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ASYMPTOTIC EXPANSIONS AND POSITIVITY OF COEFFICIENTS FOR LARGE POWERS OF ANALYTIC FUNCTIONS

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We derive an asymptotic expansion as $n \to \infty$ for a large range of coefficients of $(f(z))^n$, where $f(z)$ is a power series satisfying $|f(z)| < f(|z|)$ for $z \in \mathbb{C}$, $z \notin \mathbb{R}^+$. When f is a polynomial and the two smallest and the two largest exponents appearing in f are consecutive integers, we use the expansion to generalize results of Odlyzko and Richmond (1985) on log concavity of polynomials, and we prove that a power of f has only positive coefficients.

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1. Introduction. Suppose that $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is a power series with radius of convergence $R \in (0, \infty]$. Given any positive $r < R$, define $\hat{f} : \mathbb{R} \to \mathbb{C}$ by

$$\hat{f}(k) = \frac{1}{2\pi r^k} \int_{-\pi}^{\pi} f(re^{i\theta}) e^{-ik\theta} d\theta. \tag{1.1}$$

Then, if $k \in \mathbb{Z}^+$, we have

$$\hat{f}(k) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}} dz = c_k, \tag{1.2}$$

independently of the choice of r.

Replacing f by f^n, we obtain an integral depending on the two variables n and k, namely,

$$\widehat{f^n}(k) = \frac{1}{2\pi r^k} \int_{-\pi}^{\pi} f^n(re^{i\theta}) e^{-ik\theta} d\theta. \tag{1.3}$$

Under suitable conditions on f, the integral representation (1.3), combined with the freedom of choice in the radius of the circle of integration, can be used to estimate the coefficients $\widehat{f^n}(k)$ when n or k (or both) are large. This method can be used to derive various asymptotic estimates. See the papers of Hayman [8], Moser and Wyman [13, 14], and Odlyzko and Richmond [15] for previous related work in this area. Handelman [7] used different methods to derive asymptotic properties of the coefficients of $p^n q$, where p is a polynomial without negative coefficients and q is a polynomial without positive zeros. The assumption that all coefficients of the polynomial involved are positive is used also for the unimodality results in [15].

The object of this paper is to derive a (suitably defined) asymptotic expansion for "many" coefficients of f^n when n is large, under the weaker condition

$$|f(re^{i\theta})| < f(r) \quad \text{for } 0 < \theta < 2\pi. \tag{1.4}$$

We say that f is *strongly positive at* r if (1.4) holds. We also say that f is *numerically positive* if $f(r) > 0$ for $r > 0$. Condition (1.4) arises in the study of finite state Markov chains, and is discussed in [3]. A simple criterion to check (1.4) for polynomial f at all sufficiently small and sufficiently large r is given in [4].

Our main result on positivity of polynomials (a consequence of the asymptotic expansion in Section 5) is the following theorem (see Theorem 6.2).

Theorem 1.1. *Let $p \in \mathbb{R}[x]$ and suppose that*

(i) *p is strongly positive at every $r > 0$,*

(ii) *the two smallest and the two largest exponents appearing in p are consecutive integers.*

Then every sufficiently large power of p has only positive coefficients.

Before proceeding, we recall here the notion of asymptotic expansion. Let $f(x)$ be a (real- or complex-valued) function of a real variable x. An *asymptotic expansion* for $f(x)$ as $x \to \infty$ is a sequence $\{c_m : m \geq 0\}$ with the following property: for each $n \geq 0$, if we write

$$f(x) = \sum_{m=0}^{n} c_m x^{-m} + R_n(x), \tag{1.5}$$

then $R_n(x) = O(x^{-n-1})$ as $x \to \infty$ (i.e., $x^{n+1}R_n(x)$ is bounded as $x \to \infty$). In general, the infinite series $\sum_{m=0}^{\infty} c_m x^{-m}$ need not converge for any value of x. Nonetheless, the partial sums provide increasingly better approximations to $f(x)$ in the limit $x \to \infty$. We write

$$f(x) \sim \sum_{m=0}^{\infty} c_m x^{-m} \quad \text{as } x \longrightarrow \infty \tag{1.6}$$

to mean that the sequence $\{c_m : m \geq 0\}$ is an asymptotic expansion for $f(x)$ as $x \to \infty$. Using the obvious notation, $f(x) \sim g(x)\sum_{m=0}^{\infty} c_m x^{-m}$ as $x \to \infty$ means that $\{c_m : m \geq 0\}$ is an asymptotic expansion for $f(x)/g(x)$ as $x \to \infty$.

We now briefly describe the content of this paper. In Section 2, we derive some preliminary asymptotic expansions for certain integrals. In Section 3, we show that if f is strongly positive at r, it is possible to derive an asymptotic expansion (in inverse powers of n) for $\widehat{f^n}(n\mu_f(r))$, where $\mu_f(r) = rf'(r)/f(r)$.

This corresponds to the case that the ratio k/n remains constant as $n \to \infty$. The simplest examples are obtained by taking $f(z) = e^z$, $r = 1$, which gives the asymptotic expansion for $1/n!$, and $f(z) = 1+z$, $r = 1$, which gives the expansion for the middle binomial coefficients $\binom{2n}{n}$.

In Section 4, we derive a (suitably generalized) asymptotic expansion for $\widehat{f^n}(n\mu_f(r))$ when r remains in a compact subset of the positive reals, and f is strongly positive on the same subset. In the polynomial case, this corresponds to the case that the ratio k/n remains bounded away from zero and from the degree of the polynomial.

In Section 5, we discuss the case that $f'(0) \neq 0$. We show that, in this case, the same expansion is valid by replacing the restriction that r being bounded away from zero with the weaker one $n\mu_f(r) \to \infty$ as $n \to \infty$.

In Section 6, we discuss questions of eventual global positivity and log concavity in k of the coefficients $\widehat{f^n}(k)$ as $n \to \infty$ (see below for the definition of log concavity). We find that if p is a polynomial with real coefficients such that the two smallest and the two largest exponents appearing in p are consecutive integers, and p is strongly positive at every $r > 0$, then for all large enough n, p^n has no negative coefficients and is log concave. This generalizes Odlyzko and Richmond's log-concavity result [15], obtained by taking p with positive coefficients.

In order to motivate some of the questions and notions discussed here, we now give a brief survey of previous related work for log concavity of probability distributions.

A discrete probability distribution $\pi = \{\pi_k : k \in \mathbb{Z}\}$ on the integers is said to be *unimodal* if there is some k such that $\pi_{i+1} - \pi_i \geq 0$ for $i \leq k$ and $\pi_{i+1} - \pi_i \leq 0$ for $i > k$, and *strongly unimodal* if $\pi * \sigma$ is unimodal for any unimodal discrete distribution σ, where $*$ denotes convolution. A strongly unimodal distribution is unimodal, but the converse is false.

Questions on unimodality of discrete distribution have been studied by numerous authors. Keilson and Gerber [9] showed that π is strongly unimodal if and only if $\pi_n^2 \geq \pi_{n-1}\pi_{n+1}$ holds for all n. We refer to this last condition by saying that the sequence (π_n) is *log concave*, and we say that a polynomial is *log concave* if the sequence of its coefficient is log concave.

Rényi conjectured that for all large enough n, the n-fold convolution π^{*n} is unimodal. See [12, pages 37–48] for unimodal discrete distributions and for Rényi's conjecture. This conjecture is significant only if the support of π is infinite because immediate counterexamples exist in the finite case (e.g., if the support is $\{0,2,3\}$).

For infinite support, counterexamples were found by Brockett and Kemperman [2], and by Ushakov [16]. Brockett and Kemperman then showed that Rényi's conjecture is true if the support of π is the three-point set $\{0,1,2\}$, and conjectured that it is true for distributions with finite-connected support $\{0,1,\ldots,N\}$.

Sharper results were then obtained by Odlyzko and Richmond [15]. They proved that if both the two smallest and the two largest exponents appearing in p are consecutive integers, then for all sufficiently large n, π^{*n} is even strongly unimodal. As mentioned above, the condition on the support is clearly necessary. In addition, Odlyzko and Richmond showed that for any discrete distribution with support contained in $[0,M]$, with $M > 0$, $\pi_0 > 0$, $\pi_M > 0$, and with $\gcd\{k > 0 : \pi_k > 0\} = 1$, the inequality $((\pi^{*n})_k)^2 \geq (\pi^{*n})_{k-1}(\pi^{*n})_{k+1}$ holds for all large enough n and $\delta n \leq k \leq (M-\delta)n$, where δ is any preassigned (small) positive number. As a by-product of their proof, they also obtained an estimate for the value of $(\pi^{*n})_k$ as $n \to \infty$ and $\delta n \leq k \leq (M-\delta)n$. This estimate corresponds to the first term of the expansion in Theorem 6.2.

In order to relate this to polynomials, note that if p is a polynomial without negative coefficients and we normalize so that $p(1) = 1$, then $p(e^t)$ is the moment-generating function of the distribution $\hat{p}(k)$ with support $\{k \in \mathbb{Z} : \hat{p}(k) \neq 0\}$, and convolution of distributions corresponds to multiplication of polynomials.

2. Asymptotic expansions for certain integrals. In this section, we derive an asymptotic expansion for integrals of the form $\int_0^\varepsilon e^{-\lambda^2 t^2} g(\lambda t, t)\,dt$ as $\lambda \to \infty$. To describe this informally, if g is analytic in its second entry, and does not grow too fast in its first entry, then we obtain an asymptotic expansion for the above integral, simply by integrating the Taylor series for g termwise. We take special care to derive explicit, although cumbersome, bounds for the coefficients and for the remainder of the asymptotic expansion because we will later need to keep track of how they depend on the parameter r (see Theorem 3.1).

For $n \geq 0$ and $y \geq 0$, let

$$I_n(y) = \int_y^\infty x^n e^{-x^2/2}\,dx. \tag{2.1}$$

Then, $I_n(y)$ is exponentially small as $y \to \infty$, and $I_m(y) \leq I_n(y)$ for $m \leq n$, and $y \geq 1$. Let

$$M_n = \sup\{y^{n+1} I_n(y) : y \geq 1\}. \tag{2.2}$$

Lemma 2.1. *Suppose W is an open neighborhood of 0 in the complex plane, $g : \mathbb{R}^+ \times W \longrightarrow \mathbb{C}$ is continuous, and for each $x \geq 0$, the function $z \longmapsto g(x,z)$ is analytic on W. Let $\varepsilon > 0$ be such that $\{z : |z| \leq 2\varepsilon\} \subset W$, and suppose further that there is a constant K such that $|g(x,z)| \leq K e^{x^2/2}$ holds for all $x \geq 0$ and $|z| \leq 2\varepsilon$. Define*

$$G(\lambda) = \int_0^\varepsilon e^{-\lambda^2\theta^2} g(\lambda\theta, \theta)\,d\theta, \quad \lambda \geq 0. \tag{2.3}$$

Then,

$$G(\lambda) \sim \sum_{m=0}^{\infty} b_m \lambda^{-m-1} \quad \text{as } \lambda \longrightarrow \infty, \tag{2.4}$$

where

$$\begin{aligned} b_m &= \frac{1}{m!} \int_0^\infty x^m e^{-x^2} \frac{\partial^m g}{\partial z^m}\Big|_{z=0} dx \\ &= \frac{1}{m!} \left[\frac{d^m}{dz^m} \int_0^\infty x^m e^{-x^2} g(x,z)\, dx \right]_{z=0}. \end{aligned} \tag{2.5}$$

Moreover,

$$|b_m| \le \frac{K I_m(0)}{(2\varepsilon)^m} \quad \forall m; \tag{2.6}$$

and if $\varepsilon\lambda \ge 1$,

$$\lambda^{n+2} \left| G(\lambda) - \sum_{m=0}^{n} b_m \lambda^{-m-1} \right| \le \frac{K I_{n+1}(0)}{2^n \varepsilon^{n+1}} + \frac{2KM_n}{\varepsilon^{n+1}} \quad \forall n. \tag{2.7}$$

Proof. Write

$$g(x,z) = \sum_{m=0}^{\infty} g_m(x) z^m, \quad z \in W,\ x \ge 0, \tag{2.8}$$

where

$$g_m(x) = \frac{1}{m!} \left[\frac{\partial^m g}{\partial z^m}(x,z) \right]_{z=0} = \frac{1}{2\pi i} \int_{|w|=2\varepsilon} \frac{g(x,\omega)}{\omega^{m+1}}\, d\omega, \tag{2.9}$$

so that $b_m = \int_0^\infty x^m e^{-x^2} g_m(x)\,dx$. For each n, we can express the remainder $R_n(x,z) = g(x,z) - \sum_{m=0}^n g_m(x) z^m$ as

$$R_n(x,z) = \frac{z^{n+1}}{2\pi i} \int_{|w|=2\varepsilon} \frac{g(x,\omega)}{\omega^{n+1}(\omega - z)}\, d\omega, \quad |z| \le \varepsilon. \tag{2.10}$$

The usual Cauchy's estimates take the form

$$\begin{aligned} |R_n(x,z)| &\le 2^{-n} \left(\frac{|z|}{\varepsilon} \right)^{n+1} \sup\{ |g(x,\omega)| : |\omega| = 2\varepsilon \} \\ &\le 2^{-n} \left(\frac{|z|}{\varepsilon} \right)^{n+1} K e^{x^2/2} \quad \text{for } |z| \le \varepsilon, \end{aligned} \tag{2.11}$$

and similarly

$$\left| g_m(x) \right| \le (2\varepsilon)^{-m} K e^{x^2/2} \quad \forall m \ge 0. \tag{2.12}$$

We find, for each n,

$$\begin{aligned}
G(\lambda) &= \int_0^{\varepsilon\lambda} e^{-x^2} g\left(x, \frac{x}{\lambda}\right) \frac{dx}{\lambda} \\
&= \int_0^{\varepsilon\lambda} e^{-x^2} \left(\sum_{m=0}^{n} g_m(x) x^m \lambda^{-m} + R_n\left(x, \frac{x}{\lambda}\right)\right) \frac{dx}{\lambda} \\
&= \sum_{m=0}^{n} \left(\int_0^{\varepsilon\lambda} x^m e^{-x^2} g_m(x)\,dx \right) \lambda^{-m-1} + \int_0^{\varepsilon\lambda} e^{-x^2} R_n\left(x, \frac{x}{\lambda}\right) \frac{dx}{\lambda} \\
&= \sum_{m=0}^{n} b_m \lambda^{-m-1} - \sum_{m=0}^{n} \left(\int_{\varepsilon\lambda}^{\infty} x^m e^{-x^2} g_m(x)\,dx \right) \lambda^{-m-1} \\
&\quad + \int_0^{\varepsilon\lambda} e^{-x^2} R_n\left(x, \frac{x}{\lambda}\right) \frac{dx}{\lambda}.
\end{aligned} \tag{2.13}$$

So, to prove that $G(\lambda) \sim \sum_{m=0}^{\infty} b_m \lambda^{-m-1}$, we need to show that, for each n,

$$\lambda^{n+2} \left(\int_0^{\varepsilon\lambda} e^{-x^2} R_n\left(x, \frac{x}{\lambda}\right) \frac{dx}{\lambda} - \sum_{m=0}^{n} \left(\int_{\varepsilon\lambda}^{\infty} x^m e^{-x^2} g_m(x)\,dx \right) \lambda^{-m-1} \right) \tag{2.14}$$

is bounded as $\lambda \to \infty$.

Using (2.11), we find

$$\begin{aligned}
\lambda^{n+2} \left| \int_0^{\varepsilon\lambda} e^{-x^2} R_n\left(x, \frac{x}{\lambda}\right) \frac{dx}{\lambda} \right| &\le \frac{K}{2^n \varepsilon^{n+1}} \int_0^{\varepsilon\lambda} x^{n+1} e^{-x^2/2}\,dx \\
&\le \frac{K}{2^n \varepsilon^{n+1}} \int_0^{\infty} x^{n+1} e^{-x^2/2}\,dx \\
&= \frac{K I_{n+1}(0)}{2^n \varepsilon^{n+1}}.
\end{aligned} \tag{2.15}$$

Also, using (2.12), we find

$$\left| \int_{\varepsilon\lambda}^{\infty} x^m e^{-x^2} g_m(x)\,dx \right| \le \frac{K}{(2\varepsilon)^m} \int_{\varepsilon\lambda}^{\infty} x^m e^{-x^2/2}\,dx = \frac{K}{(2\varepsilon)^m} I_m(\varepsilon\lambda). \tag{2.16}$$

So, as long as $\varepsilon\lambda \geq 1$, we have

$$\begin{aligned}
\lambda^{n+2}&\left|\sum_{m=0}^{n}\left(\int_{\varepsilon\lambda}^{\infty} x^m e^{-x^2} g_m(x)\,dx\right)\lambda^{-m-1}\right| \\
&\leq \sum_{m=0}^{n}\frac{K}{(2\varepsilon)^m} I_m(\varepsilon\lambda)\lambda^{n-m+1} \\
&= \frac{K}{\varepsilon^{n+1}}\sum_{m=0}^{n}(\varepsilon\lambda)^{n-m+1} I_m(\varepsilon\lambda)2^{-m} \\
&\leq \frac{K}{\varepsilon^{n+1}}\sum_{m=0}^{n}(\varepsilon\lambda)^{n+1} I_n(\varepsilon\lambda)2^{-m} \\
&\leq \frac{2KM_n}{\varepsilon^{n+1}}.
\end{aligned} \tag{2.17}$$

This proves that $G(\lambda) \sim \sum_{m=0}^{\infty} b_m \lambda^{-m-1}$, and also establishes the bound for the remainder. To find the bound for the coefficients b_m, we use (2.12) again to estimate that

$$|b_m| = \left|\int_0^{\infty} x^m e^{-x^2} g_m(x)\,dx\right| \leq \frac{K}{(2\varepsilon)^m}\int_0^{\infty} x^m e^{-x^2/2}\,dx = \frac{KI_m(0)}{(2\varepsilon)^m}. \tag{2.18}$$

□

Proposition 2.2. *Suppose that $\phi(z)$ and $\psi(z)$ are analytic in a neighborhood of $z = 0$, with $\phi(0) = 1$, and let $\varepsilon > 0$ be such that $|\phi(z) - 1| \leq 1/2$, $|\psi(z)| \leq K$ for $|z| \leq 2\varepsilon$. Let*

$$G(\lambda) = \int_{-\varepsilon}^{\varepsilon} e^{-\lambda^2\theta^2\phi(i\theta)}\psi(i\theta)\,d\theta. \tag{2.19}$$

Then,

$$G(\lambda) \sim \frac{\sqrt{\pi}}{\lambda}\left(\psi(0) + \sum_{m=1}^{\infty} a_m\lambda^{-2m}\right) \quad \text{as } \lambda \longrightarrow \infty, \tag{2.20}$$

where

$$a_m = \frac{(-1)^m}{4^m m!}\left[\frac{d^{2m}}{dt^{2m}}\left(\psi(t)\phi(t)^{-m-1/2}\right)\right]_{t=0}. \tag{2.21}$$

Moreover,

$$\begin{gathered}
|a_m| \leq \frac{2I_{2m}(0)K}{\sqrt{\pi}(2\varepsilon)^{2m}} \quad \forall m, \\
\lambda^{2n+3}\left|G(\lambda) - \frac{\sqrt{\pi}}{\lambda}\sum_{m=0}^{n} a_m\lambda^{-2m}\right| \leq \frac{KI_{2n+2}(0)}{(2\varepsilon)^{2n+2}} + \frac{4KM_{2n+1}}{\varepsilon^{2n+2}} \quad \forall n.
\end{gathered} \tag{2.22}$$

Proof. Define

$$\begin{aligned} h(x,z) &= e^{-x^2(\phi(z)-1)}\psi(z), \\ g(x,z) &= h(x,iz)+h(x,-iz). \end{aligned} \tag{2.23}$$

Then,

$$G(\lambda) = \int_0^{\varepsilon} e^{-\lambda^2\theta^2} g(\lambda\theta,\theta)\,d\theta, \tag{2.24}$$

and $|g(x,z)| \le 2Ke^{x^2/2}$ for $|z| \le 2\varepsilon$.

By Lemma 2.1, we have

$$G(\lambda) \sim \sum_{m=0}^{\infty} b_m \lambda^{-m-1}, \tag{2.25}$$

where

$$b_m = \frac{1}{m!}\left[\frac{d^m}{dz^m}\int_0^{\infty} x^m e^{-x^2} g(x,z)\,dx\right]_{z=0}. \tag{2.26}$$

Since $g(x,z)$ is an even function of z, we have $b_m = 0$ for odd m, and

$$\frac{\partial^{2m}}{\partial z^{2m}} g(x,z)\Big|_{z=0} = 2(-1)^m \frac{\partial^{2m}}{\partial z^{2m}} h(x,z)\Big|_{z=0}. \tag{2.27}$$

So,

$$\begin{aligned} b_{2m} &= \frac{2(-1)^m}{(2m)!}\left[\frac{d^{2m}}{dz^{2m}}\int_0^{\infty} x^{2m}e^{-x^2}h(x,z)\,dx\right]_{z=0} \\ &= \frac{2(-1)^m}{(2m)!}\left[\frac{d^{2m}}{dz^{2m}}\psi(z)\int_0^{\infty} x^{2m}e^{-x^2\phi(z)}\,dx\right]_{z=0}. \end{aligned} \tag{2.28}$$

Using the formula (see, e.g., [5, page 337, formula 3.461.2.])

$$\int_0^{\infty} e^{-cx^2}x^{2m}\,dx = c^{-m-1/2}\frac{\sqrt{\pi}(2m)!}{2^{2m+1}m!}, \quad \operatorname{Re} c > 0, \tag{2.29}$$

the last integral can be evaluated and we find

$$b_{2m} = \frac{(-1)^m\sqrt{\pi}}{4^m m!}\left[\frac{d^{2m}}{dz^{2m}}\psi(z)\phi(z)^{-m-1/2}\right]_{t=0} = \sqrt{\pi}a_m. \tag{2.30}$$

Using the bounds on the coefficients and the remainder given by Lemma 2.1, we also find

$$|b_{2m}| \le \frac{2I_{2m}(0)K}{(2\varepsilon)^{2m}}, \tag{2.31}$$

and (since $b_{2n+1} = 0$),

$$\lambda^{2n+3}\left|G(\lambda) - \sum_{m=0}^{n} b_{2m}\lambda^{-2m-1}\right| \le \frac{KI_{2n+2}(0)}{(2\varepsilon)^{2n+2}} + \frac{4KM_{2n+1}}{\varepsilon^{2n+2}}, \tag{2.32}$$

and so we get the bounds on a_m and $G(\lambda) - (\sqrt{\pi}/\lambda)\sum_{m=0}^{n} a_m\lambda^{-2m}$ given in the statement. $\square$

3. Asymptotics for $\widehat{f^n}(k)$ with k/n fixed. Let $f(z)$ be a power series with real coefficients, radius of convergence $R \in (0,\infty]$, and $f(0) \neq 0$. This last condition can be arranged dividing by a suitable power of z without any loss of generality in what follows.

Recall the definition of strong positivity from Section 1. If X is a subset of $(0,R)$, we say that f is *strongly positive on B* if it is strongly positive at every $r \in X$, that is,

$$|f(re^{i\theta})| < f(r) \quad \text{for } r \in X,\ 0 < \theta < 2\pi. \tag{3.1}$$

In this section, we fix $r \in (0,R)$ and assume that f is strongly positive on a neighborhood of r. Let $W = W(r) = \{z \in \mathbf{C} : \operatorname{Re} f(re^z) > 0\}$. Then, W is an open neighborhood of 0 in the complex plane and the function $z \mapsto \log f(re^z)$ is analytic on W. We define

$$\begin{aligned} \mu = \mu_f(r) &= \frac{\partial}{\partial z}\left[\log f(re^z)\right]_{z=0}, \\ \sigma = \sigma_f(r) &= \frac{\partial^2}{\partial z^2}\left[\log f(re^z)\right]_{z=0}. \end{aligned} \tag{3.2}$$

We find $\mu_f(r) = rf'(r)/f(r)$ and $\sigma_f(r) = r\mu_f'(r)$. In particular, both $\mu_f(r)$ and $\sigma_f(r)$ are continuous functions of r.

Some standard computations show that $\sigma = \sum_k (k-\mu)^2\hat{f}(k)r^k/f(r)$ (see, e.g., [3, page 25]), and so if all coefficients $\hat{f}(k)$ are nonnegative and at least two of them are nonzero, we clearly have $\sigma > 0$. In fact, it is shown in [3, Theorem 6.4] that if f is strongly positive on a neighborhood of r, then $\sigma > 0$ (that theorem was stated and proved for polynomials, but the same proof can be used verbatim for power series).

Define, for $z \in W$, $z \neq 0$,

$$\phi(z) = \phi_f(r,z) = \frac{2}{z^2\sigma}\left[\log\frac{f(re^z)}{f(r)} - \mu z\right]. \tag{3.3}$$

By setting $\phi(0) = 1$, it is easily checked that $\phi(z)$ is analytic on W.

The functions μ_f, σ_f, and ϕ_f will play an important role in what follows. It can be shown that if f is the pointwise spectral radius of an irreducible matrix whose entries are polynomials with positive integral coefficients (the *beta function* of Tuncel, see [10]), then μ_f and σ_f are closely related to the entropy and information variance of the associated Markov chain. The function μ_f is the *Legendre transformation* of f (see, e.g., [3]).

The definition of $\phi(z)$ can be rewritten as

$$\frac{f(re^z)}{f(r)} = \exp\left(\mu z + \frac{1}{2}\sigma z^2\phi(z)\right). \tag{3.4}$$

We now use the integral representation (1.3) (in conjunction with Proposition 2.2) to obtain an asymptotic expansion for some of the coefficients of f^n in inverse powers of n.

From (1.3), we get

$$2\pi r^k \frac{\widehat{f^n}(k)}{(f(r))^n} = \int_{-\pi}^{\pi}\left(\frac{f(re^{i\theta})}{f(r)}\right)^n e^{-ik\theta}\,d\theta. \tag{3.5}$$

Choose $\varepsilon = \varepsilon(r) > 0$ such that $\{z : |z| \le 2\varepsilon\} \subset W$, and $|\phi(z) - 1| \le 1/2$ for $|z| \le 2\varepsilon$. Let

$$\begin{aligned} A_n = A_n(k,r,\varepsilon) &= \int_{-\varepsilon}^{\varepsilon}\exp\left(n\mu i\theta - n\frac{\sigma}{2}\theta^2\phi(i\theta) - ik\theta\right)d\theta,\\ B_n = B_n(k,r,\varepsilon) &= \int_{\varepsilon\le|\theta|\le\pi}\left(\frac{f(re^{i}\theta)}{f(r)}\right)^n e^{-ik\theta}\,d\theta. \end{aligned} \tag{3.6}$$

So, we have

$$\widehat{f^n}(k) = \frac{(f(r))^n}{2\pi r^k}(A_n + B_n). \tag{3.7}$$

The asymptotic expansion will be obtained from A_n, while B_n will be shown exponentially small with respect to every term of the same expansion.

We now set $k = n\mu$ so that the integrand of A_n is simplified to

$$\exp\left(-n\frac{\sigma}{2}\theta^2\phi(i\theta)\right). \tag{3.8}$$

This of course will result in an expansion only for $k = n\mu_f(r)$, where f is strongly positive at r. The question of describing the range of k obtainable in this way (or equivalently, the range of the function $r \mapsto \mu_f(r)$) will be discussed in Section 6.

Theorem 3.1. *Let $f(z)$ be a power series with real coefficients and positive radius of convergence R, and let $0 < r < R$. Suppose f is strongly positive on a neighborhood of r, and let μ, σ, and $\phi(z)$ be as defined in (3.2) and (3.3). Then,*

$$\widehat{f^n}(n\mu) \sim \frac{f^n(r)}{r^{n\mu}\sqrt{2\pi n\sigma}}\left(1 + \sum_{m=1}^{\infty} c_m n^{-m}\right), \tag{3.9}$$

where

$$c_m = \frac{(-1)^m}{(2\sigma)^m m!}\left[\frac{d^{2m}}{dz^{2m}}\left(\phi(z)^{-m-1/2}\right)\right]_{z=0} \quad \text{for } m \ge 1. \tag{3.10}$$

Proof. Recall that $\varepsilon = \varepsilon(r)$ is such that $|\phi(z) - 1| \le 1/2$ for $|z| \le 2\varepsilon$. Let $\delta = \delta(r) > 0$ be such that $\max\{|f(re^{i\theta})/f(r)| : \varepsilon \le |\theta| \le \pi\} \le e^{-\delta}$. With $k = n\mu$, we have

$$A_n = A_n(r, \varepsilon) = \int_{-\varepsilon}^{\varepsilon} \exp\left(-n\frac{\sigma}{2}\theta^2\phi(i\theta)\right)d\theta. \tag{3.11}$$

Let $\lambda = \sqrt{n\sigma/2}$ (here is where we need the fact that f has real coefficients, so that λ is a positive real number).

Then,

$$A_n = G(\lambda) = \int_{-\varepsilon}^{\varepsilon} e^{-\lambda^2\theta^2\phi(i\theta)}d\theta. \tag{3.12}$$

By Proposition 2.2 (with $\psi(z) = 1$), we have

$$A_n \sim \frac{\sqrt{\pi}}{\lambda}\left(1 + \sum_{m=1}^{\infty} a_m \lambda^{-2m}\right) = \sqrt{\frac{2\pi}{n\sigma}}\left(1 + \sum_{m=1}^{\infty} a_m \left(\frac{2}{\sigma}\right)^m n^{-m}\right), \tag{3.13}$$

where a_m is given by (2.21).

If $\delta > 0$ is as in the statement of the theorem, we find that

$$|B_n| = \left|\int_{\varepsilon \le |\theta| \le \pi}\left(\frac{f(re^{i\theta})}{f(r)}\right)^n e^{-in\mu\theta}d\theta\right| \le 2\pi e^{-n\delta}, \tag{3.14}$$

so that B_n is exponentially small as $n \to \infty$.

Since $\widehat{f^n}(n\mu) = [(f(r)^n)/(2\pi r^{n\mu})](A_n + B_n)$, we obtain the desired expansion. $\square$

Perhaps, the simplest example is obtained by taking $f(z) = e^z$ (which is strongly positive at every r). We then find

$$\begin{aligned} \mu(r) = \sigma(r) = r, \qquad \widehat{f^n}(k) = \frac{n^k}{k!}, \\ \phi(z) = 2\frac{e^z - 1 - z}{z^2} = 2\sum_{j=0}^{\infty} \frac{z^j}{(j+2)!}. \end{aligned} \tag{3.15}$$

Note that $\phi(z)$ is independent of r. Choosing $r = 1$ so that $k = n\mu = n$, we obtain the expansion

$$\frac{1}{n!} \sim \frac{e^n}{n^n\sqrt{2\pi n}}\left(1 + \sum_{m=1}^{\infty} c_m n^{-m}\right) \tag{3.16}$$

whose leading term is Stirling's formula $n! = n^n e^{-n}\sqrt{2\pi n}(1+O(1/n))$.

To calculate higher-order terms, one needs to find the derivatives

$$\frac{d^{2m}}{dz^{2m}}\phi(z)^{-m-1/2}\bigg|_{z=0}, \quad \text{where } \frac{d^j}{dz^j}\phi(z)\bigg|_{z=0} = \frac{2}{(j+1)(j+2)}. \tag{3.17}$$

The first few terms of the expansion are

$$\begin{aligned} \frac{1}{n!} \sim \frac{e^n}{n^n\sqrt{2\pi n}}\bigg(1 - \frac{1}{12n} + \frac{1}{288n^2} + \frac{139}{51840n^3} \\ - \frac{571}{2488320n^4} - \frac{163879}{209018880n^5} + \cdots\bigg). \end{aligned} \tag{3.18}$$

Observe that comparing the beginning of the above expansion for $1/n!$ with the expansion for $n!$ (as given, e.g., in [11]), it is apparent that the coefficients of one are obtained from the other simply by multiplication by $(-1)^m$,

$$\begin{aligned} n! \sim \frac{n^n\sqrt{2\pi n}}{e^n}\bigg(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} \\ - \frac{571}{2488320n^4} + \frac{163879}{209018880n^5} + \cdots\bigg). \end{aligned} \tag{3.19}$$

A proof of this is easily obtained by noticing that the well-known asymptotic expansion for $\log n!$ in terms of the Bernoulli numbers (see, e.g., [1, page 205]) involves only odd powers of $1/n$. As a consequence, the numbers c_m defined by

$$c_0 = 1, \qquad c_m = \frac{(-1)^m}{4^m m!\sqrt{2}} \frac{d^{2m}}{dz^{2m}}\left(\frac{z^2}{e^z - 1 - z}\right)^{m+1/2}\bigg|_{z=0} \quad \text{for } m \geq 1 \tag{3.20}$$

must satisfy the relation

$$\sum_{k=0}^{m} (-1)^k c_k c_{m-k} = 0 \quad \text{for } m \geq 1. \tag{3.21}$$

It is natural to ask whether a proof of the above relation for the coefficients c_m could be obtained directly from their definition. This would in turn supply yet one more derivation of the asymptotic expansion for $n!$.

As a second example, take $f(z) = 1+z$, strongly positive at every r. Then,

$$\mu = \frac{r}{1+r}, \qquad \sigma = \frac{r}{(1+r)^2}, \qquad \widehat{f^n}(k) = \binom{n}{k}, \tag{3.22}$$

and choosing $r = 1$ (so that $\mu = 1/2$, $\sigma = 1/4$), we find

$$\begin{aligned}
\phi(z) = \phi_f(1,z) &= \frac{8}{z^2}\left(\log(1+e^z) - \log 2 - \frac{z}{2}\right), \\
\binom{2n}{n} &= \widehat{f^{2n}}(2n\mu) \\
&\sim \frac{4^n}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \frac{21}{32768n^4} - \frac{399}{262144n^5} + \cdots\right).
\end{aligned} \tag{3.23}$$

We remark that choosing $r = 1/(a-1)$ in this example (with $a > 1$), we obtain an expansion for the generalized binomial coefficients

$$\binom{an}{n} = \frac{an(an-1)\cdots(an-n+1)}{n!}. \tag{3.24}$$

4. Asymptotics for $\widehat{f^n}(k)$ with k/n in a compact subset. We take another look at the main result of the previous section, the expansion (3.9). Recall that r was fixed at the beginning. Hence, the expansion for $\widehat{f^n}(k)$ is valid "pointwise" in the ratio k/n, in the sense that the ratio $k/n = \mu_f(r)$ remains constant as $n \to \infty$. In order to obtain results of eventual positivity of all coefficients in a given range, we need to discuss the uniformity of the expansion in k/n that translates (via the map μ_f) into uniformity in r.

So, we now consider r as a parameter ranging in an appropriate subset of $(0,R)$ (depending on n) as $n \to \infty$, and discuss the necessary restrictions on the range of r in order for the expansion to remain valid.

There are three modifications to be made:

(1) the definition of ε in the proof of Theorem 3.1 depends on r. We need to see if and when ε can be chosen independently of r;

(2) since r may now depend on n, (3.9) is not even formally an asymptotic expansion as defined in the introduction, because the coefficients will depend on the variable n. We consider more general types of expansions, with coefficients depending on a parameter (r in our case) and with n replaced by an asymptotic variable $s = s(r,n)$. A precise definition is given below. Instead of n, we use $s = n\sigma_f(r)$;

(3) in the proof of Theorem 3.1, δ also depends on r. We need to make sure that $e^{-n\delta}$ is exponentially small with respect to every term of the asymptotic expansion.

It turns out that (1) causes no restrictions at all when f is a polynomial, or even for power series if r remains bounded away from the radius of convergence.

As for (2), we need to require that $s \to \infty$ as $n \to \infty$. This restricts r to remain suitably "away from zero" as n grows.

The third point is responsible for more serious restrictions. In order to ensure that B_n in (3.7) is exponentially small, r will have to stay away from zero even more than is required by (2), so that taking care of (3) will also take care of (2). However, there is a crucial special case, discussed in Section 5. If $f'(0) \neq 0$, then δ can be chosen uniformly in r even if r approaches zero, and then the only restriction on the size of r is that required by (2). This special case (for polynomials without negative coefficients) is the one treated by Odlyzko and Richmond in [15].

We now reformulate the notion of asymptotic expansion. We discuss a generalization that is suitable for our purposes.

Let $F(n,r)$ be a real-valued function of the two variables n and r. We think of r as a parameter, with values in a parameter space X_n, say, which may depend on n. We say that $F(n,r)$ has an *asymptotic expansion* with *asymptotic variable* $s = s(n,r)$ if $\lim_{n\to\infty}(\inf_{r\in X_n} s(n,r)) = \infty$ and if there are coefficients $c_m(r)$ such that for each m, $\sup\{|c_m(r)| : r \in X_n,\ n \geq 1\} < \infty$, and for each N, the remainder

$$R_N(n,r) = F(n,r) - \sum_{m=0}^{N} c_m(r) s^{-m} \tag{4.1}$$

is such that $\sup\{s^{N+1} R_N(n,r) : r \in X_n\}$ is bounded as $n \to \infty$. When $F(n,r)$ admits such an expansion, we also write, as in the classical case, $F(n,r) = \sum_{m=0}^{N} c_m(r) s^{-m} + O(1/s^{N+1})$, where $O(1/s^m)$ indicates any function R of n and r such that $s^m R$ is bounded as $r \in X_n$ and $n \to \infty$.

We remark that while uniform boundedness of the remainder will be essential in Section 6 to derive our global positivity results, the uniform boundedness of the coefficients is not needed for that purpose. However, we include it in the definition because it is easily proved for our expansions, and it may be of independent interest in specific examples.

The asymptotic expansion derived in the previous section corresponds to $F(n,r) = \widehat{f^n}(n\mu_f(r))$, $s = n$, $X_n = \{r\}$, and c_m is independent of r.

We write

$$F(n,r) \sim \sum_{m=0}^{\infty} c_m s^{-m}, \quad r \in X_n \tag{4.2}$$

to mean that $F(n,r)$ has an asymptotic expansion with asymptotic variable $s = s(n,r)$ as described above.

Note. Unless we explicitly specify also the asymptotic variable, this notation is ambiguous because, for example, $F(n,r) \sim \sum_{m=0}^{\infty} r^m s^{-m}$ could mean that the coefficients are r^m and the asymptotic variable is s, or it could mean that the coefficients are 1 and the asymptotic variable is s/r. However, we use it to avoid more cumbersome notation, and since we always denote the asymptotic variable by s in the following, it should cause no confusion.

The factor e^{-jz} appearing in the following theorem will be needed later to derive results on global positivity and log concavity.

Theorem 4.1. *Suppose that $f(z)$ is a power series with real coefficients and positive radius of convergence R, and let X be a compact subset of $(0,R)$. Let $j \in \mathbb{R}$. If f is strongly positive on an open set containing X, then*

$$\widehat{f^n}(n\mu + j) \sim \frac{(f(r))^n}{r^{n\mu+j}\sqrt{2\pi s}}\left(1 + \sum_{m=1}^{\infty} d_m(j)s^{-m}\right), \quad r \in X, \tag{4.3}$$

where $s = n\sigma$, and

$$d_m(j) = \frac{(-1)^m}{2^m m!}\left[\frac{d^{2m}}{dz^{2m}}\left(e^{-jz}\phi(z)^{-m-1/2}\right)\right]_{z=0}. \tag{4.4}$$

Proof. Let $V = \{(w,z) \in \mathbb{C}^2 : \operatorname{Re} f(we^z) > 0\}$ and $K = \{(r,0) \in \mathbb{C}^2 : r \in X\}$. Then, V is open in $\mathbb{C}^2$, K is compact, and $K \subset V$. So, we can find some $\varepsilon > 0$ such that $(r,z) \in V$ for all $r \in X$ and $|z| \le 2\varepsilon$, and so $\log f(re^z)$ is defined and analytic for $r \in X$ and $|z| \le 2\varepsilon$. Since f is strongly positive on a neighborhood of X, we have $\sigma_f(r) > 0$ on X, and by compactness of X, $\sigma_f(r)$ is bounded away from zero on X. Hence, $\lim_{n\to\infty}(\inf_{r\in X} n\sigma_f(r)) = \infty$. The function $\phi_f(r,z)$ of Section 3 is defined and analytic (in z) for each $r \in X$. Since $\phi_f(r,0) = 1$ and by decreasing ε if necessary, we may assume that $|\phi_f(r,z) - 1| \le 1/2$ for $|z| \le 2\varepsilon$ and $r \in X$. As in Section 3, we then find

$$2\pi r^{n\mu+j}\frac{\widehat{f^n}(n\mu + j)}{(f(r))^n} = A_n(r,\varepsilon) + B_n(r,\varepsilon), \tag{4.5}$$

where

$$\begin{aligned} A_n(r,\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} \exp\left(-n\frac{\sigma}{2}\theta^2\phi(i\theta) - ij\theta\right)d\theta, \\ B_n(r,\varepsilon) &= \int_{\varepsilon\le|\theta|\le\pi}\left(\frac{f(re^{i\theta})}{f(r)}\right)^n e^{-in\mu\theta}d\theta. \end{aligned} \tag{4.6}$$

From Proposition 2.2 (with $\psi(z)=e^{-jz}$), we get

$$A_n(r,\varepsilon)\sim\sqrt{\frac{2\pi}{s}}\left(1+\sum_{m=1}^{\infty}a_m\left(\frac{s}{2}\right)^{-m}\right), \tag{4.7}$$

where a_m is given by (2.21). This gives the expansion (4.3). To show that B_n is exponentially small with respect to all terms of the asymptotic expansion, we use compactness of X to find $\delta>0$ such that $|f(re^{i\theta})/f(r)|\le e^{-\delta}$ for $\varepsilon\le|\theta|\le\pi$ and $r\in X$. Since $\sigma_f(r)$ is bounded away from zero on X, we can find a positive constant c such that $|B_n|\le\pi e^{-cs}$. It remains to show that the coefficients $d_m=2^m a_m$ and the remainder are bounded uniformly in r. Using the bounds given by Proposition 2.2, we find $|d_m|=2^m|a_m|\le 2^m(2I_{2m}(0)/\sqrt{\pi}(2\varepsilon)^{2m})$, and, for each N,

$$\begin{aligned} s^{N+1}&\left|\frac{\widehat{f^n}(n\mu)}{(f(r))^n/(r^{n\mu+j}\sqrt{2\pi s})}-1-\sum_{m=1}^{N}d_m s^{-m}\right| \\ &\qquad\le\frac{I_{2N+2}(0)+4^{N+2}M_{2N+1}}{(2\varepsilon^2)^{N+1}\sqrt{\pi}}+s^{N+1}e^{-cs}\sqrt{2\pi s}, \end{aligned} \tag{4.8}$$

where $I_m(0)$ and M_N are constants (independent of r) defined in (2.1) and (2.2). $\square$

5. The case $f'(0)\neq 0$. We now drop the restriction that r is bounded away from zero, but only under the assumption that $f'(0)\neq 0$.

We assume throughout this section that f is a power series with real coefficients and $f(0)>0$, $f'(0)>0$. It can be easily checked that if $f(0)\neq 0$ and $f'(0)\neq 0$, positivity of $f(0)$ and $f'(0)$ is equivalent to strong positivity of $f(z)$ at all sufficiently small r (cf. [4, Theorem 1]).

Lemma 5.1. *Suppose that $f(0)>0$ and $f'(0)>0$. Then, there is some $r_0>0$ such that the function $\theta\to|f(re^{i\theta})/f(r)|$ is strictly decreasing on $[0,\pi]$ for all $r\in(0,r_0]$.*

Proof. This is [4, Proposition 2(i)], stated for polynomials, but no modifications of the proof are needed for power series. $\square$

Lemma 5.2. *Suppose $f(0)>0$ and $f'(0)>0$. As r becomes small,*

$$\begin{aligned} \mu_f(r)&=\frac{f'(0)}{f(0)}r+O(r^2), \\ \sigma_f(r)&=\frac{f'(0)}{f(0)}r+O(r^2). \end{aligned} \tag{5.1}$$

Proof. We have

$$\begin{aligned}\mu_f(r) &= \frac{\sum k\hat{f}(k)r^k}{f(r)} = \frac{f'(0)r+O(r^2)}{f(0)+O(r^2)} = \frac{f'(0)}{f(0)}r+O(r^2),\\ \sigma_f(r) &= \frac{1}{f(r)}\sum k^2\hat{f}(k)r^k-(\mu_f(r))^2 = \frac{f'(0)}{f(0)}r+O(r^2).\end{aligned}\tag{5.2}$$

□

We now derive the behavior of $\phi_f(r,z)$ for small r.

Let $V=\{(w,z)\in\mathbb{C}^2 : \operatorname{Re} f(we^z)>0\}$. Clearly, V is open in $\mathbb{C}^2$ and $(0,z)\in V$ for all $z\in\mathbb{C}$.

Lemma 5.3. *There is an analytic function $H(r,z)$ on V such that*

$$\begin{aligned}\phi_f(w,z) &= \frac{2}{z^2}(e^z-1-z)+wzH(w,z)\\ &= 1+2\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+3)!}+wzH(w,z)\end{aligned}\tag{5.3}$$

Proof. Consider the function $\Phi(w,z)=\log(f(we^z)/f(w))-\mu z-(1/2)\sigma z^2$ analytic in the two variables w and z on V. We can write

$$\Phi(w,z) = \sum_{j=0}^{\infty}A_j(z)w^j = \sum_{m=0}^{\infty}B_m(w)z^m \tag{5.4}$$

for $(w,z)\in V$, where $A_j(z)$ and $B_m(w)$ are analytic. By definition of μ and σ, we have $B_0(w)=B_1(w)=B_2(w)=0$ for all w. Also, by Lemma 5.2, $\mu(0)=\sigma(0)=0$, and so $\Phi(0,z)=0$ for all z. Hence, $A_0(z)=0$. Since

$$j!\frac{d^mA_j}{dz^m}\bigg|_{z=0} = \frac{\partial^{j+m}\Phi}{\partial z^m\partial w^j}\bigg|_{\substack{z=0\\ w=0}} = m!\frac{d^jB_m}{dw^j}\bigg|_{w=0}, \tag{5.5}$$

we conclude that

$$\frac{d^mA_j}{dz^m}\bigg|_{z=0} = 0 = \frac{d^jB_m}{dw^j}\bigg|_{w=0} \tag{5.6}$$

if either $m\le 2$ or $j=0$. Since $\sigma(w)=O(w)$, we conclude that $(1/z^3)A_j(z)$ and $(1/\sigma)B_m(w)$ are analytic at $z=0$ and $w=0$, respectively, for all $j\ge 0$ and $m\ge 0$, and since $\mu'(0)=\sigma'(0)=f'(0)/f(0)$, we find

$$\begin{aligned}A_1(z) &= \frac{\partial\Phi}{\partial w}\bigg|_{w=0} = \frac{f'(0)}{f(0)}e^z-\frac{f'(0)}{f(0)}-\mu'(0)z-\frac{1}{2}\sigma'(0)z^2\\ &= \frac{f'(0)}{f(0)}\left(e^z-1-z-\frac{1}{2}z^2\right).\end{aligned}\tag{5.7}$$

So we have

$$\begin{aligned}\phi(w,z) &= 1+\frac{2}{z^2\sigma}\Phi(w,z) = 1+\frac{2}{z^2\sigma}A_1(z)w+\frac{2}{z^2\sigma}\sum_{j=2}^{\infty}A_j(z)w^j \\ &= 1+\frac{2}{z^2}A_1(z)\frac{1}{f'(0)/f(0)+O(w)}+2\sum_{j=2}^{\infty}\frac{A_j(z)}{z^2}\frac{w^j}{\sigma} \\ &= \frac{2}{z^2}(e^z-1-z)+wzH(w,z),\end{aligned} \tag{5.8}$$

where H is analytic on V. □

Theorem 5.4. *Suppose that $f(z)$ is a power series with real coefficients, positive radius of convergence R, and with $f(0)\neq 0$, $f'(0)\neq 0$. Let $0<R_0<R$, and suppose that $f(z)$ is strongly positive on $(0,R_0]$. If X_n is any sequence of subsets of $(0,R_0]$, with $\lim_{n\to\infty}(\inf_{r\in X_n}(nr))=\infty$, and $j\in\mathbb{R}$, then*

$$\widehat{f^n}(r\mu+j)\sim\frac{(f(r))^n}{r^{n\mu+j}\sqrt{2\pi s}}\left(1+\sum_{m=1}^{\infty}d_m(j)s^{-m}\right),\quad r\in X_n, \tag{5.9}$$

where $s=n\sigma$, and

$$d_m(j)=\frac{(-1)^m}{2^m m!}\left[\frac{d^{2m}}{dz^{2m}}\left(e^{-jz}\phi(z)^{-m-1/2}\right)\right]_{z=0}. \tag{5.10}$$

Proof. From the assumptions in the statement of the theorem and Lemma 5.2, we have

$$\lim_{n\to\infty}\left(\inf_{r\in X_n}s(n,r)\right)=\infty. \tag{5.11}$$

Using Lemmas 5.3 and 5.1, we find $r_0>0$ and $\varepsilon>0$ such that $|\phi(r,z)-1|\le 1/2$ for $0<r\le r_0$ and $|z|\le 2\varepsilon$, while the function $\theta\to|f(re^{i\theta})/f(r)|$ is strictly decreasing on $[0,\pi]$ for $0<r\le r_0$. As before, we write

$$2\pi r^{n\mu+j}\frac{\widehat{f^n}(n\mu+j)}{(f(r))^n}=A_n(r,\varepsilon)+B_n(r,\varepsilon), \tag{5.12}$$

where $A_n(r,\varepsilon)=\int_{-\varepsilon}^{\varepsilon}(f(re^{i\theta})/f(r))^n e^{-i(n\mu+j)\theta}d\theta$ yields the asymptotic expansion. To check that B_n is exponentially small, we consider two cases. If $r\le r_0$, we find

$$
\begin{aligned}
|B_n(r,\varepsilon)| &= \left| \int_{\varepsilon\le|\theta|\le\pi} \left(\frac{f(re^{i\theta})}{f(r)}\right)^n e^{-i(n\mu+j)\theta} d\theta \right| \le 2\int_{\varepsilon}^{\pi} \left|\frac{f(re^{i\theta})}{f(r)}\right|^n d\theta \\
&\le 2\pi \left|\frac{f(re^{i\varepsilon})}{f(r)}\right|^n = 2\pi \left|\exp\left(-\frac{n}{2}\sigma\varepsilon^2\phi(i\theta)\right)\right| \\
&\le 2\pi\exp\left(-\frac{n}{4}\sigma\varepsilon^2\right) = 2\pi\exp\left(-\frac{s}{4}\varepsilon^2\right),
\end{aligned}
\tag{5.13}
$$

so that $B_n(r,\varepsilon)$ is asymptotically small in s with respect to every term of the asymptotic expansion. If $r_0 \le r \le R_0$, we use Theorem 4.1 (with $X = [r_0, R_0]$). As in the proof of Theorem 4.1, we can use the bounds given by Proposition 2.2 to check that the coefficients and the remainder are bounded uniformly in r. $\square$

6. Log concavity of polynomial powers. We now apply the asymptotic expansion to derive results on positivity and log concavity for polynomials.

Let $p \in \mathbb{R}[x]$, $p \ne 0$, and let $m = \min\{k : \hat{p}(k) \ne 0\}$ and $d = \max\{k : \hat{p}(k) \ne 0\} = \deg(p)$. To avoid trivialities, we will assume that $m < d$. We will say that p *has no initial gap* if $\hat{p}(m+1) \ne 0$, and we say that p *has no final gap* if $\hat{p}(d-1) \ne 0$.

In order to use the asymptotic expansions of the previous sections to derive conclusions on *all* coefficients $\widehat{p^n}(k)$, $nm \le k \le nd$, we need to discuss the range of the function $r \longmapsto \mu_p(r)$, $r > 0$. Recall that strong positivity of p ensures that $\mu_p'(r) > 0$ for $r > 0$. It is then easy to see that, in the polynomial case, μ_p maps $(0,\infty)$ bijectively onto the interval (m,d) (see also [3, Corollary 6.5]). Hence, for every coefficient $\widehat{p^n}(k)$ (except the first and the last), we can find some $r > 0$ such that $n\mu_p(r) = k$. This is what allows us to use Theorem 5.4 to derive our global positivity and log-concavity results when p has no initial or final gaps.

Lemma 6.1. *Let $p \in \mathbb{R}[x]$, $p(0) \ne 0$, $p'(0) \ne 0$, and fix $M > 0$. Then, for each $k \le M$ as $n \to \infty$,*

(a) *the asymptotic behavior of $\widehat{p^n}(k)$ is given by*

$$
\widehat{p^n}(k) = \binom{n}{k}(p(0))^{n-k}(p'(0))^k\left(1 + O\left(\frac{1}{n}\right)\right),
\tag{6.1}
$$

(b) *the following asymptotic estimate holds:*

$$
\begin{aligned}
&(\widehat{p^n}(k))^2 - \widehat{p^n}(k-1)\widehat{p^n}(k+1) \\
&\quad = (p(0))^{2n-2k}(p'(0))^{2k}\frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}\left(1+O\left(\frac{1}{n}\right)\right).
\end{aligned}
\tag{6.2}
$$

Proof. For part (a), by considering the polynomial $p(x)/p(0)$, we may assume that $p(0) = 1$ without losing generality. Note that

$$\widehat{p^n}(k) = \sum \frac{n!}{u_0!u_1!\cdots u_d!}(\hat{p}(1))^{u_1}\cdots(\hat{p}(d))^{u_d}, \tag{6.3}$$

where the sum runs over all the nonnegative integral vectors $(u_0,u_1,\ldots,u_d)$ such that $\sum ju_j = k$ and $\sum u_j = n$. The term corresponding to the vector $(n-k,k,0,0,\ldots,0)$ contributes $\binom{n}{k}(p'(0))^k$ to the sum, and any other integral vector $(u_0,u_1,\ldots,u_d)$ as above is such that $u_0 > n-k$. So, the corresponding multinomial coefficient $n!/u_0!u_1!\cdots u_d!$ is not larger than $(C/n)\binom{n}{k}$ for some constant C. Since, clearly, $u_j \le k$ for $j > 0$, the result follows.

Part (b) follows from part (a) and the identity

$$\binom{n}{k}^2 - \binom{n}{k+1}\binom{n}{k-1} = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}. \tag{6.4}$$

$\square$

Theorem 6.2. *Let $p \in \mathbb{R}[x]$. Suppose that*

(i) *p is strongly positive at every $r > 0$,*

(ii) *p has no initial or final gaps.*

Then, p^n is log concave for all sufficiently large n. In particular, each coefficient $\widehat{p^n}(k)$ in the range $nm \le k \le nd$ is strictly positive for large n.

Proof. Multiplication (or division) by a monomial corresponds to a translation of the corresponding distribution of coefficients. Hence, without losing generality, we may assume that $p(0) \ne 0$ so that $m = 0$. Strong positivity of p guarantees that $p(0) > 0$, $p'(0) > 0$, $\hat{p}(d) > 0$, and $\hat{p}(d-1) > 0$. Let $L(n,k) = (\widehat{p^n}(k))^2 - \widehat{p^n}(k-1)\widehat{p^n}(k+1)$. By considering the polynomial $q(x) = x^d p(1/x)$ (so that $\hat{q}(k) = \hat{p}(d-k)$), it will be enough to prove that $\widehat{p^n}(k) > 0$ and $L(n,k) > 0$ for all large enough n, and $0 \le k \le (1/2)nd$. Since, clearly, $\widehat{p^n}(0) > 0$, we need only to consider that $0 < k \le (1/2)nd$. To prove that $\widehat{p^n}(k) > 0$ for k in this range, it will be enough to show that $\widehat{p^n}(k_n) > 0$ for every sequence $0 < k_n \le (1/2)nd$ such that $\lim_{n\to\infty} k_n = \infty$, because the case k-bounded is given by Lemma 6.1(a). So assume that $k_n \to \infty$ and find $r_n \in (0,\mu_p^{-1}(d/2)]$ such that $\mu_p(r_n) = k_n/n$. Using Lemma 5.2, we then have $\lim_{n\to\infty} nr_n = \infty$ so that Theorem 5.4 applies (with $X_n = [r_n,\mu_p^{-1}(d/2)]$ and $j = 0$), and we find $\widehat{p^n}(k_n) = \widehat{p^n}(n\mu(r_n)) = ((p(r_n))^n/r_n^{k_n}\sqrt{2\pi s})(1+O(1/s))$, where $s = n\sigma_p(r_n) \to \infty$ as $n \to \infty$. Hence, we have $\widehat{p^n}(k_n) > 0$ for large n.

To prove that $L(n,k) > 0$, we again consider a sequence $0 < k_n \le (1/2)nd$ such that $\lim_{n\to\infty} k_n = \infty$ as the case k-bounded is Lemma 6.1(b). If r_n is as before, we find that, from Theorem 5.4,

$$\begin{aligned}\left(\frac{r^{k_n}\sqrt{2\pi s}}{(p(r_n))^n}\right)^2 L(n,k_n) &= \left(1+d_1(0)\frac{1}{s}+O\left(\frac{1}{s^2}\right)\right)^2 \\ &\quad -\left(1+d_1(-1)\frac{1}{s}+O\left(\frac{1}{s^2}\right)\right)\left(1+d_1(1)\frac{1}{s}+O\left(\frac{1}{s^2}\right)\right) \\ &= (2d_1(0)-d_1(-1)-d_1(1))\frac{1}{s}+O\left(\frac{1}{s^2}\right),\end{aligned} \tag{6.5}$$

where

$$d_1(j) = -\frac{1}{2}\left[\frac{d^2}{dz^2}(e^{-jz}\phi(z)^{-3/2})\right]_{z=0} \tag{6.6}$$

Setting $\beta(z)=\phi(z)^{-3/2}$, we get

$$d_1(j) = -\frac{1}{2}(j^2-2j\beta'(0)+\beta''(0)) \tag{6.7}$$

so that (6.5) gives that

$$\begin{aligned} s\left(\frac{r^{k_n}\sqrt{2\pi s}}{(p(r_n))^n}\right)^2 L(n,k_n) &= -\beta''(0)+\frac{1}{2}(1+2\beta'(0)+\beta''(0)) \\ &\quad +\frac{1}{2}(1-2\beta'(0)+\beta''(0))+O\left(\frac{1}{s}\right) \\ &= 1+O\left(\frac{1}{s}\right)>0 \quad \text{for large } n.\end{aligned} \tag{6.8}$$

□

It is easy to check that if a polynomial p has no initial or final gaps and there is some n such that p^n has no negative coefficients, then p must be strongly positive at every r (the no-gap condition eliminates trivial cases such as $p(x)=1+x^2$ that is not strongly positive at any r because $p(re^{i\pi})=p(r)$). Hence, we obtain the following corollary.

Corollary 6.3. *Suppose that $p\in\mathbb{R}[x]$ has no initial or final gaps and $p(1)>0$. Then, the following are equivalent:*

(i) *there is some n such that p^n has no negative coefficients,*
(ii) *p^n has no negative coefficients for all sufficiently large n,*
(iii) *p is strongly positive at every r.*

The implication (i)⇒(ii) for polynomials in several variables was proved by Handelman in [6].

We conclude with some examples.

Example 6.4. Let $p(x)=b+4ax-2x^2+4ax^3+bx^4$, where a and b are positive real numbers. It is shown in [3, Example 8.4] that p is strongly positive at every r if and only if $b > 1$ and $a > b - \sqrt{b(b-1)}$. So, for any such choice of a and b, $(p(x))^n$ has no negative coefficients for large n.

Example 6.5. Let $p(x) = 2 + 2x - x^2 + 2x^3 + 2x^4$. Using the criterion of Example 6.4 applied to $2p(x)$ (with $b = 4$, $a = 1$), we see that $p(x)$ is strongly positive at every r. Since $|1 + z^3 p(z)| \le 1 + |z^3 p(z)| \le 1 + |z|^3 p(|z|)$, we conclude that $q(x) = 1 + x^3 p(x) = 1 + 2x^3 + 2x^4 - x^5 + 2x^6 + 2x^7$ is also strongly positive at every r. However, the coefficient of x^5 will clearly be negative in every power of q. So, we see that the no-gap condition is necessary in Theorem 6.2.

As a final remark, we note that the proofs of Theorems 4.1 and 5.4 can be carried out without modifications if we replace the factor e^{-jz} by $g(z)e^{-jz}$, where $g(z)$ is any continuous function. We then obtain an asymptotic expansion for the coefficients $\widehat{f^n g}(r\mu + j)$ (with leading term $f(r)^n g(r)/r^{n\mu+j}\sqrt{2\pi n\sigma}$). Then, Lemma 6.1 and Theorem 6.2 (with obvious modifications) give us the following generalization of a result of Handelman [7] (obtained by taking $p(x) = x + 1$).

Theorem 6.6. *Let $p, q \in \mathbb{R}[x]$. Suppose that*

(i) *p is strongly positive at every $r > 0$,*

(ii) *p has no initial or final gaps,*

(iii) *$q(r) > 0$ for all $r > 0$.*

Then, $p^n q$ is log concave for all sufficiently large n.

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