \tag{3.21}$$

which implies that φ is decreasing.

Similarly as before, we define

$$\begin{aligned} \hat{B}_1 &= \{(x,y) \in E_1 \setminus \{0\} \times E_1 : \|y\| < \varphi(\|x\|)\} \cup \{(0,y) \in E_1^2 : y \in E_1\}, \\ \hat{B}_2 &= \{(x,y) \in E_1^2 : \|x+y\| < d_0\}, \\ \hat{D} &= \{(0,y) \in E_1^2 : \|y\| \ge d_0\}. \end{aligned} \tag{3.22}$$

Since $\hat{B}_1 \supset \tilde{B}_1$, $\hat{B}_2 = \tilde{B}_2$, and $\hat{D} = \tilde{D}$, we may conclude that inequality (3.3) holds true for all $(x,y) \in E_1^2 \setminus (\hat{B}_1 \cup \hat{B}_2) \cup \hat{D}$.

According to Theorem 3.1, there exists a unique additive mapping $A : E_1 \to E_2$ such that inequality (3.4) is true for any $x \in E_1$. □

The author in [7] proved that it needs only to show an asymptotic property of the Jensen difference to identify a given mapping with an additive one.

Let X and Y be a real normed space and a real Banach space, respectively. A mapping $f : X \to Y$ with $f(0) = 0$ is additive if and only if

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \longrightarrow 0 \tag{3.23}$$

as $\|x\| + \|y\| \to \infty$.

By using Theorem 3.1, we now prove an asymptotic behavior of additive mappings which generalizes the above result.

Corollary 3.3. *A mapping $f : E_1 \to E_2$ with $f(0) = 0$ is additive if and only if*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \longrightarrow 0 \tag{3.24}$$

as $\|x+y\| \to \infty$.

Proof. According to our hypothesis, there exists a decreasing sequence (ε_n) with $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \le \varepsilon_n, \tag{3.25}$$

for all $(x,y) \in E_1^2$ with $\|x+y\| \ge n$.

The mapping $\varphi_1 : [0,\infty) \to [0,\infty)$ defined by $\varphi_1(s) = -s + n$ $(s \ge 0)$ is decreasing. Moreover, it holds that $\varphi_1(0) = n$. We define

$$\begin{aligned} B_1 &= \{(x,y) \in E_1 \setminus \{0\} \times E_1 : \|y\| < -\|x\| + n\} \cup \{(0,y) \in E_1^2 : y \in E_1\}, \\ B_2 &= \{(x,y) \in E_1^2 : \|x+y\| < n\}, \\ D &= \{(0,y) \in E_1^2 : \|y\| \ge n\}. \end{aligned} \tag{3.26}$$

Since $B_1 \cup B_2 = \{(x,y) \in E_1^2 : x = 0 \text{ or } \|x+y\| < n\}$ and $D = \{(x,y) \in E_1^2 : x = 0 \text{ and } \|x+y\| \ge n\}$, we have

$$E_1^2 \setminus (B_1 \cup B_2) = \{(x,y) \in E_1^2 : x \ne 0 \text{ and } \|x+y\| \ge n\}, \tag{3.27}$$

and hence

$$E_1^2 \setminus (B_1 \cup B_2) \cup D = \{(x,y) \in E_1^2 : \|x+y\| \ge n\}. \tag{3.28}$$

Therefore, inequality (3.25) holds true for all $(x,y) \in E_1^2 \setminus (B_1 \cup B_2) \cup D$.

According to Theorem 3.1, there exists a unique additive mapping $A_n : E_1 \to E_2$ such that

$$\|f(x) - A_n(x)\| \le 78\varepsilon_n, \quad \forall x \in E_1. \tag{3.29}$$

Now, let l and m be positive integers with $m > l$. Then, it follows from (3.29) that

$$\|f(x) - A_m(x)\| \le 78\varepsilon_m \le 78\varepsilon_l, \tag{3.30}$$

for $x \in E_1$. However, the uniqueness of A_n implies that $A_m = A_l$ for all positive integers l and m, that is, $A_n = A_1$ for any $n \in \mathbb{N}$. By letting $m \to \infty$ in the last inequality, we get

$$\|f(x) - A_1(x)\| = 0, \tag{3.31}$$

for each $x \in E_1$, which implies that f is an additive mapping.

The reverse assertion is trivial because every additive mapping $f : E_1 \to E_2$ is a solution of the Jensen functional equation (see [10]). □

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