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ON RESOLVING EDGE COLORINGS IN GRAPHS

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We study the relationships between the resolving edge chromatic number and other graphical parameters and provide bounds for the resolving edge chromatic number of a connected graph.

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1. Introduction. For edges e and f in a connected graph G, the *distance* $d(e,f)$ between e and f is the minimum nonnegative integer a for which there exists a sequence $e = e_0, e_1, \ldots, e_a = f$ of edges of G such that e_i and e_{i+1} are adjacent for $i = 0, 1, \ldots, a-1$. For an edge e of G and a subgraph F of positive size in G, the *distance* between e and F is defined as

$$d(e,F) = \min\{d(e,f) : f \in E(F)\}. \tag{1.1}$$

A *decomposition* of a graph G is a collection of subgraphs of G, none of which have isolated vertices, whose edge sets provide a partition of $E(G)$. A decomposition of G into k subgraphs is a *k-decomposition.* A decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ is *ordered* if the ordering $(G_1, G_2, \ldots, G_k)$ has been imposed on $\mathcal{D}$. For an ordered k-decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph G and $e \in E(G)$, the *$\mathcal{D}$-code* (or simply the *code*) of e is the k-vector

$$c_{\mathcal{D}}(e) = (d(e,G_1), d(e,G_2), \ldots, d(e,G_k)). \tag{1.2}$$

Hence exactly one coordinate of $c_{\mathcal{D}}(e)$ is 0, namely the ith coordinate if $e \in E(G_i)$. In [3], a decomposition $\mathcal{D}$ is defined to be a *resolving decomposition* for G if every two distinct edges of G have distinct $\mathcal{D}$-codes. The minimum k for which G has a resolving k-decomposition is its *decomposition dimension* $\dim_d(G)$. A resolving decomposition of G with $\dim_d(G)$ elements is a *minimum resolving decomposition* for G.

A resolving decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph G is defined in [5] to be *independent* if $E(G_i)$ is independent for each i $(1 \le i \le k)$ in G. This concept can be considered from an edge-coloring point of view. Recall that a *proper edge coloring* (or simply, an edge coloring) of a nonempty graph G is an assignment c of colors (positive integers) to the edges of G so that adjacent edges are colored differently, that is, $c : E(G) \to \mathbb{N}$ is a mapping such that $c(e) \neq c(f)$ if e and f are adjacent edges of G. The minimum k for which there is an edge coloring of G using k distinct colors is called the *edge chromatic number* $\chi_e(G)$ of G. If $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ is an independent decomposition of a graph G, then by assigning color i to all edges in G_i for each i with $1 \leq i \leq k$, we obtain an edge coloring of G using k distinct colors. On the other hand, if c is an edge coloring of a connected graph G, using the colors $1, 2, \ldots, k$ for some positive integer k, then $c(e) \neq c(f)$ for adjacent edges e and f in G. Equivalently, c produces a decomposition $\mathcal{D}$ of $E(G)$ into color classes (independent sets) $C_1, C_2, \ldots, C_k$, where the edges of C_i are colored i for $1 \leq i \leq k$. Thus, for an edge e in a graph G, the k-vector

$$c_{\mathcal{D}}(e) = \big(d(e, C_1), d(e, C_2), \ldots, d(e, C_k)\big) \tag{1.3}$$

is called the *color code* (or simply the *code*) $c_{\mathcal{D}}(e)$ of e. If distinct edges of G have distinct color codes, then c is called a *resolving edge coloring* (or *independent resolving decomposition*) of G in [5]. Thus a resolving edge coloring of G is an edge coloring that distinguishes all edges of G in terms of their distances from the resulting color classes. A *minimum resolving edge coloring* uses a minimum number of colors, and this number is the *resolving edge chromatic number* $\chi_{re}(G)$ of G. Since every resolving edge coloring is an edge coloring and every resolving edge coloring is a resolving decomposition, it follows that

$$2 \leq \max\{\dim_d(G), \chi_e(G)\} \leq \chi_{re}(G) \leq m \tag{1.4}$$

for each connected graph G of size $m \geq 2$.

To illustrate these concepts, consider the graph G of Figure 1.1. Let $\mathcal{D}_1 = \{G_1, G_2, G_3\}$ be the decomposition of G, where $E(G_1) = \{v_1v_2, v_2v_5\}$, $E(G_2) = \{v_2v_3, v_2v_6, v_3v_6\}$, and $E(G_3) = \{v_3v_4, v_3v_5\}$. Since $\mathcal{D}_1$ is a minimum resolving decomposition of G, it follows that $\dim_d(G) = 3$. Define an edge coloring c of G by assigning the color 1 to v_1v_2 and v_3v_5, the color 2 to v_2v_5 and v_3v_6, the color 3 to v_2v_3, and the color 4 to v_2v_6 and v_3v_4 (see Figure 1.1(b)). Since c is a minimum edge coloring of G, it follows that $\chi_e(G) = 4$. However, c is not a resolving edge coloring. To see that, let $\mathcal{D}_2 = \{C_1, C_2, C_3, C_4\}$ be the decomposition of G into color classes resulting from c, where the edges in C_i are colored i by c. Then $c_{\mathcal{D}_2}(v_2v_5) = (1,0,1,1) = c_{\mathcal{D}_2}(v_3v_6)$. On the other hand, define an edge coloring c^* of G by assigning the color 1 to v_1v_2 and v_3v_5, the color 2 to v_2v_3, the color 3 to v_2v_5 and v_3v_4, the color 4 to v_2v_6, and the color 5 to v_3v_6 (see Figure 1.1(c)). Let $D^* = \{C_1, C_2, \ldots, C_5\}$ be the decomposition of G into color classes of c^*. Then

$$\begin{gathered}
c_{\mathcal{D}^*}(v_1v_2) = (0,1,1,1,2), \qquad c_{\mathcal{D}^*}(v_2v_3) = (1,0,1,1,1), \\
c_{\mathcal{D}^*}(v_2v_5) = (1,1,0,1,2), \qquad c_{\mathcal{D}^*}(v_2v_6) = (1,1,1,0,1), \\
c_{\mathcal{D}^*}(v_3v_4) = (1,1,0,2,1), \qquad c_{\mathcal{D}^*}(v_3v_5) = (0,1,1,2,1), \\
c_{\mathcal{D}^*}(v_3v_6) = (1,1,1,1,0).
\end{gathered} \tag{1.5}$$

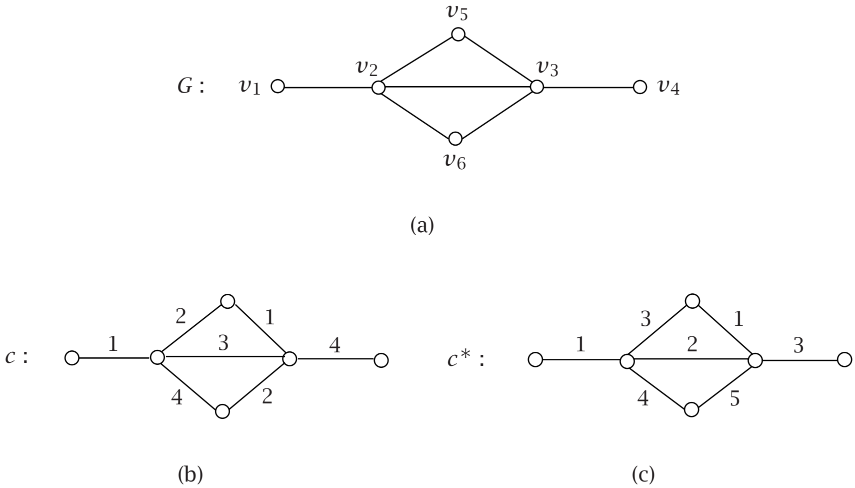

FIGURE 1.1. A graph G with $\dim_d(G) = 3$, $\chi_e(G) = 4$, and $\chi_{re}(G) = 5$.

Since the D^*-codes of the edges of G are all distinct, it follows that c^* is a resolving edge coloring. Moreover, G has no resolving edge coloring with 4 colors and so $\chi_{re}(G) = 5$.

The concept of resolvability in graphs has previously appeared in [7, 11, 12]. Slater [11, 12] introduced this concept and motivated by its application to the placement of a minimum number of sonar detecting devices in a network so that the position of every vertex in the network can be uniquely determined in terms of its distance from the set of devices. Harary and Melter [7] discovered these concepts independently as well. Resolving decompositions in graphs were introduced and studied in [3] and further studied in [6]. Resolving decompositions with prescribed properties have been studied in [5, 9, 10]. Resolving concepts were studied from the point of view of graph colorings in [1, 2]. We refer to [4] for graph theory notation and terminology not described here.

In [5], all nontrivial connected graphs of size m with resolving edge chromatic number 3 or m are characterized. Also, bounds have been established for $\chi_{re}(G)$ of a connected graph G in terms of its size, diameter, or girth, as stated below.

THEOREM 1.1. *If G is a connected graph of size $m \geq 3$ and diameter d, then*

$$2 \leq \chi_{re}(G) \leq m - d + 3. \tag{1.6}$$

Moreover, $\chi_{re}(G) = 2$ if and only if $G = P_3$, and $\chi_{re}(G) = m - d + 3$ if and only if $G = P_n$ for $n \geq 4$.

Theorem 1.2. *If G is a connected graph of size m and girth ℓ, where $m \geq \ell \geq 3$, then*

$$\chi_{re}(G) \leq m - \ell + 4. \tag{1.7}$$

Moreover, $\chi_{re}(G) = m - \ell + 4$ if and only if $G = C_n$ for some even $n \geq 4$.

In this paper, we study the relationships among the resolving edge chromatic number, edge chromatic number, and decomposition dimension of a connected graph, and provide bounds for the resolving edge chromatic number of a connected graph in terms of other graphical parameters in Section 2. We investigate the resolving edge colorings of trees in Section 3.

2. Bounds for resolving edge chromatic numbers. In this section, we establish bounds for the resolving edge chromatic number of a connected graph in terms of (1) its order and edge chromatic number; (2) its decomposition dimension and edge chromatic number. In order to this, we need some additional definitions and preliminary results. Let $\mathcal{D}$ be a decomposition of a connected graph G. Then a decomposition $\mathcal{D}^*$ of G is called a *refinement* of $\mathcal{D}$ if every element in $\mathcal{D}^*$ is a subgraph of some element of $\mathcal{D}$. First, we present two lemmas, the first of which appears in [9].

Lemma 2.1. *Let $\mathcal{D}$ be a resolving decomposition of a connected graph G. If $\mathcal{D}^*$ is a refinement of $\mathcal{D}$, then $\mathcal{D}^*$ is also a resolving decomposition of G.*

Lemma 2.2. *Let G be a connected graph of order $n \geq 5$, let T be a spanning tree of G with $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$, and let $H = G - E(T)$. Then the decomposition $\mathcal{D} = \{F_1, F_2, \ldots, F_{n-1}, H\}$, where $E(F_i) = \{e_i\}$ for $1 \leq i \leq n-1$, is a resolving decomposition of G.*

Proof. Let e and f be two edges of G. If e and f belong to distinct elements of $\mathcal{D}$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Thus we may assume that e and f belong to the same element H in $\mathcal{D}$. We show that $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Let $e = uv$, let P be the unique $u - v$ path in T, and let u' and v' be the vertices on P adjacent to u and v, respectively. If f is adjacent to at most one of uu' and vv', then either $d(e, uu') \neq d(f, uu')$ or $d(e, vv') \neq d(f, vv')$, and so $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Hence we may assume that f is adjacent to both uu' and vv'. We consider two cases according to whether $u' = v'$ or $u' \neq v'$.

Case 1 ($u' = v'$). Then f is incident with the vertex u'. Since $n \geq 5$ and T is a spanning tree, there is a vertex $x \in V(G) - \{u, v, u'\}$ such that x is adjacent in T with exactly one of u, v, and u'. If $u'x \in E(T)$, then $d(f, u'x) = 1 \neq 2 = d(e, u'x)$; otherwise, $d(e, ux) = 1 \neq 2 = d(f, ux)$ or $d(e, vx) = 1 \neq 2 = d(f, vx)$ according to whether ux or vx is an edge of T. So $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$.

Case 2 ($u' \neq v'$). Then we may assume that f is incident with u'. Let g be an edge of T distinct from uu' that is incident with u'. Then $d(e, g) = 2 \neq 1 = d(f, g)$. Thus $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. □

We now present bounds for the resolving edge chromatic number of a connected graph in terms of its order and edge chromatic number.

Theorem 2.3. *If G is a connected graph of order $n \geq 5$, then*

$$\chi_e(G) \leq \chi_{re}(G) \leq n + \chi_e(G) - 1. \tag{2.1}$$

Proof. The lower bound follows by (1.4). To verify the upper bound, let m be the size of G. If G is a tree of order n, then $m = n-1$. Since $\chi_{re}(G) \leq m$, the result is true for a tree. Thus we may assume that G is a connected graph that is not a tree. Let T be a spanning tree of G with $E(T) = \{e_1, e_2, \dots, e_{n-1}\}$. Let $H = \langle E(G) - E(T) \rangle$ be the subgraph induced by $E(G) - E(T)$. Then H is a nonempty subgraph of G. Let $\chi_e(H) = k$ and let $H_1, H_2, \dots, H_k$ be the decomposition of H into the color classes resulting from a minimum edge coloring of H. Now let

$$\mathfrak{D} = \{F_1, F_2, \dots, F_{n-1}, H\}, \qquad \mathfrak{D}^* = \{F_1, F_2, \dots, F_{n-1}, H_1, H_2, \dots, H_k\}, \tag{2.2}$$

where $E(F_i) = \{e_i\}$ for $1 \leq i \leq n-1$. Since $\mathfrak{D}$ is a resolving decomposition of G by Lemma 2.2 and D^* is a refinement of $\mathfrak{D}$, it follows by Lemma 2.1 that D^* is a resolving decomposition of G as well. Thus $\mathfrak{D}^*$ is a resolving independent decomposition of G, and so

$$\chi_{re}(G) \leq |\mathfrak{D}^*| = n + k - 1 = n + \chi_e(H) - 1 \leq n + \chi_e(G) - 1, \tag{2.3}$$

as desired. □

Next, we present bounds for the resolving edge chromatic number of a connected graph in terms of its decomposition dimension and edge chromatic number.

Theorem 2.4. *For every connected graph G of order at least 3,*

$$\dim_d(G) \leq \chi_{re}(G) \leq \chi_e(G)\dim_d(G). \tag{2.4}$$

Proof. By (1.4), it suffices to verify the upper bound: let G be a nontrivial connected graph with $\dim_d(G) = k$ and $\chi_e(G) = c$. Furthermore, let $\mathfrak{D} = \{G_1, G_2, \dots, G_k\}$ be a resolving decomposition of G. If $\mathfrak{D}$ is independent, then $\mathfrak{D}$ is a resolving independent decomposition of G and so $\chi_{re}(G) \leq |\mathfrak{D}| = k = \dim_d(G) < \chi_e(G)\dim_d(G)$ since $\chi_e(G) \geq 2$. Thus we may assume that $\mathfrak{D}$ is not independent. Without loss of generality, assume that $E(G_i)$ is not independent in $E(G)$ for $1 \leq i \leq k_1 \leq k$ and $E(G_i)$ is independent in $E(G)$ for $k_1 + 1 \leq i \leq k$ if $k_1 < k$. Let $c_i = \chi_e(G_i)$ for $1 \leq i \leq k$ and so $1 \leq c_i \leq \chi_e(G)$. Define a decomposition $\mathfrak{D}'$ of G from $\mathfrak{D}$ by (1) decomposing each G_i $(1 \leq i \leq k_1)$ into c_i color classes resulting from an edge coloring of G_i; (2) retaining each G_i for $k_1 + 1 \leq i \leq k$. Certainly, $\mathfrak{D}'$ is an independent decomposition of G with at most $\sum_{i=1}^{k} c_i \leq ck$ elements. Since $\mathfrak{D}'$ is a refinement of $\mathfrak{D}$, it follows by virtue of Lemma 2.1 that $\mathcal{D}'$ is also an independent resolving decomposition of G. Therefore, $\chi_{re}(G) \le |\mathcal{D}'| \le ck = \chi_e(G)\dim_d(G)$. □

3. On resolving edge chromatic numbers of trees. The decomposition dimension of a tree T was studied in [3, 6]. It was shown in [3] that P_n is the only connected graph of order n with decomposition dimension 2. Although there is no general formula for the decomposition dimension of a nonpath tree, several bounds have been established for $\dim_d(T)$ for such trees in [3, 6]. In this section, we investigate the resolving edge chromatic number of trees. Since $\chi_{re}(P_3) = 2$ and $\chi_{re}(P_n) = 3$ for $n \ge 4$, we consider trees that are not paths. First, we need some additional definitions and notation.

A vertex of degree at least 3 in a graph G is called a *major vertex*. An end-vertex u of G is said to be a *terminal vertex of a major vertex* v of G if $d(u,v) < d(u,w)$ for every other major vertex w of G. The *terminal degree* $\operatorname{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\sigma(G)$ denote the sum of the terminal degrees of the major vertices of G and let $\operatorname{ex}(G)$ denote the number of exterior major vertices of G. In fact, $\sigma(G)$ is the number of end-vertices of G. For an ordered set $W = \{e_1, e_2, \dots, e_k\}$ of edges in a connected graph G and an edge e of G, let

$$c_W(e) = (d(e,e_1), d(e,e_2), \dots, d(e,e_k)). \tag{3.1}$$

The following two results are useful to us, the first of which appeared in [9] and the second of which is due to König [8].

Lemma 3.1. *Let T be a tree that is not a path, having order $n \ge 4$ and p exterior major vertices $v_1, v_2, \dots, v_p$. For $1 \le i \le p$, let $u_{i1}, u_{i2}, \dots, u_{ik_i}$ be the terminal vertices of v_i, let P_{ij} be the $v_i - u_{ij}$ path $(1 \le j \le k_i)$, and let x_{ij} be a vertex in P_{ij} that is adjacent to v_i. Let*

$$W = \{v_i x_{ij} : 1 \le i \le p,\ 2 \le j \le k_i\}. \tag{3.2}$$

Then $c_W(e) \ne c_W(f)$ for each pair e, f of distinct edges of T that are not edges of P_{ij} for $1 \le i \le p$ and $2 \le j \le k_i$.

König's theorem. *If G is a bipartite graph, then $\chi_e(G) = \Delta(G)$. In particular, if T is a tree, then $\chi_e(T) = \Delta(T)$.*

For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H with the vertex v, together with all edges joining v and $V(H)$ in G, is called a *branch of G at v*. For a bridge e in a connected graph G and a component F of $G - e$, the subgraph F, together with the bridge e, is called a *branch of G at e*. For two edges $e = u_1u_2$ and $f = v_1v_2$ in G, an *$e - f$ path* in G is a path with its initial edge e and terminal edge f.

We are now prepared to present an upper bound for the resolving edge chromatic number of a tree that is not a path.

Theorem 3.2. *Let T be a tree that is not a path, having order $n \geq 4$ and p exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of v_i, let P_{ij} be the $v_i - u_{ij}$ path $(1 \leq j \leq k_i)$, and let x_{ij} be a vertex in P_{ij} that is adjacent to v_i. Let W be the set described in (3.2). Then*

$$\chi_{re}(T) \leq \Delta(T-W) + \sigma(T) - \text{ex}(T). \tag{3.3}$$

Proof. Let $U = \{v_1, u_{11}, u_{21}, \ldots, u_{p1}\}$ and let T_0 be the subtree of T of smallest size that contains U. For each pair i, j of integers with $1 \leq i \leq p$ and $1 \leq j \leq k_i$, let $Q_{ij} = P_{ij} - v_i$ be the $x_{ij} - u_{ij}$ path in T. Thus $T - W$ is the union of the tree T_0 and the paths Q_{ij} for all i, j with $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Since $T - W$ is a forest, it follows by König's theorem that $\chi_e(T-W) = \Delta(T-W)$. We define an edge coloring c of T by assigning (1) the colors to the edges in $T - W$ from the set $\{1, 2, \ldots, \Delta(T-W)\}$; (2) the color

$$c_{ij} = \Delta(T-W) + [k_1 + k_2 + \cdots + k_{i-1} - (i-1)] + (j-1) \tag{3.4}$$

to the edge $v_i x_{ij}$ in W for all i, j with $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Thus the maximum color assigned to the vertices of G by c is

$$\begin{aligned} c_{p,k_p} &= c(v_p x_{p,k_p}) \\ &= \Delta(T-W) + [k_1 + k_2 + \cdots + k_{p-1} - (p-1)] + (k_p - 1) \\ &= \Delta(T-W) + (k_1 + k_2 + \cdots + k_p - p) \\ &= \Delta(T-W) + \sigma(T) - \text{ex}(T). \end{aligned} \tag{3.5}$$

Certainly, adjacent edges are colored differently by c and so c is an edge coloring of T. It remains to show that c is a resolving edge coloring of T. Let

$$k = \Delta(T-W) + \sigma(T) - \text{ex}(T) \tag{3.6}$$

and let $\mathcal{D} = \{C_1, C_2, \ldots, C_k\}$ be the decomposition of G into the color classes resulting from c. Since all edges in W are colored differently, it suffices to show that if $e, f \in E(T-W)$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. We consider three cases.

Case 1 ($e, f \in E(T_0)$). By Lemma 3.1, it follows that $c_W(e) \neq c_W(f)$, which implies that $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$.

Case 2 ($e, f \notin E(T_0)$). There are two subcases.

Subcase 2.1 ($e, f \in E(Q_{ij})$ for some i, j with $1 \leq i \leq p$ and $2 \leq j \leq k_i$). Since $v_i x_{ij} \in W$ and $d(e, v_i x_{ij}) \neq d(f, v_i x_{ij})$, this implies that $c_W(e) \neq c_W(f)$ and so $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$.

Subcase 2.2 ($e \in E(Q_{ij})$ and $f \in E(Q_{rs})$, where $1 \le i, r \le p$, $2 \le j$, and $s \le k_i$). Notice that if $i = r$, then $j \ne s$. Again, $v_ix_{ij}, v_rx_{rs} \in W$. If $d(e, v_ix_{ij}) \ne d(f, v_ix_{ij})$, then $c_{\mathcal{D}}(e) \ne c_{\mathcal{D}}(f)$. On the other hand, if $d(e, v_ix_{ij}) = d(f, v_ix_{ij})$, then $d(f, v_rx_{rs}) < d(e, v_rx_{rs})$, implying that $c_{\mathcal{D}}(e) \ne c_{\mathcal{D}}(f)$.

Case 3 (exactly one of e and f belongs to T_0, say $f \in E(T_0)$ and $e \in E(Q_{ij})$ for some i, j with $1 \le i \le p$ and $2 \le j \le k_i$). If there is an edge $w \in W$ such that f lies on the $e - w$ path, then $d(f, w) < d(e, w)$ and so $c_{\mathcal{D}}(e) \ne c_{\mathcal{D}}(f)$. Thus we may assume that every path between e and any edge $w \in W$ does not contain f. Then f lies on some path $P_{\ell 1}$ in T for some ℓ with $1 \le \ell \le p$. We consider two subcases.

Subcase 3.1 ($i = \ell$). If $d(e, v_ix_{ij}) \ne d(f, v_ix_{ij})$, then $c_{\mathcal{D}}(e) \ne c_{\mathcal{D}}(f)$. Thus we may assume that $d(e, v_ix_{ij}) = d(f, v_ix_{ij})$. Since v_i is an exterior vertex of T, it follows that $\deg v_i \ge 3$ and so there exists a branch B at v_i that does not contain v_ix_{ij}. Necessarily, B must contain an edge w of W. Then $d(f, w) < d(e, w)$ and so $c_{\mathcal{D}}(e) \ne c_{\mathcal{D}}(f)$.

Subcase 3.2 ($i \ne \ell$). Since v_i and v_ℓ are exterior major vertices, it follows that $\deg v_i \ge 3$ and $\deg v_\ell \ge 3$. Thus there exists a branch B_1 at v_i that does not contain v_ix_{ij} and a branch B_2 at v_ℓ that does not contain $v_\ell x_{\ell 1}$. Necessarily, each of B_1 and B_2 must contain an edge of W. Let w_1 and w_2 be two edges of T such that w_i belongs to B_i for $i = 1, 2$. If $d(e, w_2) \ne d(f, w_2)$, then $c_W(e) \ne c_W(f)$ and so $c_{\mathcal{D}}(e) \ne c_{\mathcal{D}}(f)$. Thus we may assume that $d(e, w_2) = d(f, w_2)$. However, then, $d(e, w_1) < d(f, w_1)$, implying that $c_W(e) \ne c_W(f)$ and so $c_{\mathcal{D}}(e) \ne c_{\mathcal{D}}(f)$.

Thus, in any case, $c_{\mathcal{D}}(e) \ne c_{\mathcal{D}}(f)$ and so $\mathcal{D}$ is a resolving edge coloring of G. Therefore, $\chi_{re}(T) \le \Delta(T - W) + \sigma(T) - \operatorname{ex}(T)$. □

The upper bound in Theorem 3.2 is sharp. To see this, let $K_{1,n}$, $n \ge 3$, be the star with $V(K_{1,n}) = \{v, v_1, v_2, \dots, v_n\}$, where v is the central vertex of $K_{1,n}$, and let T be the tree obtained from $K_{1,n}$ by subdividing each edge vv_i into vx_i and x_iv_i for $2 \le i \le n$. Let $W = \{vx_i : 2 \le i \le n\}$. Then it can be verified that $\chi_{re}(T) = \Delta(T - W) + \sigma(T) - \operatorname{ex}(T) = n$.

Next, we present another upper bound for $\chi_{re}(T)$ in terms of the maximum degree of a tree T. A major vertex of a tree T is a *superior major vertex* of T if its terminal degree is at least 2. Let $\sup(T)$ denote the number of superior major vertices of T. Thus every superior major vertex of T is also an exterior major vertex. Hence, if T is a tree that is not a path, then $1 \le \sup(T) \le \operatorname{ex}(T)$.

Theorem 3.3. *If T is a tree that is not a path, then*

$$\chi_{re}(T) \le \Delta(T) + \sup(T). \tag{3.7}$$

Proof. Suppose that T contains $q \ge 1$ superior major vertices $v_1, v_2, \dots, v_q$. For $1 \le i \le q$, let $u_{i1}, u_{i2}, \dots, u_{ik_i}$ be the terminal vertices of v_i, where $k_i \ge 2$. For each i, j with $1 \le i \le q$ and $1 \le j \le k_i$, let P_{ij} be the $v_i - u_{ij}$ path in T, let x_{ij} be the vertex in P_{ij} that is adjacent to v_i, and let $Q_{ij} = P_{ij} - v_i$ be the $x_{ij} - u_{ij}$ path in T. Furthermore, let

$$W^* = \{v_i x_{i2} : 1 \le i \le q\} \tag{3.8}$$

and let T_1 be the subgraph of T obtained by removing all vertices in each set $V(Q_{ij}) - \{x_{ij}\}$ from T for all i, j with $1 \le i \le q$ and $1 \le j \le k_i$; that is,

$$T_1 = T - \left(\cup \{V(Q_{ij}) - \{x_{ij}\} : 1 \le i \le q,\ 1 \le j \le k_i\}\right). \tag{3.9}$$

Let Q be the linear forest whose components are the paths Q_{ij} ($1 \le i \le q$ and $1 \le j \le k_i$) in T; that is,

$$Q = \cup \{Q_{ij} : 1 \le i \le q,\ 1 \le j \le k_i\}. \tag{3.10}$$

Let

$$T_0 = T_1 - \{x_{i2} : 1 \le i \le q\}. \tag{3.11}$$

Then $E(T_0) = E(T_1) - W^*$ and

$$E(T) = E(T_0) \cup W^* \cup E(Q). \tag{3.12}$$

Hence $E(T)$ is partitioned into $E(T_0)$, W^*, and $E(Q)$. We define an edge coloring c of T by coloring the edges in each of the sets $E(T_0)$, W^*, and $E(Q)$ in the following three steps:

(1) if T has only one exterior major vertex, then this exterior major vertex is also a superior major vertex since T is not a path. Thus $\Delta(T_0) = \Delta(T) - 1$ and so $\chi_e(T_0) = \Delta(T) - 1$. Let c_1 be an edge coloring of T_0 using $\Delta(T) - 1$ colors and define $c(e) = c_1(e)$ for all $e \in E(T_0)$. If T has more than one exterior major vertex, then $\Delta(T_0) \le \Delta(T)$ and so $\chi_e(T_0) \le \Delta(T)$. Let c_1' be an edge coloring of T_0 using $\Delta(T)$ colors and define $c(e) = c_1'(e)$ for all $e \in E(T_0)$;

(2) define $c(v_i x_{i2}) = \Delta(T) + i$ for each edge $v_i x_{i2}$ in W^*, where $1 \le i \le q$;

(3) define $c(e)$ for each edge e in Q. For each pair i, j with $1 \le i \le q$ and $1 \le j \le k_i$, let $m_{ij} = |E(Q_{ij})|$ and

$$E(Q_{ij}) = \left\{e_{ij}^1, e_{ij}^2, \dots, e_{ij}^{m_{ij}}\right\}, \tag{3.13}$$

where (1) e_{ij}^1 is incident with x_{ij}, (2) $e_{ij}^{m_{ij}}$ is incident with u_{ij}, (3) e_{ij}^s is adjacent to e_{ij}^{s+1} in Q_{ij} for all s with $1 \le s \le m_{ij} - 1$. Let

$$T_0^* = T_1 - \{x_{ij} : 1 \le i \le q,\ 1 \le j \le k_i\}. \tag{3.14}$$

For each i with $1 \le i \le q$, let $d_i = \deg_{T_0^*} v_i$, and so the degree of v_i in T is

$$\deg v_i = d_i + k_i \le \Delta(T). \tag{3.15}$$

We consider two cases according to whether $d_i = 0$ or $d_i > 0$.

Case 1 ($d_i = 0$). Thus $N_{T_0^*}(v_i) = \emptyset$. This implies that T has only one exterior major vertex that is also a superior major vertex. Notice that if $j_1, j_2 \in \{1,3,4,\dots,k_1\}$ and $j_1 \ne j_2$, then $v_1x_{1j_1}$ and $v_1x_{1j_2}$ are adjacent edges in T_0 and so $c(v_1x_{1j_1}) \ne c(v_1x_{1j_2})$. There are two subcases.

Subcase 1.1 ($k_1 = 3$). Define

$$c(e_{11}^s) = c(v_1x_{13}) \quad \text{if } s \text{ is odd, } 1 \le s \le m_{11}, \tag{3.16}$$

$$c(e_{11}^s) = c(v_1x_{11}) \quad \text{if } s \text{ is even, } 2 \le s \le m_{11}, \tag{3.17}$$

$$\begin{aligned} c(e_{12}^s) &= \Delta(T) \quad \text{if } s \text{ is odd, } 1 \le s \le m_{12}, \\ c(e_{12}^s) &= c(v_1x_{11}) \quad \text{if } s \text{ is even, } 2 \le s \le m_{12}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} c(e_{13}^s) &= \Delta(T) \quad \text{if } s \text{ is odd, } 1 \le s \le m_{13}, \\ c(e_{13}^s) &= c(v_1x_{13}) \quad \text{if } s \text{ is even, } 2 \le s \le m_{13}. \end{aligned} \tag{3.19}$$

Subcase 1.2 ($k_1 \ge 4$). For s is even and $2 \le s \le m_{11}$, define $c(e_{11}^s)$ as in (3.17); for $1 \le s \le m_{12}$, define $c(e_{12}^s)$ as in (3.18); for $1 \le s \le m_{13}$, define $c(e_{13}^s)$ as in (3.19). Furthermore, define

$$\begin{aligned} c(e_{11}^s) &= c(v_1x_{1k_1}) \quad \text{if } s \text{ is odd, } 1 \le s \le m_{11}, \\ c(e_{1j}^s) &= c(v_1x_{1,j-1}) \quad \text{if } s \text{ is odd, } 1 \le s \le m_{1j},\ 4 \le j \le k_1, \\ c(e_{1j}^s) &= c(v_1x_{1j}) \quad \text{if } s \text{ is even, } 2 \le s \le m_{1j},\ 4 \le j \le k_1. \end{aligned} \tag{3.20}$$

Case 2 ($d_i > 0$). Thus $N_{T_0^*}(v_i) \ne \emptyset$. Let $x \in N_{T_0^*}(v_i)$. Then v_ix and v_ix_{ij} ($1 \le j \le k_1$) are adjacent edges in T_0 and so all colors $c(v_ix)$ and $c(v_ix_{ij})$, $1 \le j \le k_1$, are distinct. There are three subcases.

Subcase 2.1 ($k_i = 2$). Define

$$c(e_{i1}^s) = c(v_ix) \quad \text{if } s \text{ is odd, } 1 \le s \le m_{i1}, \tag{3.21}$$

$$c(e_{i1}^s) = c(v_ix_{i1}) \quad \text{if } s \text{ is even, } 2 \le s \le m_{i1}, \tag{3.22}$$

$$\begin{aligned} c(e_{i2}^s) &= c(v_ix) \quad \text{if } s \text{ is odd, } 1 \le s \le m_{i2}, \\ c(e_{i2}^s) &= c(v_ix_{i1}) \quad \text{if } s \text{ is even, } 2 \le s \le m_{i2}. \end{aligned} \tag{3.23}$$

Subcase 2.2 ($k_i = 3$). For s is even and $2 \le s \le m_{i1}$, define $c(e_{i1}^s)$ as in (3.22); for $1 \le s \le m_{i2}$, define $c(e_{i2}^s)$ as in (3.23), and define

$$c(e_{i1}^s) = c(v_ix_{i3}) \quad \text{if } s \text{ is odd, } 1 \le s \le m_{i1}, \tag{3.24}$$

$$\begin{aligned} c(e_{i3}^s) &= c(v_ix) \quad \text{if } s \text{ is odd, } 1 \le s \le m_{i3}, \\ c(e_{i3}^s) &= c(v_ix_{i3}) \quad \text{if } s \text{ is even, } 2 \le s \le m_{i3}. \end{aligned} \tag{3.25}$$

Subcase 2.3 ($k_i \geq 4$)**.** For s is even and $2 \leq s \leq m_{i1}$, define $c(e_{i1}^s)$ as in (3.22); for $1 \leq s \leq m_{i2}$, define $c(e_{i2}^s)$ as in (3.23); for $1 \leq s \leq m_{i3}$, define $c(e_{i3}^s)$ as in (3.25). Furthermore, define

$$\begin{aligned} c(e_{i1}^s) &= c(v_i x_{ik_i}) \quad \text{if } s \text{ is odd, } 1 \leq s \leq m_{i1}, \\ c(e_{ij}^s) &= c(v_i x_{i,j-1}) \quad \text{if } s \text{ is odd, } 1 \leq s \leq m_{ij},\ 4 \leq j \leq k_i, \\ c(e_{ij}^s) &= c(v_i x_{ij}) \quad \text{if } s \text{ is even, } 2 \leq s \leq m_{ij},\ 4 \leq j \leq k_i. \end{aligned} \tag{3.26}$$

Since adjacent edges of T are colored differently by c, it follows that c is an edge coloring of T using $\Delta(T)+q$ colors. It remains to show that c is a resolving edge coloring of T. Let $\mathcal{D} = \{C_1, C_2, \ldots, C_{\Delta(T)+q}\}$ be the decomposition of T into the color classes of c. Since all edges in W^* are colored differently by c, it suffices to show that if $e, f \in E(T - W^*)$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. We consider two cases.

Case 1 (there is some exterior major vertex z of T and a terminal vertex x of z such that e lies on the $z-x$ path of T)**.** Let y be a vertex in the $z-x$ path that is adjacent to z. There are two subcases.

Subcase 1(a) ($yz \in W$)**.** First, assume that f lies on some $z-x^*$ path of T for some terminal vertex x^* of z. If $x = x^*$, then either $d(e, yz) < d(f, yz)$ or $d(f, yz) < d(e, yz)$, implying that $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Thus we may assume that $x \neq x^*$. If $d(e, yz) \neq d(f, yz)$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. If $d(e, yz) = d(f, yz)$, then $c(e) \neq c(f)$ by the definition of c and so $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$.

Next, assume that f does not lie on any $z-x^*$ path of T for all terminal vertices x^* of z. If there is an edge $w \in W^*$ such that either f lies on the $e-w$ path or e lies on the $f-w$ path, then $d(f, w) < d(e, w)$ or $d(e, w) < d(f, w)$, respectively. In either case, $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Thus, we may assume that every path between e and an edge of W^* does not contain f and every path between f and an edge of W^* does not contain e. Necessarily, then, there exist an exterior major vertex z' and a terminal vertex x' of z' such that f lies on the $z'-x'$ path of T. Since f does not lie on any $z-x^*$ path of T for all terminal vertices x^* of z, it follows that $z \neq z'$. Since z' is an exterior major vertex of T, it follows that the degree of z' is at least 3 and so there exists a branch B at z' that does not contain f. Necessarily, B must contain an edge of W^*. Let w^* be an edge of W^* that belongs to B. If $d(e, yz) \neq d(f, yz)$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Thus we may assume that $d(e, yz) = d(f, yz)$. This implies that $d(f, w^*) < d(e, w^*)$ and so $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$.

Subcase 1(b) ($yz \notin W$)**.** By the argument used in Subcase 1.1, we may assume that every path between e and an edge of W^* does not contain f and every path between f and an edge of W^* does not contain e. Thus there exist an exterior major vertex z' and a terminal vertex x' of z' such that f lies on the $z'-x'$ path of T. If $z = z'$, then there exists $w \in W^*$ such that w is incident with z. If $d(e, w) \neq d(f, w)$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$, while if $d(e, w) = d(f, w)$, then $c(e) \neq c(f)$ by the definition of c and so $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Thus we may assume that $z \neq z'$. Since the degrees of z and z' are at least 3, there exists a branch B_1 at z that does not contain e and a branch B_2 at z' that does not contain f. Necessarily, B_1 must contain an edge w_1 of W^* and B_2 must contain an edge w_2 of W^*. If $d(e,w_1) \neq d(f,w_1)$, then $c_{\mathfrak{D}}(e) \neq c_{\mathfrak{D}}(f)$, while if $d(e,w_1) = d(f,w_1)$, then $d(f,w_2) < d(e,w_2)$ and so $c_{\mathfrak{D}}(e) \neq c_{\mathfrak{D}}(f)$.

Case 2 (for every exterior major vertex z of T and every terminal vertex x of z, e does not lie on the $z-x$ path of T)**.** Then there are at least two branches at e, say B_1' and B_2', each of which contains some superior major vertex. Therefore, each of B_1' and B_2' contains an edge of W^*. Let w_1' and w_2' be the edges of W^* in B_1' and B_2', respectively. First assume that $f \in E(B_1')$. Then the $f-w_2'$ path of T contains e, so $d(e,w_2') < d(f,w_2')$ and $c_{\mathfrak{D}}(e) \neq c_{\mathfrak{D}}(f)$. We now assume that $f \notin E(B_1')$. Then the $f-w_1'$ path of T contains e. Hence $d(e,w_1') < d(f,w_1')$, so $c_{\mathfrak{D}}(e) \neq c_{\mathfrak{D}}(f)$.

Therefore, $\mathfrak{D}$ is a resolving edge coloring of T and so $\chi_{re}(T) \le |\mathfrak{D}| = \Delta(T) + \sup(T)$, as desired. □

In the proof of Theorem 3.3, if T is a tree with $\sup(T) \ge 2$ such that $\deg v \le \Delta(T) - 1$ for every major vertex v of T that is not a superior major vertex, then $\Delta(T_0) \le \Delta(T) - 1$. Hence $\chi_e(T_0) \le \Delta(T) - 1$. Thus, T_0 has an edge coloring c^* using $\Delta(T) - 1$ colors. Define an edge coloring c such that $c(e) = c^*(e)$ for all $e \in E(T_0)$ and define $c(e)$ for each $e \in V(T) - E(T_0)$ as described in the proof of Theorem 3.3. Then an argument similar to the one used in the proof of Theorem 3.3 shows that c is a resolving edge coloring of T. Thus, we have the following corollary.

Corollary 3.4. *Let T be a tree with $\sup(T) \ge 2$. If every major vertex v of T that is not a superior major vertex has $\deg v < \Delta(T)$, then*

$$\chi_{re}(T) \le \Delta(T) + \sup(T) - 1. \tag{3.27}$$

The upper bound in Corollary 3.4 is sharp. To see this, let T be a tree having two superior major vertices v_1 and v_2 with $\deg v_1 = \deg v_2 = \Delta(T)$ and $\deg v < \Delta(T)$ for every major vertex v of T that is not a superior major vertex. By Corollary 3.4, $\chi_{re}(T) \le \Delta(T) + \sup(T) - 1 = \Delta(T) + 1$. Assume, to the contrary, that $\chi_{re}(T) = \Delta(T)$. Let c be a resolving edge coloring of T with $\Delta(T)$ colors and let $\mathfrak{D} = \{C_1, C_2, \dots, C_{\Delta(T)}\}$ be the decomposition of T into the color classes of c. Let $N(v_i) = \{x_{i1}, x_{i2}, \dots, x_{i\Delta(T)}\}$ for $i = 1,2$. Without loss of generality, assume that $x_{ij} \in C_j$ for $i = 1,2$ and $1 \le j \le \Delta(T)$. However, then, $c_{\mathfrak{D}}(v_1x_{11}) = (0,1,1,\dots) = c_{\mathfrak{D}}(v_2x_{21})$, which is a contradiction. Therefore, $\chi_{re}(T) = \Delta(T) + 1 = \Delta(T) + \sup(T) - 1$.

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