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REAL GEL'FAND-MAZUR DIVISION ALGEBRAS

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We show that the complexification $(\tilde{A},\tilde{\tau})$ of a real locally pseudoconvex (locally absorbingly pseudoconvex, locally multiplicatively pseudoconvex, and exponentially galbed) algebra (A,τ) is a complex locally pseudoconvex (resp., locally absorbingly pseudoconvex, locally multiplicatively pseudoconvex, and exponentially galbed) algebra and all elements in the complexification $(\tilde{A},\tilde{\tau})$ of a commutative real exponentially galbed algebra (A,τ) with bounded elements are bounded if the multiplication in (A,τ) is jointly continuous. We give conditions for a commutative strictly real topological division algebra to be a commutative real Gel'fand-Mazur division algebra.

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1. Introduction. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers. A *topological algebra* A is a topological vector space over $\mathbb{K}$ in which the multiplication is separately continuous. Herewith, A is called a *real topological algebra* if $\mathbb{K}=\mathbb{R}$ and a *complex topological algebra* if $\mathbb{K}=\mathbb{C}$. We classify topological algebras in a similar way as topological vector spaces. For example, a topological algebra A is

(a) a *Fréchet algebra* if it is complete and metrizable;

(b) an *exponentially galbed algebra* (see [3, 13]) if its underlying topological vector space is *exponentially galbed*, that is, for each neighborhood O of zero in A, there exists another neighborhood U of zero such that

$$\left\{\sum_{k=0}^{n}\frac{a_k}{2^k} : a_0,\dots,a_n\in U\right\}\subset O \tag{1.1}$$

for each $n\in\mathbb{N}$;

(c) a *locally pseudoconvex algebra* (see [5, 7]) if its underlying topological vector space is *locally pseudoconvex*, that is, A has a base $\{U_\alpha,\ \alpha\in\mathcal{A}\}$ of neighborhoods of zero in which every set U_α is *balanced* (i.e., $\lambda U_\alpha\in U_\alpha$ whenever $|\lambda|\leqslant 1$) and *pseudoconvex* (i.e., $U_\alpha+U_\alpha\subset 2^{1/k_\alpha}U_\alpha$ for some $k_\alpha\in(0,1]$). Herewith, every locally pseudoconvex algebra is an exponentially galbed algebra.

In particular, when $k_\alpha=k$ ($k_\alpha=1$) for each $\alpha\in\mathcal{A}$, then a locally pseudoconvex algebra A is called a *locally k-convex algebra* (resp., *locally convex* *algebra*). It is well known (see [14, page 4]) that the topology of a locally pseudoconvex algebra A can be given by means of a family $\mathcal{P} = \{p_\alpha : \alpha \in A\}$ of k_α-homogeneous seminorms, where $k_\alpha \in (0,1]$ for each $\alpha \in A$. A locally pseudoconvex algebra is called a *locally absorbingly pseudoconvex* (shortly, *locally A-pseudoconvex*) algebra (see [5]) if every seminorm $p \in \mathcal{P}$ is *A-multiplicative*, that is, for each $a \in A$ there are positive numbers $M_p(a)$ and $N_p(a)$ such that

$$p(ab) \leqslant M_p(a)p(b), \qquad p(ba) \leqslant N_p(a)p(b), \tag{1.2}$$

for each $b \in A$. In particular, when $M_p(a) = N_p(a) = p(a)$ for each $a \in A$ and $p \in \mathcal{P}$, then A is called a *locally multiplicatively pseudoconvex* (shortly, *locally m-pseudoconvex*) algebra.

Moreover, a topological algebra A over $\mathbb{K}$ with a unit element is a *Q-algebra* (see [10, 15, 16]) if the set of all invertible elements of A is open in A and a Q-algebra A is a *Waelbroeck algebra* (see [4, 10]) or a *topological algebra with continuous inverse* (see [9, 11]) if the inversion $a \to a^{-1}$ in A is continuous.

An element a of a topological algebra A is said to be *bounded* (see [6]) if for some nonzero complex number λ_a, the set

$$\left\{\left(\frac{a}{\lambda_a}\right)^n : n \in \mathbb{N}\right\} \tag{1.3}$$

is bounded in A. A topological algebra, in which all elements are bounded, will be called a *topological algebra with bounded elements*.

Let now A be a topological algebra over $\mathbb{K}$ and $m(A)$ the set of all closed regular two-sided ideals of A, which are maximal as left or right ideals. In case when the quotient algebra A/M (in the quotient topology) is topologically isomorphic to $\mathbb{K}$ for each $M \in m(A)$, then A is called a *Gel'fand-Mazur algebra* (see [1, 4, 2]). Herewith, A is a *real Gel'fand-Mazur algebra* if $\mathbb{K} = \mathbb{R}$ and a *complex Gel'fand-Mazur algebra* if $\mathbb{K} = \mathbb{C}$. Main classes of complex Gel'fand-Mazur algebras have been given in [4, 2, 5]. Several classes of real Gel'fand-Mazur division algebras are described in the present paper.

2. Complexification of real algebras. Let A be a (not necessarily topological) real algebra and let $\tilde{A} = A + iA$ be the complexification of A. Then every element $\tilde{a}$ of $\tilde{A}$ is representable in the form $\tilde{a} = a + ib$, where $a, b \in A$ and $i^2 = -1$. If the addition, scalar multiplication, and multiplication in $\tilde{A}$ are to be defined by

$$\begin{aligned}(a+ib)+(c+id) &= (a+c)+i(b+d),\\ (\alpha+i\beta)(a+ib) &= (\alpha a-\beta b)+i(\alpha b+\beta a),\\ (a+ib)(c+id) &= (ac-bd)+i(ad+bc),\end{aligned} \tag{2.1}$$

for all $a, b, c, d \in A$ and $\alpha, \beta \in \mathbb{R}$, then $\tilde{A}$ is a complex algebra with zero element $\theta_{\tilde{A}} = \theta_A + i\theta_A$ (here and later on θ_A denotes the zero element of A). In case when A has the unit element e_A, then $e_{\tilde{A}} = e_A + i\theta_A$ is the unit element of $\tilde{A}$. Herewith, $\tilde{A}$ is an associative (commutative) algebra if A is an associative (resp., commutative) algebra. Therefore, we can consider A as a real subalgebra of $\tilde{A}$ under the imbedding ν from A into $\tilde{A}$ defined by $\nu(a) = a + i\theta_A$ for each $a \in A$.

A real (not necessarily topological) algebra A is called a *formally real algebra* if from $a, b \in A$ and $a^2 + b^2 = \theta_A$ that follows that $a = b = \theta_A$ and is called a *strictly real algebra* if $\mathrm{sp}_{\tilde{A}}(a + i\theta_A) \subset \mathbb{R}$ (here $\mathrm{sp}_A(a)$ denotes the spectrum of $a \in A$ in A). It is known (see, e.g., [7, Proposition 1.9.14]) that every formally real division algebra is strictly real and every commutative strictly real division algebra is formally real.

Let now (A, τ) be a real topological algebra and $\{U_\alpha : \alpha \in \mathcal{A}\}$ a base of neighborhoods of zero of (A, τ). As usual (see [7, 17]), we endow $\tilde{A}$ with the topology $\tilde{\tau}$ in which $\{U_\alpha + iU_\alpha : \alpha \in \mathcal{A}\}$ is a base of neighborhoods of zero. It is easy to see that $(\tilde{A}, \tilde{\tau})$ is a topological algebra and the multiplication in $(\tilde{A}, \tilde{\tau})$ is jointly continuous if the multiplication in (A, τ) is jointly continuous (see [7, Proposition 2.2.10]). Moreover, the underlying topological space of $(\tilde{A}, \tilde{\tau})$ is a Hausdorff space if the underlying topological space of (A, τ) is a Hausdorff space.

3. Complexification of real locally pseudoconvex algebras.

Let (A, τ) be a real locally pseudoconvex algebra and $\{p_\alpha : \alpha \in \mathcal{A}\}$ a family of k_α-homogeneous seminorms on A (where $k_\alpha \in (0,1]$ for each $\alpha \in \mathcal{A}$), which defines the topology τ on A and $\tilde{A}$, the complexification of A,

$$\begin{gathered}\Gamma_{k_\alpha}(U_\alpha + i\theta_A) \\ = \left\{ \sum_{k=1}^{n} \lambda_k (u_k + i\theta_A) : n \in \mathbb{N}, u_1, \ldots, u_n \in U_\alpha, \lambda_1, \ldots, \lambda_n \in \mathbb{C} \text{ and } \sum_{k=1}^{n} |\lambda_k|^{k_\alpha} \leqslant 1 \right\}, \\ q_\alpha(a + ib) = \inf\{|\lambda|^{k_\alpha} : (a + ib) \in \lambda \Gamma_{k_\alpha}(U_\alpha + i\theta_A)\}\end{gathered} \tag{3.1}$$

for each $a + ib \in \tilde{A}$. Then $\Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ is the absolutely k_α-convex hull of $U_\alpha + i\theta_A$ for each $\alpha \in \mathcal{A}$ and q_α is a k_α-homogeneous Minkowski functional of $\Gamma_{k_\alpha}(U_\alpha + i\theta_A)$. (For real normed algebras the following result has been proved in [8, pages 68–69] (see also [12, page 8]) and for k-seminormed algebras with $k \in (0,1]$ in [7, pages 183–184]).

Theorem 3.1. *Let (A, τ) be a real locally pseudoconvex algebra, let $\{p_\alpha, \alpha \in \mathcal{A}\}$ be a family of k_α-homogeneous seminorms on A (with $k_\alpha \in (0,1]$ for each $\alpha \in \mathcal{A}$), which defines the topology τ on A, and let $U_\alpha = \{a \in A : p_\alpha(a) < 1\}$.*

Then the following statements are true for each $\alpha \in \mathcal{A}$:

(a) *q_α is a k_α-homogeneous seminorm on $\tilde{A}$;*

(b) $\max\{p_\alpha(a), p_\alpha(b)\} \leqslant q_\alpha(a + ib) \leqslant 2\max\{p_\alpha(a), p_\alpha(b)\}$ *for each $a, b \in A$;*

(c) $q_\alpha(a+i\theta_A) = p_\alpha(a)$ *for each* $a \in A$;
(d) $\Gamma_{k_\alpha}(U_\alpha + i\theta_A) = \{a+ib \in \tilde{A} : q_\alpha(a+ib) < 1\}$.

Proof. (a) Let $\alpha \in \mathcal{A}$, $(a+ib) \in \tilde{A} \setminus \{\theta_{\tilde{A}}\}$, and $\mu_\alpha^{k_\alpha} > \max\{p_\alpha(a), p_\alpha(b)\}$. Then $a/\mu_\alpha, b/\mu_\alpha \in U_\alpha$. Since

$$\begin{gathered} 2^{-1/k_\alpha}\left(\frac{a}{\mu_\alpha} + i\frac{b}{\mu_\alpha}\right) = 2^{-1/k_\alpha}\left(\frac{a}{\mu_\alpha} + i\theta_A\right) + i2^{-1/k_\alpha}\left(\frac{b}{\mu_\alpha} + i\theta_A\right), \\ \left|2^{-1/k_\alpha}\right|^{k_\alpha} + \left|i2^{-1/k_\alpha}\right|^{k_\alpha} = 1, \end{gathered} \tag{3.2}$$

then

$$(a+ib) \in 2^{1/k_\alpha}\mu_\alpha \Gamma_{k_\alpha}(U_\alpha + i\theta_A). \tag{3.3}$$

Hence $(a+ib) \in \lambda_\alpha \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ for each $\alpha \in \mathcal{A}$ if $|\lambda_\alpha| \geqslant 2^{1/k_\alpha}\mu_\alpha$. It means that the set $\Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ is absorbing. Consequently (see [7, Proposition 4.1.10]), q_α is a k_α-homogeneous seminorm on $\tilde{A}$.

(b) Let again $(a+ib) \in \tilde{A} \setminus \{\theta_{\tilde{A}}\}$. Then from (3.3), it follows that $q_\alpha(a+ib) \leqslant 2\mu_\alpha^{k_\alpha}$. Since this inequality is valid for each $\mu_\alpha^{k_\alpha} > \max\{p_\alpha(a), p_\alpha(b)\}$, then

$$q_\alpha(a+ib) \leqslant 2\max\{p_\alpha(a), p_\alpha(b)\}. \tag{3.4}$$

Let now $a+ib \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$. Then

$$a+ib = \sum_{k=1}^{n} (\lambda_k + i\mu_k)(a_k + i\theta_A) = \sum_{k=1}^{n} \lambda_k a_k + i\sum_{k=1}^{n} \mu_k a_k \tag{3.5}$$

for some $a_1, \dots, a_n \in U_\alpha$ and real numbers $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$ such that

$$\sum_{k=1}^{n} \left|\lambda_k + i\mu_k\right|^{k_\alpha} \leqslant 1. \tag{3.6}$$

Since $|\lambda_k| \leqslant |\lambda_k + i\mu_k|$ and $|\mu_k| \leqslant |\lambda_k + i\mu_k|$ for each $k \in \{1, \dots, n\}$, then

$$a = \sum_{k=1}^{n} \lambda_k a_k, \qquad b = \sum_{k=1}^{n} \mu_k a_k \tag{3.7}$$

belong to $\Gamma_{k_\alpha}(U_\alpha) = U_\alpha$.

Let now $\varepsilon > 0$ and

$$\mu_\alpha > \left(\frac{1}{q_\alpha(a+ib) + \varepsilon}\right)^{1/k_\alpha}. \tag{3.8}$$

Then from $\mu_\alpha(a+ib) \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$ follows that $\mu_\alpha a, \mu_\alpha b \in U_\alpha$ or $p_\alpha(\mu_\alpha a) < 1$ and $p_\alpha(\mu_\alpha b) < 1$. Therefore

$$\max\{p_\alpha(a), p_\alpha(b)\} < \mu_\alpha^{-k_\alpha} < q_\alpha(+ib) + \varepsilon. \tag{3.9}$$

Since ε is arbitrary, then from (3.9) follows that $\max\{p_\alpha(a), p_\alpha(b)\} \leqslant q_\alpha(a+ib)$ for each $a, b \in A$. Taking this and inequality (3.4) into account, it is clear that statement (b) holds.

(c) Let $a \in A$, $\alpha \in \mathcal{A}$, and $\rho^{k_\alpha} > q_\alpha(a+i\theta_A)$. Then from

$$\left(\frac{a}{\rho} + i\theta_A\right) \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A), \tag{3.10}$$

it follows that $a \in \rho U_\alpha$ or $p_\alpha(a) < \rho^{k_\alpha}$. It means that the set of numbers ρ^{k_α} for which $\rho^{k_\alpha} > q_\alpha(a+i\theta_A)$ is bounded below by $p_\alpha(a)$. Therefore $p_\alpha(a) \leqslant q_\alpha(a+i\theta_A)$.

Let now $\rho^{k_\alpha} > p_\alpha(a)$. Then $a \in \rho U_\alpha$ and from

$$\left(\frac{a}{\rho} + i\theta_A\right) \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A), \tag{3.11}$$

it follows that $q_\alpha(a+i\theta_A) < \rho^{k_\alpha}$. Hence $q_\alpha(a+i\theta_A) \leqslant p_\alpha(a)$. Thus $q_\alpha(a+i\theta_A) = p_\alpha(a)$ for each $a \in A$ and $\alpha \in \mathcal{A}$.

(d) It is clear that the set $\{a+ib \in \tilde{A} : q_\alpha(a+ib) < 1\} \subset \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$. Let now $a+ib \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A)$. Then

$$a+ib = \sum_{k=1}^{n} (\lambda_k + i\mu_k)(a_k + i\theta_A) \tag{3.12}$$

for some $a_1, \dots, a_n \in U_\alpha$ and real numbers $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$ such that

$$\sum_{k=1}^{n} |\lambda_k + i\mu_k|^{k_\alpha} \leqslant 1. \tag{3.13}$$

Since $p_\alpha(a_k) < 1$ for each $k \in \{1, \dots, n\}$, we can choose $\varepsilon_\alpha > 0$ so that

$$\max\{p_\alpha(a_1), \dots, p_\alpha(a_n)\} < \varepsilon_\alpha^{k_\alpha} < 1. \tag{3.14}$$

Then $a_k \in \varepsilon_\alpha U_\alpha$ for each $\alpha \in \mathcal{A}$ and each $k \in \{1, \dots, n\}$. Therefore

$$\frac{a+ib}{\varepsilon_\alpha} \in \sum_{k=1}^{n} (\lambda_k + i\mu_k)\left(\frac{a_k}{\varepsilon_\alpha} + i\theta_A\right) \in \Gamma_{k_\alpha}(U_\alpha + i\theta_A). \tag{3.15}$$

Hence

$$(a+ib) \in \varepsilon_\alpha \Gamma_{k_\alpha}(U_\alpha + i\theta_A) \tag{3.16}$$

or $q_\alpha(a+ib) \leqslant \varepsilon_\alpha^{k_\alpha} < 1$. It means that statement (d) holds. □

Corollary 3.2. *If (A, τ) is a real locally pseudoconvex Fréchet algebra, then $(\tilde{A}, \tilde{\tau})$ is a complex locally pseudoconvex Fréchet algebra.*

Proof. Let (A,τ) be a real locally pseudoconvex Fréchet algebra and let $\{p_n, n \in \mathbb{N}\}$ be a countable family of k_n-homogeneous seminorms (with $k_n \in (0,1]$ for each $n \in \mathbb{N}$), which defines the topology τ on A. Then $\{q_n : n \in \mathbb{N}\}$ defines on $\tilde{A}$ a metrizable locally pseudoconvex topology $\tilde{\tau}$ (see Theorem 3.1). If $(a_n + ib_n)$ is a Cauchy sequence in $(\tilde{A},\tilde{\tau})$, then (a_n) and (b_n) are Cauchy sequences in (A,τ) by Theorem 3.1(b). Because (A,τ) is complete, then (a_n) converges to $a_0 \in A$ and (b_n) converges to $b_0 \in A$. Hence $(a_n + ib_n)$ converges in $(\tilde{A},\tilde{\tau})$ to $a_0 + ib_0 \in \tilde{A}$ by the same inequality (b). Thus $(\tilde{A},\tilde{\tau})$ is a complex locally pseudoconvex Fréchet algebra. □

Theorem 3.3. *Let (A,τ) be a real locally A-pseudoconvex (locally m-pseudoconvex) algebra and $\{p_\alpha,\ \alpha \in \mathcal{A}\}$ a family of k_α-homogeneous A-multiplicative (resp., submultiplicative) seminorms on A (with $k_\alpha \in (0,1]$ for each $\alpha \in \mathcal{A}$), which defines the topology τ on A. Then $(\tilde{A},\tilde{\tau})$ is a complex locally A-pseudoconvex (resp., locally m-pseudoconvex) algebra. (Here $\tilde{\tau}$ denotes the topology on $\tilde{A}$ defined by the system $\{q_\alpha : \alpha \in \mathcal{A}\}$.)*

Proof. Let p_α be an A-multiplicative seminorm on A. Then for each fixed element $a_0 \in A$, there are numbers $M_\alpha(a_0) > 0$ and $N_\alpha(a_0) > 0$ such that

$$p_\alpha(a_0 a) \leqslant M_\alpha(a_0)p_\alpha(a), \qquad p_\alpha(a a_0) \leqslant N_\alpha(a_0)p_\alpha(a), \tag{3.17}$$

for each $a \in A$. If $a_0 + ib_0$ is a fixed element and $a + ib$ an arbitrary element of $\tilde{A}$, then

$$\begin{aligned} q_\alpha\big((a_0+ib_0)(a+ib)\big) &= q_\alpha\big((a_0a - b_0b) + i(a_0b + b_0a)\big) \\ &\leqslant 2\max\{p_\alpha(a_0a-b_0b), p_\alpha(a_0b+b_0a)\} \end{aligned} \tag{3.18}$$

by Theorem 3.1(b). If now $p_\alpha(a_0a - b_0b) \geqslant p_\alpha(a_0b + b_0a)$, then

$$\begin{aligned} &\max\{p_\alpha(a_0a-b_0b), p_\alpha(a_0b+b_0a)\} \\ &\quad = p_\alpha(a_0a-b_0b) \\ &\quad \leqslant M_\alpha(a_0)p_\alpha(a) + M_\alpha(b_0)p_\alpha(b) \\ &\quad \leqslant \max\{p_\alpha(a), p_\alpha(b)\}\big(M_\alpha(a_0) + M_\alpha(b_0)\big) \\ &\quad \leqslant \frac{1}{2}M_\alpha(a_0,b_0)q_\alpha(a+ib) \end{aligned} \tag{3.19}$$

by Theorem 3.1(b) (here $M_\alpha(a_0,b_0) = 2(M_\alpha(a_0) + M_\alpha(b_0))$). Hence

$$q_\alpha\big((a_0+ib_0)(a+ib)\big) \leqslant M_\alpha(a_0,b_0)q_\alpha(a+ib) \tag{3.20}$$

for each $a + ib \in \tilde{A}$.

The proof for the case when $p_\alpha(a_0a - b_0b) < p_\alpha(a_0b + b_0a)$ is similar. Thus inequality (3.20) holds for both cases. In the same way, it is easy to show that the inequality

$$q_\alpha\big((a+ib)(a_0+ib_0)\big) \leqslant N_\alpha(a_0,b_0)q_\alpha(a+ib) \tag{3.21}$$

holds for each $a+ib\in\tilde{A}$. Consequently, $(\tilde{A},\tilde{\tau})$ is a complex locally A-pseudoconvex algebra.

Let now p_α be a submultiplicative seminorm on A. Then $p_\alpha(ab)\leqslant p_\alpha(a)p_\alpha(b)$ for each $a,b\in A$. If $a+ib,a'+ib'\in\tilde{A}$, then

$$q_\alpha\big((a+ib)(a'+ib')\big)\leqslant 2\max\{p_\alpha(aa'-bb'),p_\alpha(ab'+ba')\} \tag{3.22}$$

by Theorem 3.1(b). If now $p_\alpha(aa'-bb')\geqslant p_\alpha(ab'+ba')$, then

$$\begin{aligned}\max&\{p_\alpha(aa'-bb'),p_\alpha(ab'+ba')\}\\ &=p_\alpha(aa'-bb')\leqslant p_\alpha(a)p_\alpha(a')+p_\alpha(b)p_\alpha(b')\\ &\leqslant 2\max\{p_\alpha(a),p_\alpha(b)\}\max\{p_\alpha(a'),p_\alpha(b')\}\\ &\leqslant 2q_\alpha(a+ib)q_\alpha(a'+ib')\end{aligned} \tag{3.23}$$

by Theorem 3.1(b). Hence

$$q_\alpha\big((a+ib)(a'+ib')\big)\leqslant 4q_\alpha(a+ib)q_\alpha(a'+ib'). \tag{3.24}$$

Putting $r_\alpha=4q_\alpha$ for each $\alpha\in\mathcal{A}$, we see that

$$r_\alpha\big((a+ib)(a'+ib')\big)\leqslant r_\alpha(a+ib)r_\alpha(a'+ib') \tag{3.25}$$

for each $a+ib,a'+ib'\in\tilde{A}$.

The proof for the case when $p_\alpha(aa'-bb')<p_\alpha(ab'+ba')$ is similar. Hence inequality (3.25) holds for both cases. Since the families $\{q_\alpha:\alpha\in\mathcal{A}\}$ and $\{r_\alpha:\alpha\in\mathcal{A}\}$ define on $\tilde{A}$ the same topology, then $(\tilde{A},\tilde{\tau})$ is a complex locally m-pseudoconvex algebra. □

4. Complexification of real exponentially galbed algebras. Next, we will show that the complexification $(\tilde{A},\tilde{\tau})$ of (A,τ) is a complex exponentially galbed algebra if (A,τ) is a real exponentially galbed algebra, and all elements of $(\tilde{A},\tilde{\tau})$ are bounded in $(\tilde{A},\tilde{\tau})$ if (A,τ) is a commutative exponentially galbed algebra in which all elements are bounded and the multiplication in (A,τ) is jointly continuous.

Theorem 4.1. *Let (A,τ) be a real exponentially galbed algebra (commutative real exponentially galbed algebra with jointly continuous multiplication and bounded elements). Then $(\tilde{A},\tilde{\tau})$ is a complex exponentially galbed algebra (resp., commutative complex exponentially galbed algebra with bounded elements).*

Proof. Let (A,τ) be a real exponentially galbed algebra and $\tilde{O}$ a neighborhood of zero in $(\tilde{A},\tilde{\tau})$. Then there are a neighborhood O of zero of (A,τ) such that $O+iO\subset\tilde{O}$ and another neighborhood U of zero of (A,τ) such that

$$\left\{\sum_{k=0}^{n}\frac{a_k}{2^k}:a_0,\dots,a_n\in U\right\}\subset O \tag{4.1}$$

for each $n \in \mathbb{N}$. Since $U + iU$ is a neighborhood of zero in $(\tilde{A}, \tilde{\tau})$ and

$$\left\{ \sum_{k=0}^{n} \frac{a_k + ib_k}{2^k} : a_0 + ib_0, \ldots, a_n + ib_n \in U + iU \right\} \subset O + iO \subset \tilde{O} \tag{4.2}$$

for each $n \in \mathbb{N}$, then $(\tilde{A}, \tilde{\tau})$ is a complex exponentially galbed algebra.

Let now (A, τ) be a commutative real exponentially galbed algebra with jointly continuous multiplication and bounded elements, $\tilde{O}$ an arbitrary neighborhood of zero of $(\tilde{A}, \tilde{\tau})$, and $a + ib \in \tilde{A}$ an arbitrary element. Then there are a neighborhood O of zero of (A, τ) such that $O + iO \subset \tilde{O}$ and $\lambda_a, \lambda_b \in \mathbb{C} \setminus \{0\}$ and the sets

$$\left\{ \left(\frac{a}{\lambda_a} \right)^n : n \in \mathbb{N} \right\}, \qquad \left\{ \left(\frac{b}{\lambda_b} \right)^n : n \in \mathbb{N} \right\} \tag{4.3}$$

are bounded in (A, τ). The neighborhood O defines now a balanced neighborhood U of zero of (A, τ) such that (4.2) holds and U defines a balanced neighborhood V of zero of (A, τ) such that $VV \subset U$ (because the multiplication in (A, τ) is jointly continuous). Now there are numbers $\mu_a, \mu_b > 0$ such that

$$\left(\frac{a}{|\lambda_a|} \right)^n \in \mu_a V, \qquad \left(\frac{b}{|\lambda_b|} \right)^n \in \mu_b V, \tag{4.4}$$

for each $n \in \mathbb{N}$. Let $\kappa = 4(|\lambda_a| + |\lambda_b|)$. Since $a + ib = (a + i\theta_A) + i(b + i\theta_A)$, then

$$\begin{aligned} \left(\frac{a + ib}{\kappa} \right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\left(\frac{a}{\kappa} \right)^k + i\theta_A \right) i^{n-k} \left(\left(\frac{b}{\kappa} \right)^{n-k} + i\theta_A \right) \\ &= \mu_a \mu_b \sum_{k=0}^{n} \frac{\tilde{x}_k}{2^k} \end{aligned} \tag{4.5}$$

for each $n \in \mathbb{N}$, where

$$\begin{aligned} \tilde{x}_k &= \varrho_{nk} \frac{1}{\mu_a \mu_b} \left(\left(\frac{a}{|\lambda_a|} \right)^k \left(\frac{b}{|\lambda_b|} \right)^{n-k} + i\theta_A \right), \\ \varrho_{nk} &= 2^k i^{n-k} \binom{n}{k} \left(\frac{|\lambda_a|}{\kappa} \right)^k \left(\frac{|\lambda_b|}{\kappa} \right)^{n-k}, \end{aligned} \tag{4.6}$$

for each $k \leqslant n$. Herewith

$$\begin{aligned} |\varrho_{nk}| &= \frac{2^k}{\kappa^n} \binom{n}{k} |\lambda_a|^k |\lambda_b|^{n-k} \leqslant \frac{2^n}{\kappa^n} (|\lambda_a| + |\lambda_b|)^n \leqslant \left(\frac{1}{2} \right)^n < 1, \\ \left(\frac{a}{|\lambda_a|} \right)^k &\left(\frac{b}{|\lambda_b|} \right)^{n-k} + i\theta_A \in \mu_a \mu_b VV + i\theta_A \subset \mu_a \mu_b (U + iU). \end{aligned} \tag{4.7}$$

Since U is a balanced set, then $\tilde{x}_k \in U + iU$ for each $k \in \{0,\dots,n\}$. Hence

$$\left(\frac{a+ib}{\kappa}\right)^n \in \mu_a\mu_b(O+iO) \subset \mu_a\mu_b\tilde{O} \tag{4.8}$$

by (4.2) for each $n \in \mathbb{N}$. It means that $a+ib$ is bounded in $(\tilde{A},\tilde{\tau})$. Consequently, $(\tilde{A},\tilde{\tau})$ is a commutative complex exponentially galbed algebra with bounded elements. □

5. Real Gel'fand-Mazur division algebras. To describe main classes of real Gel'fand-Mazur division algebras, we first describe these real topological division algebras (A,τ) for which the complexification $(\tilde{A},\tilde{\tau})$ of (A,τ) is a complex Gel'fand-Mazur division algebra.

Proposition 5.1. *If (A,τ) is a commutative strictly real topological Hausdorff division algebra with continuous inversion, then the complexification $(\tilde{A},\tilde{\tau})$ of (A,τ) is a commutative complex topological Hausdorff division algebra with continuous inversion.*

Proof. Let A be a commutative strictly real division algebra. Then $\tilde{A}$ is a complex division algebra (see [7, Proposition 1.6.20]). Since the underlying topological space of (A,τ) is a Hausdorff space, then (A,τ) is a Q-algebra. Hence (A,τ) is a commutative real Waelbroeck algebra with a unit element. Therefore $(\tilde{A},\tilde{\tau})$ is a commutative Waelbroeck algebra (see [7, Proposition 3.6.31] or [17, proposition on page 237]). Thus, $(\tilde{A},\tilde{\tau})$ is a commutative complex Hausdorff division algebra with continuous inversion. □

Proposition 5.2. *Let (A,τ) be a real topological algebra and $\tilde{A}$ the complexification of A. If the topological dual $(A,\tau)^*$ of (A,τ) is nonempty, then the topological dual $(\tilde{A},\tilde{\tau})^*$ of $(\tilde{A},\tilde{\tau})$ is also nonempty.*

Proof. If $\psi \in (A,\tau)^*$, then $\tilde{\psi}$, defined by $\tilde{\psi}(a+ib) = \psi(a) + i\psi(b)$ for each $a+ib \in \tilde{A}$, is an element of $(\tilde{A},\tilde{\tau})^*$. □

Proposition 5.3. *Let A be a commutative strictly real (not necessarily topological) division algebra and $\tilde{A}$ the complexification of A. Then*

$$\operatorname{sp}_{\tilde{A}}(a+ib) = \{\alpha + i\beta \in \mathbb{C} : \alpha \in \operatorname{sp}_A(a) \text{ and } \beta \in \operatorname{sp}_A(b)\}. \tag{5.1}$$

Proof. Let $\alpha + i\beta \in \operatorname{sp}_{\tilde{A}}(a+ib)$. Since A is a commutative strictly real division algebra, then $\tilde{A}$ is a commutative complex division algebra (see [7, Proposition 1.6.20]). Therefore

$$a+ib-(\alpha+i\beta)(e_A+i\theta) = (a-\alpha e_A) + i(b-\beta e_A) = \theta_A + i\theta_A \tag{5.2}$$

if and only if $\alpha \in \operatorname{sp}_A(a)$ and $\beta \in \operatorname{sp}_A(b)$. □

The main result of the present paper is the following theorem.

Theorem 5.4. *Let (A,τ) be a commutative strictly real topological division algebra and $\tilde{A}$ the complexification of A. If there is a topology τ' on A such that (A,τ') is*

(a) *a locally pseudoconvex Hausdorff algebra with continuous inversion;*

(b) *a Hausdorff algebra with continuous inversion for which $(A,\tau)^*$ is nonempty;*

(c) *an exponentially galbed Hausdorff algebra with jointly continuous multiplication and bounded elements;*

(d) *a topological Hausdorff algebra for which the spectrum $\mathrm{sp}_A(a)$ is nonempty for each $a \in A$,*

then (A,τ) and $\mathbb{R}$ are topologically isomorphic.

Proof. If A is a commutative strictly real division algebra, then $\tilde{A}$ is a commutative complex division algebra (by [7, Proposition 1.6.20]). In case (a) the complexification $(\tilde{A},\tilde{\tau}')$ of (A,τ') is a commutative complex locally pseudoconvex Hausdorff division algebra with continuous inversion (by Theorem 3.1 and Proposition 5.1); in case (b) $(\tilde{A},\tilde{\tau}')$ of (A,τ') is a commutative complex topological Hausdorff algebra with continuous inversion for which the set $(\tilde{A},\tilde{\tau}')^*$ is nonempty (by Propositions 5.1 and 5.2); in case (c) $(\tilde{A},\tilde{\tau}')$ of (A,τ') is a commutative complex exponentially galbed Hausdorff division algebra with bounded elements (by Theorem 4.1); and in case (d) $(\tilde{A},\tilde{\tau}')$ of (A,τ') is such a commutative topological Hausdorff division algebra for which the spectrum $\mathrm{sp}_{\tilde{A}}(a+ib)$ is nonempty for each $a+ib \in \tilde{A}$ (by Proposition 5.3), therefore $(\tilde{A},\tilde{\tau})$ and $\mathbb{C}$ are topologically isomorphic (see [4, Theorem 1] and [2, Proposition 1]). Hence every element $a+ib \in \tilde{A}$ is representable in the form $a+ib = \lambda e_{\tilde{A}}$ for some $\lambda \in \mathbb{C}$. It means that for each $a \in A$ there is a real number μ such that $a = \mu e_A$. Consequently, A is an isomorphism to $\mathbb{R}$. In the same way as in complex case (see, e.g., [4, page 122]) it is easy to show that this isomorphism is a topological isomorphism because (A,τ) is a Hausdorff space. □

Corollary 5.5. *Let A be a commutative strictly real division algebra. If A has a topology τ such that (A,τ) is*

(a) *a locally pseudoconvex Hausdorff algebra with continuous inversion;*

(b) *a locally A-pseudoconvex (in particular, locally m-pseudoconvex) Hausdorff algebra;*

(c) *a locally pseudoconvex Fréchet algebra;*

(d) *an exponentially galbed Hausdorff algebra with jointly continuous multiplication and bounded elements;*

(e) *a topological Hausdorff algebra for which the spectrum $\mathrm{sp}_A(a)$ is nonempty for each $a \in A$,*

then (A,τ) is a commutative real Gel'fand-Mazur division algebra.

Proof. It is easy to see that (A,τ) is a commutative real Gel'fand-Mazur division algebra (by Theorem 5.4) in cases (a), (d), and (e). Since the inversion is continuous in every locally m-pseudoconvex algebra and every locally A-pseudoconvex Hausdorff algebra with a unit element having a topology τ' such that (A,τ') is a locally m-pseudoconvex Hausdorff algebra (see [5, Lemma 2.2]), then (A,τ) is a commutative real Gel'fand-Mazur division algebra in case (b) by (a) and Theorem 5.4.

Let now (A,τ) be a commutative strictly real locally pseudoconvex Fréchet division algebra. Then (A,τ) is a commutative strictly real locally pseudoconvex Fréchet Q-algebra by Corollary 3.2. Therefore the inversion in (A,τ) is continuous (see [15, Corollary 7.6]). Hence (A,τ) is also a commutative real Gel'fand-Mazur division algebra by Theorem 5.4. □

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