ON A CERTAIN CLASS OF NONSTATIONARY SEQUENCES IN HILBERT SPACE

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To my Professor A. A. Yansevitch

We study the functions of correlation $K(n, m) = \langle X(n), X(m) \rangle$ of certain sequences: $X(n) = T^n x_0$, $x_0 \in H$ where T is a contraction in Hilbert space H . By using the spectral methods of the nonunitary operators, we give the general form of $K(n, m)$ and its asymptotic behaviour $\lim_{p \rightarrow +\infty} K(n+p, m+p)$.

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1. Introduction. Let $X(n)$ ($n \in A = \mathbb{N}$ or $\mathbb{Z}$) be a sequence of elements of a separable Hilbert space H . The function of correlation of $X(n)$ is given by formula

$$K(n, m) = \langle X(n), X(m) \rangle. \quad (1.1)$$

If the function of correlation depends only on the difference of arguments, that is, $K(n, m) = K(n - m)$, one calls that $X(n)$ is stationary. Kolmogorov (see [4]) showed that if $X(n)$ is stationary and $A = \mathbb{Z}$, then

$$X(n) = U^n x_0, \quad x_0 = X(0), \quad (1.2)$$

where U is a unitary operator acting in the subspace H_X which is defined as the closed linear envelope of $X = \{X(n); n \in \mathbb{Z}\}$. This representation as well as the spectral theory of the monparametric groups of unitary operators allowed to find the general form of the function $K(n, m)$ in the stationary case. More exactly, one has (see [4])

$$K(n, m) = \int_{-\pi}^{+\pi} e^{i(n-m)\lambda} dF_X(\lambda), \quad (1.3)$$

where F_X is real function, continuous on the left and nondecreasing on $[-\pi; +\pi]$ such that $F_X(-\pi) = 0$. This function is called spectral function of $X(n)$.

In this paper, we are interested in some sequences of the form

$$X(n) = T^n x_0, \quad x_0 \in H, \quad (1.4)$$

where T is a linear contraction ($\|T\| \leq 1$) in H . Such sequences are called linearly representable and were introduced by Yansevitch [8, 9]. They represent a natural generalization of the sequences of the form (1.2). But they were especially introduced like the analogue of the processes of the form

$$Y(t) = e^{itA}y_0, \quad (1.5)$$

where A is a dissipative ($(A - A^*)/i \geq 0$) operator in H . The correlation theory of these processes constituted a remarkable field of application for the spectral theory of nonselfadjoint operators [2, 3, 5, 10].

Necessary and sufficient criteria in terms of function of correlation for linear representability (1.4) are established by the following theorem [8].

THEOREM 1.1. *A given function $K(n, m)$ is the function of correlation of a certain sequence $X(n) = T^n x_0$ if and only if there exists a constant C ($0 < C < +\infty$) such that*

$$\begin{aligned} & \sum_{n,m=0}^N K(n, m) \lambda_n \bar{\lambda}_m \geq 0, \\ & \left| \sum_{n=0}^N \sum_{m=0}^M (K(n+1, m) - K(n, m)) \lambda_n \bar{\mu}_m \right|^2 \\ & \leq C \cdot \sum_{n,p=0}^N K(n, p) \lambda_n \bar{\lambda}_p \cdot \sum_{m,q=0}^M K(m, q) \mu_m \bar{\mu}_q \end{aligned} \quad (1.6)$$

for every $(\lambda_n)_{n=0}^N$ and $(\mu_m)_{m=0}^M$ in the field of complex numbers.

DEFINITION 1.2. Let $X(n) = T^n x_0$ be a linearly representable sequence. The difference of correlation of $X(n)$ is the function

$$W(n, m) = K(n, m) - K(n+1, m+1). \quad (1.7)$$

It is clear that in the stationary case, $W(n, m) = 0$.

Formula (1.7) implies that, for every natural $p \geq 1$,

$$K(n, m) = K(n+p, m+p) + \sum_{j=0}^{p-1} W(n+j, m+j), \quad (1.8)$$

what gives, for $p \rightarrow +\infty$,

$$K(n, m) = \lim_{p \rightarrow +\infty} K(n+p, m+p) + \sum_{j=0}^{+\infty} W(n+j, m+j). \quad (1.9)$$

Furthermore, since $(I - T^*T)$ is selfadjoint, then

$$(I - T^*T) = \sum_{k=1}^r \langle \cdot; g_k \rangle \cdot g_k, \quad g_k \in (I - T^*T)H, \quad r = \dim(I - T^*T)H. \quad (1.10)$$

Hence,

$$\begin{aligned} K(n, m) &= \lim_{p \rightarrow +\infty} K(n+p, m+p) + \sum_{j=0}^{+\infty} W(n+j, m+j) \\ &= \lim_{p \rightarrow +\infty} K(n+p, m+p) + \sum_{j=0}^{+\infty} \langle (I - T^*T)X(n+j); X(m+j) \rangle \\ &= \lim_{p \rightarrow +\infty} K(n+p, m+p) + \sum_{j=0}^{+\infty} \sum_{k=1}^r \langle X(n+j); g_k \rangle \cdot \langle g_k; X(m+j) \rangle \quad (1.11) \\ &= \lim_{p \rightarrow +\infty} K(n+p, m+p) + \sum_{j=0}^{+\infty} \sum_{k=1}^r \Phi_k(n+j) \cdot \overline{\Phi_k(m+j)}, \\ \Phi_k(n) &= \langle X(n); g_k \rangle = \langle T^n x_0; g_k \rangle. \end{aligned}$$

Consequently, the study of linearly representable sequences can be carried out in two stages.

(a) To find the limit $\lim_{p \rightarrow +\infty} K(n+p, m+p)$.

(b) To give the explicit expression of the quantity $\Phi_k(n)$.

In [8], the case when $\dim(I - T^*T)H = 1$ was considered and the spectrum of T is made up only of eigenvalues $\{\lambda_k\}_{k=1}^{+\infty}$ such that $|\lambda_k| < 1$, $k \geq 1$. In this case, one has [8]

$$\begin{aligned} \lim_{p \rightarrow +\infty} K(n+p, m+p) &= 0, \\ K(n, m) &= \sum_{j=0}^{+\infty} \Phi(n+j) \cdot \overline{\Phi(m+j)}, \\ \Phi(n) &= \sum_{k=1}^{+\infty} f_{0k} \cdot \frac{-1}{2\pi i} \sqrt{1 - |\lambda_k|^2} \oint_{\Gamma} \frac{\lambda^n}{\lambda - \lambda_k} \prod_{j=1}^{k-1} \frac{1 - \lambda \cdot \lambda_j}{\lambda - \lambda_j} \cdot \frac{|\lambda_j|}{\lambda_j} \cdot d\lambda, \quad (1.12) \\ &\quad \sum_{k=1}^{+\infty} |f_{0k}|^2 < +\infty, \end{aligned}$$

where Γ is a closed contour containing all the spectrum of T .

Let T be a simple contraction (i.e., there is no invariant for T and T^* subspace in which, T induces a unitary operator) with spectrum $\sigma(T)$ on the circle unit. It is known [1] that there exists an increasing function α on the interval $[0, l]$ ($l > 0$) such that

$$\sigma(T) = \{e^{-i\alpha(x)} : x \in [0, l]\}. \quad (1.13)$$

DEFINITION 1.3. Say that $X(n) = T^n x_0$ belongs to the class $D^{(r)}[\alpha]$ if T is a contraction such that (1.13) holds and $\dim(I - T^*T)H \leq r$.

Throughout this paper, we will suppose that α is a continuous function. In this case, we will prove the following results.

THEOREM 1.4. Let $T^n x_0$ be an element of class $D^{(r)}[\alpha]$. Assume that T is simple. Then,

$$K(n, m) = \tilde{\mathbf{K}}_\infty(n - m) + F(n - m) + \sum_{j=0}^{+\infty} \sum_{k=1}^r \Phi_k(n + j) \cdot \overline{\Phi_k(m + j)},$$

$$\Phi_k(n) = -\frac{1}{2\pi i} \oint_{\Gamma} \lambda^n \cdot \left\{ \frac{\sqrt{2}e^{-x}}{e^{-i\alpha(x)} - \lambda} \int_0^l \Psi_{0k}(x) \cdot e^{2 \int_0^x (e^{-i\alpha(t)}/e^{-i\alpha(t)} - \lambda) dt} dx \right\} d\lambda, \quad (1.14)$$

where $\Psi_{0k} \in L^2_{[0;l]}$, Γ is any closed contour containing all the spectrum of T , $F(n - m)$ is a Hermitian nonnegative function which equals zero in the case when $\dim(I - T^*T)H = 1$, and $\tilde{\mathbf{K}}_\infty(n - m)$ is defined by the spectrum of T . Moreover, if T has a singular spectrum or the measurement of the intersection of its spectrum with the unit circle is null, then $\tilde{\mathbf{K}}_\infty(n - m) = F(n - m) = 0$.

THEOREM 1.5. If a function $K(n, m)$ admits the representation (1.14), then there exists a linearly representable sequence $X(n) = T^n x_0$ such that $X(n) \in D^{(r)}[\alpha]$ and the function of correlation of $X(n)$ equals $K(n, m)$.

Throughout this paper, H is a separable Hilbert space and $\oplus$ denotes orthogonal sum.

2. On the structure of $\lim_{p \rightarrow +\infty} \mathbf{K}(n + p, m + p)$

PROPOSITION 2.1. If T is a contraction in H , then the sequence $A_n = T^{*n} T^n$ admits a positive strong limit $R = s \cdot \lim_{n \rightarrow +\infty} T^{*n} T^n$ which verifies the relation

$$T^{*n} R T^m = R T^{m-n} \quad (n \geq m). \quad (2.1)$$

Moreover, if T is invertible, then

$$T^{*n} R = R T^{-n}. \quad (2.2)$$

PROOF. The existence and positivity of R are a consequence of the fact that the sequence A_n is a decreasing and bounded sequence of positive operators. Formulas (2.1) and (2.2) are verified easily. $\square$

COROLLARY 2.2. If $X(n) = T^n x_0 \in D^{(r)}[\alpha]$, then

$$\lim_{p \rightarrow +\infty} \mathbf{K}(n + p, m + p) = \mathbf{K}_\infty(n - m) = \langle R T^{n-m} x_0, x_0 \rangle,$$

$$\lim_{p \rightarrow +\infty} \mathbf{W}(n + p, m + p) = \mathbf{0}. \quad (2.3)$$

Consider now the sequence

$$\hat{\Psi}(x, n) = \hat{T}^n \Psi_0(x), \quad \Psi_0 \in L^2_{[0; l]} \quad (l < \infty), \quad (2.4)$$

$$(\hat{T}f)(x) = e^{-i\alpha(x)}f(x) - 2e^{-i\alpha(x)+x} \int_x^l e^{-t}f(t)dt. \quad (2.5)$$

A direct calculation shows that $\sigma(T) = \{e^{-i\alpha(x)} : x \in [0, l]\}$ and

$$(I - \hat{T}^* \hat{T}) = \langle \cdot; g \rangle \cdot g \quad (g(x) = \sqrt{2}e^{-x}). \quad (2.6)$$

Hence, the sequence $\hat{\Psi}(x, n)$ is an element of the class $D^{(1)}[\alpha]$.

For every $u \in [0, l]$, let

$$L^2_{[u; l]} = \{f \in L^2_{[0; l]} : f(x) = 0 \text{ for } x \in [0, u]\}. \quad (2.7)$$

Let also P_u be the orthoprojector of $L^2_{[0; l]}$ on $L^2_{[u; l]}$.

PROPOSITION 2.3. *The sequence $A_n(u) = T^{*n}P_uT^n$ admits a positive strong limit R_u which verifies the relation $T^{*n}R_uT^m = R_uT^{m-n}$ ($n \geq m$). Moreover, if T is invertible, then $T^{*n}R_u = R_uT^{-n}$.*

Pose that

$$\begin{aligned} L_0(n, u) &= \langle P_u(\hat{\Psi}(x, n)), \hat{\Psi}(x, n) \rangle = \int_u^l |\hat{\Psi}(t, n)|^2 dt, \\ \hat{K}(n, m, u) &= \langle P_u(\hat{\Psi}(x, n)), \hat{\Psi}(x, m) \rangle, \\ \widehat{W}(n, m, u) &= \hat{K}(n, m, u) - \hat{K}(n+1, m+1, u) = \gamma(u, n) \cdot \overline{\gamma(u, m)}, \\ \gamma(u, n) &= \sqrt{2}e^u \int_u^l e^{-t} \cdot \hat{\Psi}(t, n) dt. \end{aligned} \quad (2.8)$$

For $n \geq m$,

$$\begin{aligned} \hat{K}(n, m, u) &= \hat{K}(n-m, 0, u) - \sum_{j=1}^m \widehat{W}(n-j, m-j, u) \\ &= \hat{K}(n-m, 0, u) - \sum_{j=0}^{m-1} \widehat{W}(n-m+j, j, u). \end{aligned} \quad (2.9)$$

Thus, for $p \geq 1$,

$$\hat{K}(n+p, m+p, u) = \hat{K}(n-m, 0, u) - \sum_{j=0}^{m+p-1} \widehat{W}(n-m+j, j, u). \quad (2.10)$$

Let $\tau = n - m$ and $\hat{\mathbf{K}}_\infty(n - m, u) = \lim_{p \rightarrow +\infty} \hat{\mathbf{K}}(n + p, m + p, u)$. Then,

$$\hat{\mathbf{K}}_\infty(n - m) = \hat{K}(\tau, 0, u) - \sum_{j=0}^{+\infty} \gamma(u, \tau + j) \overline{\gamma(u, j)}. \quad (2.11)$$

Let

$$\mathbf{L}_p(\tau, u) = \hat{K}(\tau, 0, u) - \sum_{j=0}^{p-1} \gamma(u, \tau + j) \overline{\gamma(u, j)} \quad (p \geq 1). \quad (2.12)$$

Then,

$$\begin{aligned} \hat{\mathbf{K}}_\infty(\tau, u) &= \lim_{p \rightarrow +\infty} \mathbf{L}_p(\tau, u), \\ \mathbf{L}_p(0, u) &= \hat{K}(0, 0, u) - \sum_{j=0}^{p-1} \gamma(u, j) \overline{\gamma(u, j)} = \hat{K}(p, p, u) = L_0(p, u), \\ K_0(u) &= \lim_{p \rightarrow +\infty} \mathbf{L}_p(0, u) = \lim_{p \rightarrow +\infty} L_0(p, u) = \langle R_u \Psi_0, \Psi_0 \rangle. \end{aligned} \quad (2.13)$$

THEOREM 2.4. *The function $\hat{\mathbf{K}}_\infty(\tau, u)$ admits the following representation:*

$$\hat{\mathbf{K}}_\infty(\tau, u) = - \int_u^l e^{i\tau\alpha(x)} dK_0(x). \quad (2.14)$$

In particular,

$$\hat{\mathbf{K}}_\infty(n - m) = - \int_0^l e^{i(n-m)\alpha(x)} dK_0(x). \quad (2.15)$$

PROOF. Remark that K_0 is a decreasing function. Thus integrals in (2.14) and (2.15) exist. A direct but long calculation makes it possible to affirm that

$$\begin{aligned} \frac{d}{du}(\mathbf{L}_p(\tau + 1, u)) &= e^{i\alpha(u)} \left(\frac{d}{du} \mathbf{L}_p(\tau, u) \right) + \sqrt{2} \gamma(u, \tau + p) \overline{\gamma(u, p - 1)} \\ &\quad + 2\gamma(u, \tau + p) \overline{\Psi(u, p - 1)}. \end{aligned} \quad (2.16)$$

Hence,

$$\begin{aligned} \frac{d}{du}(\mathbf{L}_p(\tau, u)) &= e^{i\tau\alpha(u)} \left(\frac{d}{du} L_0(p, u) \right) \\ &\quad + \sqrt{2} \overline{\gamma(u, p - 1)} \sum_{j=1}^{\tau} e^{i(1-j)\alpha(u)} \cdot \gamma(u, \tau + p - j) \\ &\quad + 2\overline{\Psi(u, p - 1)} \sum_{j=1}^{\tau} e^{i(1-j)\alpha(u)} \cdot \gamma(u, \tau + p - j). \end{aligned} \quad (2.17)$$

Let

$$\begin{aligned} I_1 &= - \int_u^l e^{i\tau\alpha(x)} \left(\frac{d}{dx} \mathbf{L}_0(p, x) \right) dx, \\ I_2 &= -\sqrt{2} \int_u^l \overline{\gamma(x, p-1)} \sum_{j=1}^{\tau} e^{i(1-j)\alpha(x)} \cdot \gamma(x, \tau+p-j) dx, \\ I_3 &= -2 \int_u^l \overline{\Psi(x, p-1)} \sum_{j=1}^{\tau} e^{i(1-j)\alpha(x)} \cdot \gamma(x, \tau+p-j) dx, \end{aligned} \quad (2.18)$$

then

$$\mathbf{L}_p(\tau, u) = I_1 + I_2 + I_3. \quad (2.19)$$

By using the theorem of Lebesgue about dominated convergence, one can show that $I_2 = I_3 = 0$. Thus,

$$\mathbf{L}_p(\tau, u) = - \int_u^l e^{i\tau\alpha(x)} d(\mathbf{L}_0(p, x)). \quad (2.20)$$

Furthermore,

$$\mathbf{L}_0(p, x) = \int_x^l |\hat{\Psi}(t, p)|^2 dt \quad (2.21)$$

is an absolutely continuous function in x . Moreover, since operator $\hat{T}$ is a contraction, then

$$\mathbf{L}_0(p, x) = \int_x^l |\hat{\Psi}(t, p)|^2 dt \leq \int_0^l |\hat{\Psi}(t, p)|^2 dt \leq \|\hat{\Psi}_0\|^2. \quad (2.22)$$

That means that the sequence V_p ($p \geq 1$) of total variation of $\mathbf{L}_0(p, x)$ on $[0, l]$ is bounded. Moreover, function $e^{i\tau\alpha(x)}$ is continuous. Thus,

$$\begin{aligned} \lim_{p \rightarrow +\infty} \mathbf{L}_p(\tau, u) &= - \int_u^l e^{i\tau\alpha(x)} d\left(\lim_{p \rightarrow +\infty} \mathbf{L}_0(p, x)\right) \\ &= - \int_u^l e^{i\tau\alpha(x)} d\mathbf{K}_0(x). \end{aligned} \quad (2.23) \quad \square$$

It is known [6] that if the $X(n) = T^n x_0 \in D^{(1)}[\alpha]$ and T is simple, then $T = U^{-1} \hat{T} U$ where U is a unitary operator from H into $L^2_{[0, l]}$. Hence, from [Theorem 2.4](#), the following theorem follows.

THEOREM 2.5. *Let $X(n) = T^n x_0 \in D^{(1)}[\alpha]$. Suppose that T is simple. Then, there exists an increasing function β on $[0, l]$ such that*

$$\mathbf{K}_{\infty}(n-m) = \lim_{p \rightarrow +\infty} \mathbf{K}(n+p, m+p) = \int_0^l e^{i(n-m)\alpha(x)} d\beta(x). \quad (2.24)$$

Consider now the space $L_2^r = L_{[0;l]}^2 \oplus \cdots \oplus L_{[0;l]}^2$ (r times), with scalar product:

$$\langle f; g \rangle_r = \sum_{j=1}^r \int_0^l f_j(x) \cdot \overline{g_j(x)} dx, \quad f = (f_1, \dots, f_r), \quad g = (g_1, \dots, g_r). \quad (2.25)$$

In this space, define the operator $\overline{T}(r) = \hat{T} \oplus \cdots \oplus \hat{T}$ as follows:

$$(\overline{T}(r))(f_1, \dots, f_r) = (\hat{T}f_1, \dots, \hat{T}f_r). \quad (2.26)$$

Every sequence of the form

$$\begin{aligned} \tilde{\Psi}(x, n) &= (\overline{T}(r))^n(\tilde{\Psi}_0(x)) = (\hat{T}^n(\tilde{\Psi}_{01}), \dots, \hat{T}^n(\tilde{\Psi}_{0r})), \\ \tilde{\Psi}_0 &= (\tilde{\Psi}_{01}, \dots, \tilde{\Psi}_{0r}) \in L_2^r, \end{aligned} \quad (2.27)$$

is an element of class $D^{(r)}[\alpha]$ (see [1]). The following relations hold immediately:

$$\begin{aligned} \tilde{\mathbf{K}}_\infty(n-m) &= \lim_{p \rightarrow +\infty} \tilde{\mathbf{K}}(n+p, m+p) = \sum_{j=1}^r \hat{\mathbf{K}}_\infty^{(j)}(n-m), \\ \hat{\mathbf{K}}_\infty^{(j)}(n-m) &= - \int_0^l e^{i(n-m)\alpha(x)} dK_0^{(j)}(x), \\ K_0^{(j)}(x) &= \langle R_x \Psi_{0j}, \Psi_{0j} \rangle, \quad (j = 1, \dots, r). \end{aligned} \quad (2.28)$$

THEOREM 2.6. *Let $X(n) = T^n x_0 \in D^{(r)}[\alpha]$. Suppose that T is simple. Then, there exists r increasing functions $\{\beta_j\}_{j=1}^r$ on $[0, l]$ and there exists a Hermitian nonnegative function $F(n-m)$ such that*

$$\mathbf{K}_\infty(n-m) = \sum_{j=1}^r \int_0^l e^{i(n-m)\alpha(x)} d\beta_j(x) + F(n-m). \quad (2.29)$$

PROOF. According to [1], there exists a unitary operator B defined in a Hilbert space M such that operator T is unitarily equivalent to the restriction of operator $\overline{B}(r) = \overline{T}(r) \oplus B$ on a certain invariant subspace $\Theta \subset L_2^r \oplus M$, that is, $T = U^{-1} \overline{B}(r) U$ where U is a unitary operator from H into Θ . Thus,

$$\begin{aligned} K(n+p, m+p) &= \langle T^{n+p} x_0; T^{m+p} x_0 \rangle \\ &= \langle U^{-1} \overline{B}(r)^{n+p} U(x_0); U^{-1} \overline{B}(r)^{m+p} U(x_0) \rangle \\ &= \langle \overline{B}(r)^{n+p}(f_0); \overline{B}(r)^{m+p}(f_0) \rangle, \end{aligned} \quad (2.30)$$

where $f_0 = U(x_0) = \tilde{\Psi}_0 + x_M \in \Theta$ ($\tilde{\Psi}_0 \in L_2^r$, $x_M \in M$).

Since $\overline{B}(r)^n = \overline{T}(r)^n \oplus B^n$, then

$$\begin{aligned} K(n+p, m+p) &= \langle \overline{T}(r)^{n+p}(\tilde{\Psi}_0); \overline{T}(r)^{m+p}(\tilde{\Psi}_0) \rangle \\ &\quad + \langle B^{n+p}(x_M); B^{m+p}(x_M) \rangle \\ &= \langle \overline{T}(r)^{n+p}(\tilde{\Psi}_0); \overline{T}(r)^{m+p}(\tilde{\Psi}_0) \rangle \\ &\quad + \langle B^{n-m}(x_M); x_M \rangle. \end{aligned} \quad (2.31)$$

Let $F(n-m) = \langle B^{n-m}(x_M); x_M \rangle$. It is clear that function $F(n-m)$ satisfies all conditions of [Theorem 2.6](#). Finally, one has

$$\begin{aligned} \mathbf{K}_\infty(n-m) &= \lim_{p \rightarrow +\infty} K(n+p, m+p) \\ &= \lim_{p \rightarrow +\infty} \langle (\overline{T}(r))^{n+p}(\tilde{\Psi}_0); (\overline{T}(r))^{m+p}(\tilde{\Psi}_0) \rangle + F(n-m) \\ &= \tilde{\mathbf{K}}_\infty(n-m) + F(n-m). \end{aligned} \quad (2.32)$$

To complete the demonstration, it is enough to notice that

$$\tilde{\mathbf{K}}_\infty(n-m) = \sum_{j=1}^r \hat{\mathbf{K}}_\infty^{(j)}(n-m) = \sum_{j=1}^r \int_0^l e^{i(n-m)\alpha(x)} d\beta_j(x). \quad (2.33)$$

□

We now will see two situations where $\mathbf{K}_\infty(n-m) = 0$.

PROPOSITION 2.7. *Let $X(n) = T^n x_0 \in D^{(r)}[\alpha]$. If T is simple and the measurement of the intersection of its spectrum with the circle unit is null, then $\mathbf{K}_\infty(n-m) = 0$.*

PROOF. One has

$$\begin{aligned} |K(n+p, m+p)|^2 &= |\langle T^{n+p}x_0; T^{m+p}x_0 \rangle|^2 \\ &\leq \|T^{n+p}x_0\|^2 \cdot \|T^{m+p}x_0\|^2. \end{aligned} \quad (2.34)$$

Under these assumptions, one has according to [\[7\]](#)

$$\lim_{p \rightarrow +\infty} \|T^{n+p}x_0\|^2 = \lim_{p \rightarrow +\infty} \|T^{m+p}x_0\|^2 = 0. \quad (2.35)$$

□

THEOREM 2.8. *Let $T^n x_0$ be an element of class $D^{(r)}[\alpha]$ and let $\sigma(t) = \text{mes} \cdot \{x \in [0, l] : \alpha(x) < t\}$, $t \in [\alpha(0); \alpha(l)]$, be the repartition function of α . If T is simple and σ singular, then $\mathbf{K}_\infty(n-m) = 0$.*

PROOF. Under these assumptions, operator T is unitarily equivalent to operator $\overline{T}(r)$ (see [\[1\]](#)). Thus, $\mathbf{K}_\infty(n-m) = \tilde{\mathbf{K}}_\infty(n-m)$. But in this case, the

characteristical function $\tilde{S}(\lambda)$ of operator $\overline{T}(r)$ satisfies the following relations (see [1, 7]):

$$\begin{aligned}\det \tilde{S}(\lambda) &= \exp \left\{ \int_0^l (e^{i\alpha(t)} + \lambda)(e^{i\alpha(t)} - \lambda)^{-1} dt \right\} \\ &= \exp \left\{ - \int_{\alpha(0)}^{\alpha(l)} (e^{it} + \lambda)(e^{it} - \lambda)^{-1} d\sigma(t) \right\} \\ &= \exp \left\{ - \int_0^{2\pi} (e^{it} + \lambda)(e^{it} - \lambda)^{-1} d\nu(t) \right\},\end{aligned}\quad (2.36)$$

where

$$\nu(t) = \begin{cases} \sigma(t), & t \in [0, l], \\ 0, & t \notin [0, l] \end{cases} \quad (2.37)$$

is a singular function. Thus $\det \tilde{S}(\lambda)$ is an interior function and according to [1], for every $\tilde{\Psi} = (\tilde{\Psi}_1, \dots, \tilde{\Psi}_r) \in L_2^r$, $\lim_{p \rightarrow +\infty} \|T^{n+p}x_0\|^2 = 0$. By using the same reasoning that in Proposition 2.7, one shows that $\mathbf{K}_\infty(n-m) = \tilde{\mathbf{K}}_\infty(n-m) = 0$. $\square$

3. General form of $\mathbf{K}(n, m)$

THEOREM 3.1. *Let $T^n x_0$ be an element of class $D^{(r)}[\alpha]$. Assume that T is simple. Then,*

$$\begin{aligned}K(n, m) &= \tilde{\mathbf{K}}_\infty(n-m) + F(n-m) + \sum_{j=0}^{+\infty} \sum_{k=1}^r \Phi_k(n+j) \cdot \overline{\Phi_k(m+j)}, \\ \Phi_k(n) &= \frac{-1}{2\pi i} \oint_{\Gamma} \lambda^n \cdot \left\{ \frac{\sqrt{2}e^{-x}}{e^{-i\alpha(x)} - \lambda} \int_0^l \Psi_{0k}(x) \cdot e^{2 \int_0^x (e^{-i\alpha(t)}/e^{-i\alpha(t)} - \lambda) dt} dx \right\} d\lambda,\end{aligned}\quad (3.1)$$

where $\Psi_{0k} \in L_{[0;l]}^2$ and $F(n-m)$ is a Hermitian nonnegative function.

PROOF. Using the same reasoning that in Theorem 2.6, one can affirm that

$$K(n, m) = \tilde{K}(n, m) + F(n-m), \quad (3.2)$$

where $F(n-m)$ satisfies the conditions of Theorem 3.1. According to (1.11),

$$\begin{aligned}\tilde{K}(n, m) &= \lim_{p \rightarrow +\infty} K(n+p, m+p) + \sum_{j=0}^{+\infty} \sum_{k=1}^r \Phi_k(n+j) \cdot \overline{\Phi_k(m+j)}, \\ \Phi_k(n) &= \langle (\overline{T}(r))^n (\tilde{\Psi}_0); g_k \rangle, \quad (I - (\overline{T}(r))^* (\overline{T}(r))) = \sum_{k=1}^r \langle \cdot; g_k \rangle \cdot g_k.\end{aligned}\quad (3.3)$$

One has (see [1])

$$g_k = h_k \cdot e_k, \quad h_k(x) = \sqrt{2}e^{-x}, \quad (3.4)$$

where e_k ($k = 1, \dots, r$) is the canonical basic in C^r . Thus,

$$\Phi_k(n) = \langle (\overline{T}(r))^n(\tilde{\Psi}_0); g_k \rangle = \Phi_k(n) = \langle \hat{T}^n(\tilde{\Psi}_{0k}); h_k \rangle. \quad (3.5)$$

Since $\hat{T}$ is bounded, then

$$\hat{T}^n = \frac{-1}{2\pi i} \oint_{\Gamma} \lambda^n (\hat{T} - \lambda \cdot I)^{-1} d\lambda, \quad (3.6)$$

where Γ is a closed contour containing all the spectrum of $\hat{T}$.

Consequently,

$$\begin{aligned} \Phi_k(n) &= \frac{-1}{2\pi i} \oint_{\Gamma} \langle \lambda^n (\hat{T} - \lambda \cdot I)^{-1}(\tilde{\Psi}_{0k}); h_k \rangle d\lambda \\ &= \frac{-1}{2\pi i} \oint_{\Gamma} \lambda^n \langle \tilde{\Psi}_{0k}; (\hat{T}^* - \bar{\lambda} \cdot I)^{-1}(h_k) \rangle. \end{aligned} \quad (3.7)$$

A direct calculation shows that the form $\Phi_k(n)$ is as in (3.1). □

Theorem 3.1 admits the following reciprocal.

THEOREM 3.2. *If a function $K(n, m)$ admits the representation (3.1), then there exists a linearly representable sequence $X(n) = T^n x_0$ such that $X(n) \in D^{(r)}[\alpha]$ and the function of correlation of $X(n)$ equals $K(n, m)$.*

PROOF. Since $F(n - m)$ is a Hermitian nonnegative function, there exists (see [4]) a unitary operator S defined in a Hilbert space M such that

$$F(n - m) = \langle S^n x_M, S^m x_M \rangle, \quad (x_M \in M). \quad (3.8)$$

By the functions α and $\{\tilde{\Psi}_{0k}\}_{k=1}^r$ appearing in representation (3.1), construct, in the space L_2^r , the sequence

$$\tilde{\Psi}(x, n) = (\overline{T}(r))^n(\tilde{\Psi}_0(x)) = (\hat{T}^n(\tilde{\Psi}_{01}), \dots, \hat{T}^n(\tilde{\Psi}_{0r})), \quad (3.9)$$

where operator T is defined in $L_{[0;1]}^2$ by formula (2.5). Let H denotes the Hilbert space $L_2^r \oplus M$ with scalar product:

$$\langle g + Y_M, g' + Y'_M \rangle = \langle g, g' \rangle_{L_2^r} + \langle Y_M, Y'_M \rangle_M. \quad (3.10)$$

In this space, define the operator $T = \bar{T}(r) \oplus S$ by $T(g + y_M) = \bar{T}(r)(g) + S(y_M)$. Operator T is a contraction and $\dim(I - T^*T)H = r$. Thus, the sequence $X(n) = T^n(f + x_M) = \bar{X}(n) + S^n x_M$ is an element of class $D^{(r)}[\alpha]$ whose function of correlation equals the given function $K(n, m)$. $\square$

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