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SOME PROPERTIES OF LINEAR RIGHT IDEAL NEARRINGS

K. D. MAGILL JR.

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In a previous paper, we determined all those topological nearrings $\mathcal{N}_n$ whose additive groups are the n-dimensional Euclidean groups, $n > 1$, and which contain n one-dimensional linear subspaces $\{J_i\}_{i=1}^n$ which are also right ideals of the nearring with the property that for each $w \in \mathcal{N}_n$, there exist $w^i \in J_i$, $1 \le i \le n$, such that $w = w^1 + w^2 + \cdots + w^n$ and $vw = vw^n$ for each $v \in \mathcal{N}_n$. In this paper, we determine the properties of these nearrings, their ideals, and when two of these nearrings are isomorphic, and we investigate the multiplicative semigroups of these nearrings.

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1. Introduction. The nearrings considered here are right nearrings. For information about abstract nearrings, one may consult [1, 4, 5]. An n-dimensional Euclidean nearring is any topological nearring whose additive group is the n-dimensional Euclidean group. Linear right ideal nearrings were introduced in [3]. These are n-dimensional ($n \ge 2$) Euclidean nearrings which contain n distinct right ideals $\{J_i\}_{i=1}^n$, each of which is a one-dimensional linear subspace of $\mathbb{R}^n$ such that, for each $w \in \mathbb{R}^n$, there exist $w^i \in J_i$, $1 \le i \le n$, such that $w = w^1 + w^2 + \cdots + w^n$ and $vw = vw^n$ for all $v \in \mathbb{R}^n$. In the main theorem of [3], we determined, to within isomorphism, all linear right ideal nearrings. In Section 2 of this paper, we determine all the ideals of these nearrings and in Section 3 we determine when two of these nearrings are isomorphic. Finally, in Sections 4, 5, 6, and 7, we investigate the multiplicative semigroups of these nearrings.

2. The ideals of linear right ideal nearrings. We begin by recalling the main theorem from [3].

Theorem 2.1. *An n-dimensional ($n \ge 2$) Euclidean nearring $\mathcal{N}_n$ is a linear right ideal nearring if and only if $\mathcal{N}_n$ is isomorphic to one of the four types of nearrings whose multiplications follow:*

$$(vw)_i = 0 \quad \forall v, w \in \mathcal{N}_n \quad \text{or} \quad (vw)_i = v_i \quad \forall v, w \in \mathcal{N}_n \text{ for } 1 \le i \le n, \tag{2.1}$$

$$(vw)_i = v_i |w_n|^{r_i} \quad \text{for } i \ne n,\ (vw)_n = v_n w_n \text{ where } r_i > 0 \text{ for } 1 \le i < n, \tag{2.2}$$

$$(vw)_i = \begin{cases} v_i w_n^{r_i} & \text{for } w_n \geq 0, \\ -v_i |w_n|^{r_i} & \text{for } w_n < 0, \end{cases} \tag{2.3}$$

for $i \neq n$, $(vw)_n = v_n w_n$ *where* $r_i > 0$,

$$(vw)_i = \begin{cases} v_i (a w_n)^{r_i} & \text{for } w_n \leq 0, \\ v_i (b w_n)^{r_i} & \text{for } w_n > 0, \end{cases} \tag{2.4}$$

for $i \neq n$,

$$(vw)_n = \begin{cases} a v_n w_n & \text{for } w_n \leq 0, \\ b v_n w_n & \text{for } w_n > 0, \end{cases} \tag{2.5}$$

where $r_i > 0$, $a \leq 0$, $b \geq 0$, *and* $a^2 + b^2 \neq 0$.

An n-dimensional Euclidean nearring whose multiplication is given by (2.1), (2.2), (2.3), (2.4) will be referred to as a Type I, Type II, Type III, Type IV nearring, respectively.

Theorem 2.2. *Let $\mathcal{N}$ be a Type I nearring. Denote by N the collection (which may be empty) of all positive integers i such that $(vw)_i = 0$ for all $v, w \in \mathcal{N}$ and define an endomorphism of the group $(\mathbb{R}^n, +)$ by $(\varphi(v))_i = 0$ for $i \in N$ and $(\varphi(v))_i = v_i$ for $i \notin N$. The right ideals of $\mathcal{N}$ are precisely the additive subgroups J of $(\mathbb{R}^n, +)$ with the property that $\varphi[J] \subseteq J$ and the left ideals of $\mathcal{N}$ are precisely the additive subgroups J of $(\mathbb{R}^n, +)$. Consequently, the two-sided ideals of $\mathcal{N}$ are precisely the additive subgroups J of $(\mathbb{R}^n, +)$ with the property that $\varphi[J] \subseteq J$.*

Proof. Let J be any subgroup of $(\mathbb{R}^n, +)$. Let $u, v \in \mathcal{N}$ and $w \in J$.
Case 1 ($(uv)_i = 0$ for all $u, v \in \mathcal{N}$). Then

$$(u(v+w) - uv)_i = (u(v+w))_i - (uv)_i = 0. \tag{2.6}$$

Case 2 ($(uv)_i = u_i$ for all $u, v \in \mathcal{N}$). Then

$$(u(v+w) - uv)_i = (u(v+w))_i - (uv)_i = u_i - u_i = 0. \tag{2.7}$$

It follows from (2.6) and (2.7) that $(u(v+w) - uv)_i = 0$ for $1 \leq i \leq n$. Thus, $u(v+w) - uv = 0 \in J$ and we conclude that J is a left ideal of $\mathcal{N}$.

It follows immediately that $vw = \varphi(v)$ for all $v, w \in \mathcal{N}$ and this readily implies that an additive subgroup J of $\mathcal{N}$ is a right ideal if and only if $\varphi[J] \subseteq J$. This concludes the proof. □

Example 2.3. Let $\mathcal{N}$ be a Type I nearring. It follows easily that $J_i = \{v \in \mathcal{N} : v_j = 0 \text{ for } j \neq i\}$ is a right ideal (and hence, a two-sided ideal of $\mathcal{N}$) but there are many more right ideals in addition to these. Denote by $\mathcal{N}_4$ the four-dimensional Type I nearring whose multiplication is given by $vw = (v_1, 0, 0, v_4)$ for all

$v,w \in \mathcal{N}_4$. In this case, the additive endomorphism φ of Theorem 2.2 is given by $\varphi(v) = (v_1,0,0,v_4)$. Let $J = \{(x,x,x,x) : x \in \mathbb{R}\}$. In view of Theorem 2.2, J is a left ideal of $\mathcal{N}_4$ but it is not a right ideal since $\varphi[J] \nsubseteq J$. This time, let $J = \{v \in \mathcal{N}_4 : v_1 = v_4, v_2 = v_3\}$. For any $v \in J$, we have $v_1 = v_4$ which means that $\varphi(v) = (v_1,0,0,v_4) = (v_1,0,0,v_1) \in J$. Thus, J is a two-sided ideal of $\mathcal{N}_4$. Denote by $\mathcal{N}_2$ the two-dimensional Type I nearring whose multiplication is given by $vw = (v_1,0)$. It happens that $\mathcal{N}_4/J$ is isomorphic to $\mathcal{N}_2$. To see this, just note that $\psi(v) = (v_1 - v_4, v_2 - v_3)$ is a homomorphism from $\mathcal{N}_4$ onto $\mathcal{N}_2$ whose kernel is J.

Definition 2.4. For any Euclidean nearring $\mathcal{N}_n$, let $M = \{v \in \mathcal{N}_n : v_n = 0\}$.

Theorem 2.5. *Let $\mathcal{N}_n$ be either a Type II, a Type III, or a Type IV nearring. A subset J of $\mathcal{N}_n$ is a proper left ideal of $\mathcal{N}_n$ if and only if J is an additive subgroup of M.*

Proof. Suppose J is an additive subgroup of M. In order to show that J is a left ideal of $\mathcal{N}_n$, we must verify that $u(v+w) - uv \in J$ for all $u,v \in \mathcal{N}_n$ and all $w \in J$. Since $w \in J \subseteq M$, we have $w_n = 0$ so that $(v+w)_n = v_n$ and it follows that

$$\big(u(v+w) - uv\big)_i = \big(u(v+w)\big)_i - (uv)_i = 0 \quad \text{for } 1 \le i < n \tag{2.8}$$

and similarly

$$\big(u(v+w) - uv\big)_n = \big(u(v+w)\big)_n - (uv)_n = 0. \tag{2.9}$$

Consequently, $u(v+w) - uv = 0 \in J$ in view of (2.8) and (2.9).

Now suppose that J is a proper left ideal of $\mathcal{N}_n$. For this portion of the proof, we give the details only in the case of Type IV nearrings since the remaining cases are similar and even somewhat simpler in the case of Type II nearrings. Evidently, J is a proper subgroup of $\mathcal{N}_n$ but suppose that $J \nsubseteq M$. This means that $w_n \neq 0$ for some $w \in J$. Now, $u(v+w) - uv \in J$ for all $u,v \in \mathcal{N}_n$ and the vector w under consideration. Take $v = 0$ and conclude that

$$uw \in J \quad \forall u \in \mathcal{N}_n. \tag{2.10}$$

Let $x = (x_1, x_2, \dots, x_n)$ be an arbitrary element of $\mathcal{N}_n$. The multiplication here is given by (2.4) where not both a and b can be zero. There is no loss in generality in assuming that $a \neq 0$ and since $-w, w \in J$ we may assume that $w_n < 0$. For $1 \le i < n$, define $u^i \in \mathcal{N}_n$ by $u^i_j = 0$ for $j \neq i$ and $u^i_i = x_i/(aw_n)^{r_i}$. It follows from (2.10) that

$$u^i w \in J, \quad \text{where } \big(u^i w\big)_j = 0 \text{ for } j \neq i,\ \big(u^i w\big)_i = x_i \text{ for } 1 \le i < n. \tag{2.11}$$

This time, define $u^n_j = 0$ for $j \neq n$ and $u^n_n = x_n/aw_n$ and conclude that

$$u^n w \in J, \quad \text{where } (u^i w)_j = 0 \text{ for } j \neq n,\ (u^n w)_n = x_n. \tag{2.12}$$

It follows from (2.11) and (2.12) that $x = u^1 w + u^2 w + \cdots + u^n w \in J$. But this is a contradiction since J is a proper left ideal of $\mathcal{N}_n$. Consequently, we conclude that $J \subseteq M$ and the proof is complete. □

Corollary 2.6. *Let $\mathcal{N}_n$ be either a Type II, a Type III, or a Type IV nearring. Then M is the unique maximal ideal of $\mathcal{N}_n$. If $\mathcal{N}_n$ is either a Type II or a Type III nearring, then $\mathcal{N}_n/M$ is isomorphic to the field of real numbers and if $\mathcal{N}_n$ is a Type IV nearring, then $\mathcal{N}_n/M$ is isomorphic to the nearring $(\mathbb{R},+,*)$ where $(\mathbb{R},+)$ is the additive group of real numbers and the multiplication is given by*

$$x * y = \begin{cases} axy & \text{for } y \leq 0, \\ bxy & \text{for } y > 0. \end{cases} \tag{2.13}$$

Proof. The ideal M is a left ideal of $\mathcal{N}_n$ in view of Theorem 2.5 and it readily follows that $M\mathcal{N}_n \subseteq M$ so that M is also a right ideal of $\mathcal{N}_n$. It also follows from Theorem 2.5 that M is maximal. If $\mathcal{N}_n$ is either a Type II or a Type III nearring, then the mapping φ, defined by $\varphi(v) = v_n$, is easily shown to be a homomorphism from $\mathcal{N}_n$ onto the real field whose kernel is M. If $\mathcal{N}_n$ is a Type IV nearring, one verifies that the mapping φ, defined as before, is a homomorphism from $\mathcal{N}_n$ onto the nearring $(\mathbb{R},+,*)$ whose multiplication is given by (2.13). Since the kernel of φ is M, the proof is complete. □

Theorem 2.7. *Let $\mathcal{N}_n$ be either a Type II, a Type III, or a Type IV nearring and suppose that $r_i = r$ for $1 \leq i < n$. Let $v \in \mathcal{N}_n$ and let*

$$J_v = \{(cv_1, cv_2, \ldots, cv_{n-1}, dv_n)\}, \tag{2.14}$$

where $c = \sum_{k=1}^m e_k c_k$, $c_k \geq 0$, $e_k = \pm 1$ for $1 \leq k \leq m$, and $d = \sum_{k=1}^m e_k c_k^{(1\backslash r)}$. Then J_v is the smallest right ideal of $\mathcal{N}_n$ which contains the element v.

Proof. Take $c = c_1 = d = m = e_1 = 1$ and conclude that $v \in J_v$. Let $x, y \in J_v$. Then $x = (cv_1, cv_2, \ldots, cv_{n-1}, dv_n)$ and $y = (sv_1, sv_2, \ldots, sv_{n-1}, tv_n)$ where

$$c = \sum_{k=1}^m e_{c_k} c_k, \qquad d = \sum_{k=1}^m e_{c_k} c_k^{(1/r)}, \qquad s = \sum_{k=1}^j e_{s_k} s_k, \qquad t = \sum_{k=1}^j e_{s_k} s_k^{(1/r)}. \tag{2.15}$$

Define $e_k^* = e_{c_k}$ and $c_k^* = c_k$ for $1 \leq k \leq m$ and define $e_k^* = -e_{s_{k-m}}$ and $c_k^* = s_{k-m}$ for $m+1 \leq k \leq m+j$. Then

$$c - s = \sum_{k=1}^{m+j} e_k^* c_k^*, \qquad d - t = \sum_{k=1}^{m+j} e_k^* (c_k^*)^{(1/r)}, \tag{2.16}$$

and it follows from (2.16) that

$$x-y=((c-s)v_1,(c-s)v_2,\dots,(c-s)v_{n-1},(d-t)v_n)\in J_v. \tag{2.17}$$

Thus J_v is an additive subgroup of $(\mathbb{R}^n,+)$. For the remaining portion of the proof, we give the details only in the case where $\mathcal{N}_n$ is a Type IV nearring since the remaining cases are quite similar. Let $u=(cv_1,cv_2,\dots,cv_{n-1},dv_n)\in J_v$ where $c=\sum_{k=1}^m e_kc_k$ and $d=\sum_{k=1}^m e_kc_k^{(1/r)}$ and let $w\in\mathcal{N}_n$. Suppose $w_n\le 0$. According to (2.4), $(uw)_i=cv_i(aw_n)^r$ for $1\le i\le n-1$ and $(uw)_n=daw_nv_n$. Let $c_k^*=(aw_n)^rc_k$, $c^*=\sum_{k=1}^m e_kc_k^*$, and $d^*=\sum_{k=1}^m e_k(c_k^*)^{(1/r)}$. Then

$$\begin{aligned}(uw)_i&=cv_i(aw_n)^r=v_i\sum_{k=1}^m e_k(aw_n)^rc_k\\&=v_i\sum_{k=1}^m e_kc_k^*=c^*v_i\quad\text{for } 1\le i<n,\\(uw)_n&=daw_nv_n=aw_nv_n\sum_{k=1}^m e_kc_k^{(1/r)}\\&=v_n\sum_{k=1}^m e_k((aw_n)^rc_k)^{(1/r)}=v_n\sum_{k=1}^m e_kc_k^*=d^*v_n.\end{aligned} \tag{2.18}$$

Consequently, $uw=(c^*v_1,c^*v_2,\dots,c^*v_{n-1},d^*v_n)\in J_v$. One verifies, in a similar manner, that $vw\in J_v$ whenever $w_n>0$. Thus J_v is a right ideal of $\mathcal{N}_n$. Next, let J be any right ideal of $\mathcal{N}_n$ which contains v. Choose any $w\in\mathcal{N}_n$ with $w_n\ge 0$. According to (2.4), $(vw)_i=v_i(bw_n)^r$ for $1\le i\le n-1$ and $(vw)_n=bv_nw_n$. Let $c=(bw_n)^r$. Then $bw_n=c^{(1/r)}$ and we conclude that

$$vw=(cv_1,cv_2,\dots,cv_{n-1},c^{(1/r)})\in J\quad\forall c\ge 0. \tag{2.19}$$

Since finite sums and differences of elements of the form (2.19) must belong to J, we conclude that if $c=\sum_{k=1}^m e_kc_k$ and $d=\sum_{k=1}^m e_kc_k^{(1/r)}$ where $c_k\ge 0$ and $e_k=\pm 1$ for $1\le k\le m$, then $(cv_1,cv_2,\dots,cv_{n-1},dv_n)\in J$. Thus $J_v\subseteq J$ and J_v is, indeed, the smallest right ideal of $\mathcal{N}_n$ which contains the element v. □

If we take $v_n=0$ in the previous theorem, we immediately get the following corollary.

Corollary 2.8. *Let $\mathcal{N}_n$ be either a Type II, a Type III, or a Type IV nearring and suppose that $r_i=r$ for $1\le i<n$. Let $v\in\mathcal{N}_n$ and suppose that $v_n=0$. Then $J_v=\{cv:c\in\mathbb{R}\}$ is the smallest right ideal of $\mathcal{N}_n$ which contains the element v.*

Theorem 2.9. *Let $\mathcal{N}_n$ be either a Type II, a Type III, or a Type IV nearring. Then each proper ideal of $\mathcal{N}_n$ is a linear subspace of M. Moreover, the proper ideals of $\mathcal{N}_n$ are precisely the linear subspaces of M if and only if there exists a positive number r such that $r = r_i$ for $1 \le i < n$.*

Proof. Once again, we give the details only in the case where $\mathcal{N}_n$ is a Type IV nearring. Suppose $r = r_i$ for $1 \le i < n$ and J is a proper ideal of $\mathcal{N}_n$. Then $J \subseteq M$ according to Theorem 2.5. Choose a maximal linearly independent subset $\{v^i\}_{i=1}^m$ of vectors from J. Then $J_{v^i} \subseteq J$ for $1 \le i \le m$ by Corollary 2.8 and it follows easily that the vector space V which is generated by the vectors $\{v^i\}_{i=1}^m$ is contained in J. On the other hand, for any $w \in J$, we have $w = a_1v^1 + a_2v^2 + \cdots + a_mv^m$ since $\{v^i\}_{i=1}^m$ is a maximal linearly independent collection of vectors from J and we see that $w \in V$. That is, J is a linear subspace V of M.

Now suppose $r = r_i$ for $1 \le i < n$. We already know that each proper ideal of $\mathcal{N}_n$ is a linear subspace of M. Let L be any linear subspace of M which is a left ideal of $\mathcal{N}_n$ according to Theorem 2.5. Let $v \in L$ and $w \in \mathcal{N}_n$. There is no loss of generality in assuming that $w_n \le 0$. Then, according to (2.4), $(vw)_i = (aw_n)^r v_i$ and $(vw)_n = av_nw_n = 0$ since $v_n = 0$. Consequently, $vw = (aw_n)^r v \in L$ so that L is also a right ideal. We have now shown that if $r = r_i$ for $1 \le i < n$, then the proper ideals of $\mathcal{N}_n$ are precisely the linear subspaces of M.

Suppose, conversely, that the proper ideals of $\mathcal{N}_n$ are precisely the linear subspaces of M. Let $1 \le i, j < n$ and let $J = \{v \in M : v_i = v_j \text{ and } v_k = 0 \text{ for } k \ne i, j\}$. Let v be the vector such that $v_i = v_j = 1$ and $v_k = 0$ for $k \ne i, j$. Not both a and b can be zero and there is no loss of generality if we suppose $b \ne 0$. Let w be any vector in $\mathcal{N}_n$ such that $bw_n > 1$. Now, $v \in J$ and since J is an ideal of $\mathcal{N}_n$, we must have $vw \in J$. Thus, we have $(bw_n)^{r_i} = (vw)_i = (vw)_j = (bw_n)^{r_j}$ which readily implies that $r_i = r_j$. □

Theorem 2.10. *Let $\mathcal{N}_n$ be either a Type II, a Type III, or a Type IV nearring. Let J be a proper ideal of $\mathcal{N}_n$ and let $N(J)$ consist of all i, $1 \le i < n$, such that $v_i \ne 0$ for some $v \in J$ and suppose that $r_i \ne r_j$ for all distinct $i, j \in N(J)$. Then $J = \{v \in M : v_i = 0 \text{ for } i \notin N(J)\}$.*

Proof. In this case we give the details only in the case where $\mathcal{N}_n$ is a Type II nearring since, again, the remaining cases are similar. If $N(J) = \emptyset$, then $J = \{\mathbf{0}\} = \{v \in M : v_i = 0 \text{ for } i \notin N(J)\}$. We next consider the case where $N(J) \ne \emptyset$. Choose any $i \in N(J)$. According to the hypothesis, there exists a $v \in J$ with $v_i \ne 0$. We want to show that there exists a $w \in J$ with $w_i \ne 0$ and $w_j = 0$ for $j \ne i$. Suppose $v_j \ne 0$ where $j \ne i$. Let u be the vector in $\mathcal{N}_n$ such that $u_i = 0$ for $1 \le i < n$ and $u_n = e^{(\ln 2)/r_j}$. Then $vu - 2v \in J$. Note that $(vu - 2v)_j = 0$ while $(vu - 2v)_i = (e^{(r_i \ln 2)/r_j} - 2)v_i \ne 0$ since $r_i \ne r_j$. We have now shown that, for each $i \in N(J)$, there exists a vector $w^i \in J$ such that $w_i^i \ne 0$ while $w_j^i = 0$ for $j \ne i$. There is no loss of generality if we suppose that $w_i^i > 0$. Let $a \geq 0$ and let u be the vector such that $u_j = 0$ for $j \neq n$ and $u_n = (a/w_i^i)^{1/r_i}$. Then $w^i u \in J$. Note that $(w^i u)_i = a$ while $(w^i u)_j = 0$ for $j \neq i$. That is, $\{v \in \mathcal{N}_n : v_i \geq 0 \text{ and } v_j = 0 \text{ for } j \neq i\} \subseteq J$. Since the negative of a vector in J also belongs to J, we conclude that

$$V_i = \{v \in \mathcal{N}_n : v_j = 0 \text{ for } j \neq i\} \subseteq J. \tag{2.20}$$

Evidently, $J \subseteq \{v \in M : v_i = 0 \text{ for } i \notin N(J)\}$. Suppose $v_i = 0$ for $i \notin N(J)$. We want to show that $v \in J$. For each $i \in N(J)$, there exists a $w^i \in V_i \subseteq J$ such that $w_i^i = v_i$ and $w_i^j = 0$ for $i \neq j$. Thus, $v = \sum_{i \in N(J)} w^i \in J$ in view of (2.20). Consequently, $J = \{v \in M : v_i = 0 \text{ for } i \notin N(J)\}$ and the proof is complete. □

Our next result is an easy consequence of the previous theorem.

Corollary 2.11. *Let $\mathcal{N}_n$ be either a Type II, a Type III, or a Type IV nearring where $r_i \neq r_j$ for $i \neq j$. Let $K \subseteq \{1,2,\dots,n\}$, let $n \in K$, and let $J(K) = \{v \in \mathcal{N}_n : v_i = 0$ for all $i \in K\}$. Then each $J(K)$ is a proper ideal of $\mathcal{N}_n$ and every proper ideal of $\mathcal{N}_n$ is of this form. Consequently, $\mathcal{N}_n$ has exactly 2^{n-1} proper ideals.*

Corollary 2.12. *Let $n \geq 3$ and let $\mathcal{N}_n$ be either a Type II, a Type III, or a Type IV nearring where $r_i \neq r_j$ for $i \neq j$. Then each quotient ring of $\mathcal{N}_n$ by a proper ideal of $\mathcal{N}_n$ different from M is a Type II, a Type III, or a Type IV nearring, respectively.*

Proof. We give the details only in the case of Type II nearrings as the remaining cases are similar. Let $\mathcal{N}_n$ be a Type II nearring. According to Corollary 2.11, each proper ideal of $\mathcal{N}_n$ is of the form $J(K)$ where $K = \{i_1, i_2, \dots, i_m\}$, $i_m = n$. Moreover $|K| \geq 2$ since $J(K) \neq M$. Define a multiplication $*$ on $\mathbb{R}^m$ by

$$\begin{aligned} (v * w)_j &= v_j |w_m|^{r_{i_j}} \quad \text{for } 1 \leq j < m, \\ (v * w)_m &= v_m w_m. \end{aligned} \tag{2.21}$$

Evidently, $(\mathbb{R}^m, +, *)$ is a Type II nearring where $m \geq 2$. Define a map φ from $\mathcal{N}_n$ to $(\mathbb{R}^m, +, *)$ by $\varphi(v) = (v_{i_1}, v_{i_2}, \dots, v_{i_m})$. The map φ is evidently an additive epimorphism. For any $v, w \in \mathcal{N}_n$ and any $j < m$, we have

$$\begin{gathered} (\varphi(vw))_j = (vw)_{i_j} = v_{i_j} |w_n|^{r_{i_j}}, \\ (\varphi(v) * \varphi(w))_j = (\varphi(v))_j |(\varphi(w))_m|^{r_{i_j}} = v_{i_j} |w_{i_m}|^{r_{i_j}} = v_{i_j} |w_n|^{r_{i_j}} \end{gathered} \tag{2.22}$$

since $n = i_m$. It follows from (2.22) that $(\varphi(vw))_j = (\varphi(v) * \varphi(w))_j$ for $1 \leq j < m$.

In addition to this, we have

$$\begin{aligned}(\varphi(vw))_m &= (vw)_{i_m} = (vw)_n = v_n w_n,\\ (\varphi(v)*\varphi(w))_m &= (\varphi(v))_m(\varphi(w))_m = v_{i_m}w_{i_m} = v_n w_n.\end{aligned} \tag{2.23}$$

Thus, $(\varphi(vw))_m = (\varphi(v)*\varphi(w))_m$ as well and we conclude that $\varphi(vw) = \varphi(v)*\varphi(w)$ for all $v,w \in \mathcal{N}_n$. That is, φ is an epimorphism from $\mathcal{N}_n$ onto $(\mathbb{R}^m,+,*)$. Since the kernel of φ is $J(K)$, the proof is complete. □

3. Isomorphisms between linear right ideal nearrings. By a nonassociative Euclidean nearring we mean any triple $(\mathbb{R}^n,+,\cdot)$ where $(\mathbb{R}^n,+)$ is the n-dimensional Euclidean group, multiplication is continuous and right distributive over addition but may or may not be associative. We first prove a result about isomorphisms for two nonassociative Euclidean nearrings.

Theorem 3.1. *Let $\{f_i\}_{i=1}^n$ be n distinct, nonconstant, continuous self-maps of $\mathbb{R}$. Similarly, let $\{g_i\}_{i=1}^n$ be n distinct, nonconstant, continuous self-maps of $\mathbb{R}$ and define two binary operations $*$ and $\circ$ on $\mathbb{R}^n$ by $(v*w)_i = v_i f_i(w_n)$ and $(v\circ w)_i = v_i g_i(w_n)$ for $1 \le i \le n$. Then both $(\mathbb{R}^n,+,*)$ and $(\mathbb{R}^n,+,\circ)$ are nonassociative Euclidean nearrings. Moreover, they are isomorphic if and only if there exist a permutation p of $\{1,2,\dots,n\}$ such that $p(n)=n$ and a nonzero real number c such that $g_i(cx) = f_{p(i)}(x)$ for $1 \le i \le n$ and all $x \in \mathbb{R}$.*

Proof. Suppose first that there exist a permutation p of $\{1,2,\dots,n\}$ such that $p(n)=n$ and a nonzero real number c such that $g_i(cx) = f_{p(i)}(x)$ for all $x \in \mathbb{R}$. Define a self-map φ of $\mathbb{R}^n$ by $(\varphi(v))_i = v_{p(i)}$ for $1 \le i < n$ and $(\varphi(v))_n = cv_n$. Then φ is a linear automorphism which implies that it is an additive automorphism of $\mathbb{R}^n$. For $1 \le i < n$, we have

$$\begin{aligned}(\varphi(v*w))_i &= (v*w)_{p(i)} = v_{p(i)}f_{p(i)}(w_n),\\ (\varphi(v)\circ\varphi(w))_i &= (\varphi(v))_i g_i((\varphi(w))_n) = v_{p(i)}g_i(cw_n) = v_{p(i)}f_{p(i)}(w_n).\end{aligned} \tag{3.1}$$

Thus $(\varphi(v*w))_i = (\varphi(v)\circ\varphi(w))_i$ for $1 \le i < n$.

Furthermore, we have

$$(\varphi(v*w))_n = c(v*w)_n = cv_n f_n(w_n) \tag{3.2}$$

and since $p(n) = n$, we also have

$$\begin{aligned}(\varphi(v)\circ\varphi(w))_n &= (\varphi(v))_n g_n((\varphi(w))_n) = cv_n g_n(cw_n)\\ &= cv_n f_{p(n)}(w_n) = cv_n f_n(w_n).\end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that $(\varphi(v*w))_n = (\varphi(v)\circ\varphi(w))_n$. Consequently,

$(\varphi(v*w))_i = (\varphi(v)\circ\varphi(w))_i$ for $1 \le i \le n$. That is, $\varphi(v*w) = \varphi(v)\circ\varphi(w)$ and we conclude that φ is an isomorphism from $(\mathbb{R}^n,+,*)$ onto $(\mathbb{R}^n,+,\circ)$.

Suppose, conversely, that $(\mathbb{R}^n,+,*)$ and $(\mathbb{R}^n,+,\circ)$ are isomorphic and that φ is an isomorphism from $(\mathbb{R}^n,+,*)$ onto $(\mathbb{R}^n,+,\circ)$. Since φ is an additive automorphism, it is a linear automorphism which means that there exists an $n\times n$ nonsingular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{3.4}$$

such that

$$\varphi(v) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{1j}v_j \\ \sum_{j=1}^n a_{2j}v_j \\ \vdots \\ \sum_{j=1}^n a_{nj}v_j \end{bmatrix}. \tag{3.5}$$

Recall that $(v*w)_i = v_i f_i(w_n)$ and $(v\circ w)_i = v_i g_i(w_n)$. It follows from this and (3.5) that

$$(\varphi(v*w))_i = \sum_{j=1}^n a_{ij}(v*w)_j = \sum_{j=1}^n a_{ij}v_j f_j(w_n). \tag{3.6}$$

Similarly,

$$(\varphi(v)\circ\varphi(w))_i = (\varphi(v))_i g_i((\varphi(w))_n) = \left(\sum_{j=1}^n a_{ij}v_j\right) g_i\left(\sum_{j=1}^n a_{nj}w_j\right), \tag{3.7}$$

and it follows from (3.6) and (3.7) that

$$\sum_{j=1}^n a_{ij}v_j f_j(w_n) = \left(\sum_{j=1}^n a_{ij}v_j\right) g_i\left(\sum_{j=1}^n a_{nj}w_j\right) \tag{3.8}$$

for $1 \le i \le n$. Now fix i. Since the matrix A is nonsingular, there exists a j such that $a_{ij} \ne 0$. Let $v_k = 0$ for $k \ne j$ and $v_j = 1/a_{ij}$ in (3.8) and conclude that

$$f_j(w_n) = g_i\left(\sum_{k=1}^n a_{nk}w_k\right). \tag{3.9}$$

Suppose $a_{ir} \ne 0$. The technique used previously yields the fact that

$$f_r(w_n) = g_i\left(\sum_{k=1}^n a_{nk}w_k\right), \tag{3.10}$$

and since w_n can be any real number, it follows from (3.9) and (3.10) that $f_r = f_j$. This means that $r = j$ since the functions $\{f_i\}_{i=1}^n$ are all distinct. We therefore conclude that for each i, there exists exactly one j such that $a_{ij} \neq 0$ and we define a self-map p of $\{1,2,\dots,n\}$ by $p(i) = j$ where i and j are the subscripts in (3.9). Again, we appeal to the fact that A is nonsingular to conclude that $a_{nk} \neq 0$ for some k. Suppose $k \neq n$. Let $w_j = 0$ for $j \neq k$ and from (3.9) conclude that $g_i(a_{nk}w_k) = f_j(0)$ for all $w_k \in \mathbb{R}$. But this is a contradiction since each g_i is nonconstant. Consequently, $k = n$, $a_{nn} \neq 0$, and we appeal again to (3.9) to conclude that

$$f_{p(i)}(w_n) = g_i(a_{nn}w_n) \quad \text{for } 1 \le i \le n,\ \forall w_n \in \mathbb{R}. \tag{3.11}$$

Since the functions $\{g_i\}_{i=1}^n$ are distinct, it follows from (3.11) that the function p is injective and, consequently, is a permutation of $\{1,2,\dots,n\}$. It remains for us to show that $p(n) = n$. Since there is exactly one j such that $a_{nj} \neq 0$ and $a_{nn} \neq 0$, it follows from (3.5) that

$$(\varphi(v))_n = a_{nn}v_n \quad \forall v \in \mathbb{R}^n. \tag{3.12}$$

It follows from (3.5) and (3.12) that

$$(\varphi(v*w))_n = a_{nn}v_n f_n(w_n) \tag{3.13}$$

and it follows from (3.5), (3.11), and (3.12) that

$$\begin{aligned}(\varphi(v)\circ\varphi(w))_n &= (\varphi(v))_n g((\varphi(w))_n) = a_{nn}v_n g_n(a_{nn}w_n)\\ &= a_{nn}v_n f_{p(n)}(w_n).\end{aligned} \tag{3.14}$$

It follows from (3.13) and (3.14) that $f_n = f_{p(n)}$ and since the functions $\{f_i\}_{i=1}^n$ are all distinct, we conclude that $p(n) = n$. Take c in the statement of the theorem to be a_{nn} and it follows from (3.11) that the proof is complete. □

We will see in Section 7 that if two linear right ideal nearrings are isomorphic, then they must be of the same type.

For a Euclidean nearring $\mathcal{N}_n$, let $I(\mathcal{N}_n) = \{i : (vw)_i = 0 \text{ for all } v,w \in \mathcal{N}_n\}$. The cardinality of a set A will be denoted by $|A|$. The constant function which maps all of $\mathbb{R}$ into the real number a will be denoted by $\langle a\rangle$ and the range of a function f will be denoted by $\operatorname{Ran} f$.

Theorem 3.2. *Two n-dimensional Type I nearrings $\mathcal{N}_{n_1} = (\mathbb{R}^n,+,*)$ and $\mathcal{N}_{n_2} = (\mathbb{R}^n,+,\circ)$ are isomorphic if and only if $|I(\mathcal{N}_{n_1})| = |I(\mathcal{N}_{n_2})|$ and either $n \in I(\mathcal{N}_{n_1}) \cap I(\mathcal{N}_{n_2})$ or $n \notin I(\mathcal{N}_{n_1}) \cup I(\mathcal{N}_{n_2})$.*

Proof. Let $N = \{1,2,\dots,n\}$. For each $i \in I(\mathcal{N}_{n_1})$ define the constant self-map f_i of $\mathbb{R}$ by $f_i = \langle 0\rangle$ and for each $i \in N\backslash I(\mathcal{N}_{n_1})$ define $f_i = \langle 1\rangle$. Similarly, for each $i \in I(\mathcal{N}_{n_2})$, define $g_i = \langle 0\rangle$ and for each $i \in N\backslash I(\mathcal{N}_{n_2})$ define $g_i = \langle 1\rangle$. Note that $(v*w)_i = v_i f_i(w_n)$ and $(v\circ w)_i = v_i g_i(w_n)$ for $1 \le i \le n$. Therefore it follows from Theorem 3.1 that $\mathcal{N}_{n_1}$ and $\mathcal{N}_{n_2}$ are isomorphic if and only if there exist a permutation p of $\{1,2,\dots,n\}$ such that $p(n) = n$ and a nonzero real number c such that $g_i(cx) = f_{p(i)}(x)$ for all $x \in \mathbb{R}$. It readily follows that the latter holds if and only if $|I(\mathcal{N}_{n_1})| = |I(\mathcal{N}_{n_2})|$ and either $n \in I(\mathcal{N}_{n_1}) \cap I(\mathcal{N}_{n_2})$ or $n \notin I(\mathcal{N}_{n_1}) \cup I(\mathcal{N}_{n_2})$. □

Definition 3.3. A Type II, III, or IV nearring $\mathcal{N}_n$ is said to be *distinguished* if $r_i \neq r_j$ whenever $i \neq j$.

Theorem 3.4. *Let* $\mathcal{N}_* = (\mathbb{R}^n, +, *)$ *and* $\mathcal{N}_\circ = (\mathbb{R}^n, +, \circ)$ *be two distinguished Type II nearrings where* $(v * w)_i = v_i |w_n|^{r_i}$ *for* $1 \le i < n$ *and* $(v * w)_n = v_n w_n$ *and where* $(v \circ w)_i = v_i |w_n|^{s_i}$ *for* $1 \le i < n$ *and* $(v * w)_n = v_n w_n$. *Then* $\mathcal{N}_*$ *and* $\mathcal{N}_\circ$ *are isomorphic if and only if there exist a permutation* p *of* $\{1,2,\dots,n-1\}$ *such that* $s_i = r_{p(i)}$ *for* $1 \le i \le n-1$.

Proof. Suppose first that there exist a permutation p of $\{1,2,\dots,n-1\}$ such that $s_i = r_{p(i)}$ for $1 \le i \le n-1$. Define a self-map φ of $\mathbb{R}^n$ by $(\varphi(v))_i = v_{p(i)}$ for $1 \le i < n$ and $(\varphi(v))_n = v_n$. For $1 \le i < n$, we have

$$\begin{aligned}(\varphi(v * w))_i &= (v * w)_{p(i)} = v_{p(i)} |w_n|^{r_{p(i)}} = v_{p(i)} |w_n|^{s_i},\\ (\varphi(v) \circ \varphi(w))_i &= (\varphi(v))_i |(\varphi(w))_n|^{s_i} = v_{p(i)} |w_n|^{s_i}.\end{aligned} \tag{3.15}$$

Since

$$(\varphi(v * w))_n = (v * w)_n = v_n w_n = (\varphi(v))_n (\varphi(w))_n = (\varphi(v) \circ \varphi(w))_n, \tag{3.16}$$

it follows from (3.15) and (3.16) that φ is a multiplicative isomorphism from $\mathcal{N}_*$ onto $\mathcal{N}_\circ$. This proves that $\mathcal{N}_*$ is isomorphic to $\mathcal{N}_\circ$ since it is evident that φ is also an additive automorphism.

Suppose, conversely, that $\mathcal{N}_*$ and $\mathcal{N}_\circ$ are isomorphic. Define continuous self-maps of $\mathbb{R}$ by $f_i(x) = |x|^{r_i}$ and $g_i(x) = |x|^{s_i}$ for $1 \le i < n$ and $f_n(x) = x = g_n(x)$. It readily follows that $(v * w)_i = v_i f_i(w_n)$ and $(v \circ w)_i = v_i g_i(w_n)$ for $1 \le i \le n$. According to Theorem 3.1, there exist a permutation p of $\{1,2,\dots,n\}$ such that $p(n) = n$ and a nonzero real number c such that $g_i(cx) = f_{p(i)}(x)$ for $1 \le i \le n$. For $1 \le i < n$, we have

$$|c|^{s_i} |x|^{s_i} = |cx|^{s_i} = g_i(cx) = f_{p(i)}(x) = |x|^{r_{p(i)}} \tag{3.17}$$

which implies that $|c|^{s_i} = |x|^{r_{p(i)} - s_i}$. Since x can be any real number, this, in turn, implies that $s_i = r_{p(i)}$ and the proof is complete. □

The proof of the following result is quite similar to the preceding proof and, for that reason, will be omitted.

Theorem 3.5. *Let $\mathcal{N}_* = (\mathbb{R}^n, +, *)$ and $\mathcal{N}_\circ = (\mathbb{R}^n, +, \circ)$ be two distinguished Type III nearrings where*

$$(v * w)_i = \begin{cases} v_i (w_n)^{r_i} & \text{for } w_n \geq 0, \\ -v_i |w_n|^{r_i} & \text{for } w_n < 0, \end{cases} \tag{3.18}$$

for $i \neq n$, $(v * w)_n = v_n w_n$ where $r_i > 0$,

$$(v \circ w)_i = \begin{cases} v_i (w_n)^{s_i} & \text{for } w_n \geq 0, \\ -v_i |w_n|^{s_i} & \text{for } w_n < 0, \end{cases} \tag{3.19}$$

for $i \neq n$, $(v \circ w)_n = v_n w_n$ where $s_i > 0$.

Then $\mathcal{N}_*$ and $\mathcal{N}_\circ$ are isomorphic if and only if there exist a permutation p of $\{1, 2, \ldots, n-1\}$ such that $s_i = r_{p(i)}$ for $1 \leq i \leq n-1$.

Theorem 3.6. *Let $\mathcal{N}_* = (\mathbb{R}^n, +, *)$ and $\mathcal{N}_\circ = (\mathbb{R}^n, +, \circ)$ be two distinguished Type IV nearrings where*

$$(v * w)_i = \begin{cases} v_i (a_1 w_n)^{r_i} & \text{for } w_n \leq 0, \\ v_i (b_1 w_n)^{r_i} & \text{for } w_n > 0, \end{cases} \tag{3.20}$$

for $i \neq n$,

$$(v * w)_n = \begin{cases} a_1 v_n w_n & \text{for } w_n \leq 0, \\ b_1 v_n w_n & \text{for } w_n > 0, \end{cases} \tag{3.21}$$

where $r_i > 0$, $a_1 \leq 0$, $b_1 \geq 0$, and $a_1^2 + b_1^2 \neq 0$

$$(v \circ w)_i = \begin{cases} v_i (a_2 w_n)^{s_i} & \text{for } w_n \leq 0, \\ v_i (b_2 w_n)^{s_i} & \text{for } w_n > 0, \end{cases} \tag{3.22}$$

for $i \neq n$,

$$(v \circ w)_n = \begin{cases} a_2 v_n w_n & \text{for } w_n \leq 0, \\ b_2 v_n w_n & \text{for } w_n > 0, \end{cases} \tag{3.23}$$

where $s_i > 0$, $a_2 \leq 0$, $b_2 \geq 0$, and $a_2^2 + b_2^2 \neq 0$.

Then $\mathcal{N}_*$ and $\mathcal{N}_\circ$ are isomorphic if and only if there exists a positive number c such that $a_1 = ca_2$ and $b_1 = cb_2$ or a negative number c such that $a_1 = cb_2$ and $b_1 = ca_2$ and there exist a permutation p of $\{1, 2, \ldots, n-1\}$ such that $s_i = r_{p(i)}$ for $1 \leq i \leq n-1$.

Proof. Suppose that there exists a negative number c such that $a_1 = cb_2$ and $b_1 = ca_2$ and there exist a permutation p of $\{1,2,\dots,n-1\}$ such that $s_i = r_{p(i)}$ for $1 \le i \le n-1$. Define a linear automorphism φ of $\mathbb{R}^n$ by $(\varphi(v))_i = v_{p(i)}$ for $1 \le i < n$ and $(\varphi(v))_n = cv_n$. For $w_n \le 0$ and for $1 \le i < n$, we have

$$\begin{aligned}(\varphi(v * w))_i &= (v * w)_{p(i)} = v_{p(i)}(a_1 w_n)^{r_{p(i)}} = v_{p(i)}(a_1 w_n)^{s_i},\\ (\varphi(v) \circ \varphi(w))_i &= (\varphi(v))_i (cb_2 w_n)^{s_i} = v_{p(i)}(a_1 w_n)^{s_i}.\end{aligned} \tag{3.24}$$

In addition to this, we have

$$(\varphi(v * w))_n = c(v * w)_n = ca_1 v_n w_n = b_2 c v_n c w_n = (\varphi(v) \circ \varphi(w))_n. \tag{3.25}$$

It follows from (3.24) and (3.25) that $\varphi(v * w) = \varphi(v) \circ \varphi(w)$ whenever $w_n \le 0$. One verifies, in a similar manner, that $\varphi(v * w) = \varphi(v) \circ \varphi(w)$ whenever $w_n > 0$ and we conclude that φ is an isomorphism from $\mathcal{N}_*$ onto $\mathcal{N}_\circ$ whenever there exists a negative number c such that $a_1 = cb_2$ and $b_1 = ca_2$ and there exist a permutation p of $\{1,2,\dots,n-1\}$ such that $s_i = r_{p(i)}$ for $1 \le i \le n-1$. The remaining case is similar to the preceding one so we omit the details.

Now suppose that $\mathcal{N}_*$ and $\mathcal{N}_\circ$ are isomorphic. For $1 \le i < n$ define continuous self-maps f_i and g_i of $\mathbb{R}$ by

$$f_i(x) = \begin{cases} (a_1 x)^{r_i} & \text{for } x \le 0,\\ (b_1 x)^{r_i} & \text{for } x > 0,\end{cases} \qquad g_i(x) = \begin{cases} (a_2 x)^{s_i} & \text{for } x \le 0,\\ (b_2 x)^{s_i} & \text{for } x > 0\end{cases} \tag{3.26}$$

and define continuous self-maps f_n and g_n by

$$f_n(x) = \begin{cases} a_1 x & \text{for } x \le 0,\\ b_1 x & \text{for } x > 0,\end{cases} \qquad g_n(x) = \begin{cases} a_2 x & \text{for } x \le 0,\\ b_2 x & \text{for } x > 0.\end{cases} \tag{3.27}$$

It readily follows that $(v * w)_i = v_i f_i(w_n)$ and $(v \circ w)_i = v_i g_i(w_n)$ for $1 \le i \le n$. Consequently, Theorem 3.1 assures that there exist a permutation p of $\{1,2,\dots,n\}$ such that $p(n) = n$ and a nonzero real number c such that $g_i(cx) = f_{p(i)}(x)$ for $1 \le i \le n$ and for all $x \in \mathbb{R}$. We consider the case where $c < 0$. For $x < 0$, we have $cb_2 x = g_n(cx) = f_n(x) = a_1 x$ and it follows that $a_1 = cb_2$. In a similar manner, one chooses $x > 0$ and shows that $b_1 = ca_2$. For all $x < 0$, we have $f_{p(i)}(x) = (a_1 x)^{r_{p(i)}}$ and $g_i(cx) = (cb_2 x)^{s_i} = (a_1 x)^{s_i}$, and since $g_i(cx) = f_{p(i)}(x)$, it follows that $s_i = r_{p(i)}$ for $1 \le i < n$. The case where $c > 0$ is similar so we omit the details. □

4. The multiplicative semigroups of Type I nearrings. For any semigroup S, we denote by S^1 the semigroup S with an identity adjoined when S has no identity. We take $S^1 = S$ when S does have an identity. For a detailed discussion of Green's relations and related concepts, one may consult [2, Chapter 2]. We now recall the definitions of Green's five equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ for any semigroup S. Two elements v and w of a semigroup S are $\mathcal{L}$-equivalent if $S^1v = S^1w$. They are $\mathcal{R}$-equivalent if $vS^1 = wS^1$ and $\mathcal{J}$-equivalent if $S^1vS^1 = S^1wS^1$. The $\mathcal{H}$ and $\mathcal{D}$ relations are defined by $\mathcal{H} = \mathcal{L}\cap\mathcal{R}$ and $\mathcal{D} = \mathcal{L}\circ\mathcal{R}$ which is also an equivalence relation since $\mathcal{L}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{L}$. Types I, II, III, and IV semigroups will be the multiplicative semigroups of Types I, II, III, and IV nearrings, respectively. For a Type I semigroup S, we let $I(S) = \{i : (vw)_i = 0 \text{ for all } v,w \in \mathbb{R}^n\}$.

Theorem 4.1. *Let S be a Type I semigroup and let $v,w \in S$. Then $v\mathcal{L}w$ if and only if $v = w$ or $v_i = 0 = w_i$ for all $i \in I(S)$.*

Proof. Suppose $v_i = 0 = w_i$ for all $i \in I(S)$. For $i \in I(S)$, we have $(vw)_i = 0 = v_i$ and for $i \notin I(S)$, we have $(vw)_i = v_i$. Thus, $v = vw$. One shows, in a similar manner, that $w = wv$ and we conclude that $v\mathcal{L}w$. Now suppose that $v\mathcal{L}w$ and $v \neq w$. Then $v = uw$ for some u, and for each $i \in I(S)$, we have $v_i = (uw)_i = 0$. Similarly, we have $w_i = 0$ for each $i \in I(S)$ and the proof is complete. □

Theorem 4.2. *Let S be a Type I semigroup and let $v,w \in S$. Then $v\mathcal{R}w$ if and only if $v = w$.*

Proof. Suppose $v\mathcal{R}w$. Then $v = wx$ and $w = vy$ for $x,y \in S^1$. If either $x = 1$ or $y = 1$, then $v = w$. Consider the case where $x \neq 1 \neq y$. Then $x,y \in S$. For $i \in I(S)$, we have $v_i = (wx)_i = 0$ and $w_i = (vy)_i = 0$. For $i \notin I(S)$, we have $v_i = (wx)_i = w_i$. Consequently, $v = w$ in this case also. □

Since $\mathcal{H} = \mathcal{L}\cap\mathcal{R}$ and $\mathcal{D} = \mathcal{L}\circ\mathcal{R}$, the following result is an immediate consequence of Theorems 4.1 and 4.2.

Theorem 4.3. *Let S be a Type I semigroup. Then $\mathcal{H} = \mathcal{R}$ and $\mathcal{D} = \mathcal{L}$.*

Theorem 4.4. *Let S be a Type I semigroup and let $v,w \in S$. Then $v\mathcal{J}w$ if and only if $v = w$ or $v_i = 0 = w_i$ for all $i \in I(S)$. In other words, $\mathcal{J} = \mathcal{L}$.*

Proof. Suppose that either $v = w$ or $v_i = 0 = w_i$ for all $i \in I(S)$. Then $v\mathcal{L}w$ by Theorem 4.1 which implies $v\mathcal{J}w$. Next, suppose that $v\mathcal{J}w$ and consider the case where $v \neq w$. Then $v = uwr$ for $u,r \in S^1$. Since $v \neq w$, we must have either $u \neq 1$ or $r \neq 1$. In either event, we have $v_i = (uwr)_i = 0$ for all $i \in I(S)$. In a similar manner, one shows that $w_i = 0$ for all $i \in I(S)$ and the proof is complete. □

It is well known that the maximal subgroups of a semigroup are precisely the $\mathcal{H}$-classes which contain idempotents. It follows from Theorems 4.2 and 4.3 that all subgroups of a Type I semigroup consist of a single element.

5. The multiplicative semigroups of Type II nearrings

Theorem 5.1. *Let S be a Type II semigroup and let $v,w \in S$. Then $v\mathcal{L}w$ if and only if $v = w$ or $v_n \neq 0 \neq w_n$.*

Proof. Suppose $v_n \neq 0 \neq w_n$. Define $u_n = v_n/w_n$ and $u_i = v_i/|w_n|^{r_i}$ for $1 \leq i < n$. One easily verifies that $v = uw$. One shows, in a similar manner, that $w = uv$ for some $u \in S$ and we conclude that $v\mathcal{L}w$.

Now suppose that $v\mathcal{L}w$ and that $v \neq w$. Then $v = uw$ for some $u \in S$. Suppose $w_n = 0$. Then $v_n = (uw)_n = u_n w_n = 0$ and $v_i = u_i|w_n|^{r_i} = 0$ for $1 \leq i < n$. Thus, $v_i = 0$ for all i. One shows, in a similar manner, that $w_i = 0$ for all i but this means that $v = w$ which is a contradiction. Thus, we conclude that $w_n \neq 0$. In the same manner, one shows that $v_n \neq 0$ and the proof is complete. □

It will be convenient to denote the element of $\mathbb{R}^n$, whose all coordinates are 0, by the symbol $\mathbf{0}$.

Theorem 5.2. *Let S be a Type II semigroup and let $v, w \in S$. Then $v\mathcal{R}w$ if and only if there exists a real number $c \neq 0$ such that $v_n = cw_n$ and $v_i = |c|^{r_i}w_i$ for $1 \leq i < n$.*

Proof. Suppose there exists a real number $c \neq 0$ such that $v_n = cw_n$ and $v_i = |c|^{r_i}w_i$ for $1 \leq i < n$. Let x_i be arbitrary for $1 \leq i < n$ and let $x_n = c$. Then

$$\begin{aligned} v_i &= |x_n|^{r_i} w_i = (wx)_i \quad \text{for } 1 \leq i < n, \\ v_n &= w_n c = w_n x_n = (wx)_n \end{aligned} \tag{5.1}$$

which means that $v = wx$. Now $w_i = |1/c|^{r_1}v_i$ for $1 \leq i < n$ and $w_n = (1/c)v_n$. Let y_i be arbitrary for $1 \leq i < n$ and let $y_n = 1/c$. It follows that $w = vy$ and we conclude that $v\mathcal{R}w$.

Now suppose $v\mathcal{R}w$. If $v = w$, take $c = 1$. Now consider the case where $v \neq w$. Then $v = wx$ for some $x \in S$. Thus, $v_n = w_n x_n$ and $v_i = (wx)_i = w_i|x_n|^{r_i}$ for $1 \leq i < n$. Suppose $x_n = 0$. Then $v = \mathbf{0}$ and since $w = vy$ for some $y \in S$, it follows that $w = \mathbf{0}$. But this contradicts the fact that $v \neq w$. Thus, $w_n \neq 0$ and we take $c = w_n$. □

The next result is an immediate consequence of Theorems 5.1 and 5.2 since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

Theorem 5.3. *Let S be a Type II semigroup, then $v\mathcal{H}w$ if and only if either $v = w$ or $v_n \neq 0 \neq w_n$ and there exists a nonzero real number c such that $v_n = cw_n$ and $v_i = |c|^{r_i}w_i$ for $1 \leq i < n$.*

As we mentioned before, the maximal subgroups of any semigroup are precisely the $\mathcal{H}$-classes which contain idempotents, so our next task is to find the idempotent elements of a Type II semigroup.

Theorem 5.4. *An element v of a Type II semigroup is idempotent if and only if $v = \mathbf{0}$ or $v_n = 1$.*

Proof. It is immediate that $\mathbf{0}$ is idempotent and if $v_n = 1$, we have $(vv)_n = v_n v_n = 1 = v_n$ and $(vv)_i = v_i|v_n|^{r_i} = v_i$ for $1 \le i < n$ so that v is idempotent as well.

Suppose, conversely, that v is idempotent. Then $v_n = (vv)_n = v_n^2$. Then $v_n = 0$ or $v_n = 1$. Suppose $v_n = 0$. Then $v_i = (vv)_i = v_i|v_n|^{r_i} = 0$ for $1 \le i < n$ and we conclude that $v = \mathbf{0}$. □

We will denote the $\mathcal{H}$-class of S containing the idempotent e by $\mathcal{H}_e$ and we will denote by $\mathbb{R}_M$ the multiplicative group of nonzero real numbers.

Theorem 5.5. *Let S be a Type II semigroup. Then $\mathcal{H}_{\mathbf{0}} = \{\mathbf{0}\}$ and if e is a nonzero idempotent of S, then $\mathcal{H}_e = \{v \in S : v_n \neq 0$ and $v_i = |v_n|^{r_i} e_i$ for $1 \le i < n\}$. Moreover, $\mathcal{H}_e$ is isomorphic to $\mathbb{R}_M$ whenever $e \neq \mathbf{0}$.*

Proof. The first two assertions are immediate consequences of Theorems 5.2, 5.3, and 5.4 and it remains for us to show that $\mathcal{H}_e$ is isomorphic to $\mathbb{R}_M$ for nonzero e. Note that $v_n \neq 0$ for all $v \in \mathcal{H}_e$. Define a surjection φ from $\mathcal{H}_e$ onto $\mathbb{R}_M$ by $\varphi(v) = v_n$. Then $\varphi(vw) = (vw)_n = v_n w_n = \varphi(v)\varphi(w)$ and we conclude that φ is an epimorphism. Furthermore, if $\varphi(v) = 1$, then $v_n = 1$ and $v_i = e_i|v_n|^{r_i} = e_i|1|^{r_i} = e_i$ for $1 \le i < n$. Thus, $v = e$ and φ is an isomorphism from $\mathcal{H}_e$ onto $\mathbb{R}_M$. □

Corollary 5.6. *Let S be a Type II semigroup. Then S is the union of its nonzero maximal subgroups, each of which is isomorphic to $\mathbb{R}_M$, together with a subsemigroup Z which has the property that $vw = \mathbf{0}$ for each $v \in S$ and each $w \in Z$. Moreover, the identity of each nonzero maximal subgroup of S is a right identity for all of S.*

Proof. Let $Z = \{w \in S : w_n = 0\}$. It is immediate that $vw = \mathbf{0}$ for each $v \in S$ and $w \in Z$. Suppose $v \notin Z$. Then $v_n \neq 0$. Define $e_n = 1$ and $e_i = v_i/|v_n|^{r_i}$. Then e is idempotent by Theorem 5.4 and $v \in \mathcal{H}_e$, which is isomorphic to $\mathbb{R}_M$ by Theorem 5.5. If e is the identity of a nonzero maximal subgroup of S, then $e_n = 1$ by Theorem 5.4 and it readily follows that $ve = v$ for all $v \in S$. □

Lemma 5.7. *Let $v, w \in S$, a Type II semigroup, and suppose that $v \neq \mathbf{0}$. Then $v \in SwS$ if and only if $w_n \neq 0$.*

Proof. Suppose that $v \in SwS$. Then $v = xwy$ where $x, y \in S$. Then $v_n = x_n w_n y_n$ and $v_i = x_i|w_n y_n|^{r_i}$ for $1 \le i < n$. Since $v \neq \mathbf{0}$, it follows that $w_n y_n \neq 0$ and hence $w_n \neq 0$. Suppose, conversely, that $w_n \neq 0$. Let $y_n = 1$ and let y_i be arbitrary for $1 \le i < n$. Define $x_n = v_n/w_n$ and let $x_i = v_i/|w_n|^{r_i}$ for $1 \le i < n$. Then $v_n = x_n w_n y_n$ and $v_i = x_i|w_n y_n|^{r_i}$ for $1 \le i < n$. Thus, $v = xwy$ and the proof is complete. □

Lemma 5.8. *Let $v, w \in S$, a Type II semigroup, and suppose that $v \neq \mathbf{0}$. Then $v \in wS$ if and only if there exists a nonzero $c \in \mathbb{R}$ such that $v_n = cw_n$ and $v_i = |c|^{r_i} w_i$ for $1 \le i < n$.*

Proof. Suppose there exists a nonzero $c \in \mathbb{R}$ such that $v_n = cw_n$ and $v_i = |c|^{r_i}$ for $1 \le i < n$. Define $y_n = c$ and let y_i be arbitrary for $1 \le i < n$. Then $v_n = cw_n = w_n y_n$ and $v_i = |c|^{r_i} w_i = w_i |y_n|^{r_i} = (wy)_i$ for $1 \le i < n$. Consequently, $v = wy \in wS$. Suppose, conversely, that $v \in wS$. Then $v = wy$ for some $y \in S$. Then $v_n = w_n y_n$ and $v_i = w_i |y_n|^{r_i}$ for $1 \le i < n$. Now $y_n \ne 0$ since $v \ne \mathbf{0}$ so we take $c = y_n$. □

Theorem 5.9. *Let $v, w \in S$, a Type II semigroup. If $v = \mathbf{0}$, then $v \in S^1 w S^1$. If $v \ne \mathbf{0}$, then $v \in S^1 w S^1$ if and only if $w_n \ne 0$ or there exists a nonzero $c \in \mathbb{R}$ such that $v_n = cw_n$ and $v_i = |c|^{r_i} w_i$ for $1 \le i < n$.*

Proof. Suppose $v \ne \mathbf{0}$. If $w_n \ne 0$, then $v \in SwS \subseteq S^1 w S^1$ by Lemma 5.7 and if there exists a nonzero number c such that $v_n = cw_n$ and $v_i = |c|^{r_i}$ for $1 \le i < n$, then $v \in wS \subseteq S^1 w S^1$ by Lemma 5.8. Suppose, conversely, that $v \in S^1 w S^1$. Since S has a right identity (any element w where $w_n = 1$), $S^1 w S^1 = S^1 w S$ and it follows that $v = xwy$ where $x \in S^1$ and $y \in S$. If $x = 1$, then $v \in wS$ and it follows from Lemma 5.8 that there exists a nonzero $c \in \mathbb{R}$ such that $v_n = cw_n$ and $v_i = |c|^{r_i} w_i$ for $1 \le i < n$. If $x \ne 1$, then $v = xwy \in SwS$ and it follows from Lemma 5.7 that $w_n \ne 0$. □

Theorem 5.10. *Let S be a Type II semigroup and let $v, w \in S$. Then $v \mathcal{J} w$ if and only if $v = w$ or one of the following two conditions is satisfied:*

(i) $v_n \ne 0 \ne w_n$ *or*

(ii) *there exists a nonzero real number c such that $v_n = cw_n$, $v_i = |c|^{r_i} w_i$ for $1 \le i < n$.*

Proof. Suppose (i) holds. Then $v \mathcal{L} w$ by Theorem 5.1 which means $v \mathcal{J} w$. Suppose (ii) holds. Then $v \in wS \subseteq wS^1$ by Lemma 5.8. Let $b = 1/c$. Then $b \ne 0$, $w_n = bv_n$, and $w_i = |b|^{r_i} v_i$ for $1 \le i < n$ and it follows from Lemma 5.8 that $w \in vS \subseteq vS^1$. Consequently, $v \mathcal{R} w$ and we conclude that $v \mathcal{J} w$ in this case as well. □

Suppose, conversely, that $v \mathcal{J} w$ and suppose also that $v \ne w$. Then either $v \ne \mathbf{0}$ or $w \ne \mathbf{0}$ and there is no loss of generality in assuming that $v \ne \mathbf{0}$. We observed in the proof of Theorem 5.9 that $S^1 u S^1 = S^1 u S$ for all $u \in S$. Consequently, $S^1 v S = S^1 w S$. Since $v \in S^1 w S$, we have $v = xwy$ where $x \in S^1$ and $y \in S$.

Case 1 ($x \in S$). Then $v \in SwS$ and $w_n \ne 0$ by Lemma 5.7. Since $w \in S^1 v S$, either $w = rvs$ or $w = vs$ where $r, s \in S$. Then $w_n = r_n v_n s_n$ in the former case and $w_n = v_n s_n$ in the latter. In either event, $v_n \ne 0$ since $w_n \ne 0$ and we have $v_n \ne 0 \ne w_n$. That is, (i) holds.

Case 2 ($x \notin S$). Then $x = 1$ and $v = wy \in wS$ and (ii) holds in this case in view of Lemma 5.8.

6. The multiplicative semigroups of Type III nearrings

Theorem 6.1. *Let S be a Type III semigroup and let $v, w \in S$. Then $v \mathcal{L} w$ if and only if $v = w$ or $v_n \ne 0 \ne w_n$.*

Proof. Suppose $v_n \neq 0 \neq w_n$ and consider first the case where $w_n > 0$. Define $u_n = v_n/w_n$ and $u_i = v_i/w_n^{r_i}$. Then $v = uw$. Now consider the case where $w_n < 0$. Again, define $u_n = v_n/w_n$ but this time, define $u_i = -v_i/|w_n|^{r_i}$. Again, we have $v = uw$. In much the same way, one shows that $w = uv$ for some $u \in S$ and we conclude that $v\mathcal{L}w$.

Now suppose $v\mathcal{L}w$ and suppose further that $v \neq w$. It follows that $v = xw$ and $w = yv$ for appropriate $x, y \in S$. Suppose $v_n = 0$. Then $w_n = y_n v_n = 0$ and $w_i = y_i v_n^{r_i} = 0$ for $1 \leq i < n$ which means that $w = \mathbf{0}$. But we also have $v_i = x_i w_n^{r_i}$ which means that $v = \mathbf{0}$. But this is a contradiction since $v \neq w$. Thus $v_n \neq 0$ and, similarly, $w_n \neq 0$. □

Theorem 6.2. *Let S be a Type III semigroup and let $v, w \in S$. Then $v\mathcal{R}w$ if and only if either $v = w$ or there exists a real number $c > 0$ such that*

$$v_n = cw_n, \qquad v_i = c^{r_i} w_i \quad \text{for } 1 \leq i < n \tag{6.1}$$

or there exists a real number $c < 0$ such that

$$v_n = cw_n, \qquad v_i = -|c|^{r_i} w_i \quad \text{for } 1 \leq i < 0. \tag{6.2}$$

Proof. Suppose there exists a positive real number c such that (6.1) is satisfied. Take x_i to be arbitrary for $1 \leq i < n$ and let $x_n = c$. It readily follows that $v = wx$. Now let y_i be arbitrary for $1 \leq i < n$ and let $y_n = 1/c$. It follows just easily that $w = vy$ and we conclude that $v\mathcal{R}w$. One shows, in the same manner, that $v\mathcal{R}w$ when (6.2) is satisfied.

Now suppose $v\mathcal{R}w$ and suppose further that $v \neq w$. Then $v = wx$ and $w = vy$ for some $x, y \in S$. Suppose $x_n = 0$. Then it follows from (2.3) that $v = \mathbf{0}$ and this, together with (2.3), implies that we also have $w = \mathbf{0}$. But this contradicts the fact that $v \neq w$. Thus $x_n \neq 0$. Take $c = x_n$. If $c > 0$, it follows from (2.3) that $v_n = cw_n$ and $v_i = (wx)_i = c^{r_i} w_i$ for $1 \leq i < n$. Consequently, (6.1) is satisfied. It follows in much the same way that if $c < 0$, then (6.2) is satisfied. □

Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, the following result is an immediate consequence of Theorems 6.1 and 6.2.

Theorem 6.3. *Let S be a Type III semigroup and let $v, w \in S$. Then $v\mathcal{H}w$ if and only if $v = w$ or $v_n \neq 0 \neq w_n$ and one of the following two conditions is satisfied:*

(i) *there exists a $c > 0$ such that $v_n = cw_n$, $v_i = c^{r_i} w_i$ for $1 \leq i < n$, or*
(ii) *there exists a $c < 0$ such that $v_n = cw_n$, $v_i = -|c|^{r_i} w_i$ for $1 \leq i < n$.*

Theorem 6.4. *An element v of a Type III semigroup is idempotent if and only if $v = \mathbf{0}$ or $v_n = 1$.*

Proof. It follows immediately from (2.3) that v is idempotent if either $v = \mathbf{0}$ or $v_n = 1$. Suppose, conversely, that v is idempotent. Then $v_n = v_n^2$ which means that either $v_n = 0$ or $v_n = 1$. If $v_n = 0$, it follows immediately from (2.3) and the fact that v is idempotent that $v = \mathbf{0}$.

As before $\mathcal{H}_e$ is the $\mathcal{H}$-class containing the idempotent e. □

Theorem 6.5. *Let S be a Type III semigroup and let e be an idempotent of S. Then $\mathcal{H}_e = \{\mathbf{0}\}$ if $e = \mathbf{0}$. If e is a nonzero idempotent of S, then $\mathcal{H}_e$ consists of all $v \in S$ such that $v_n > 0$ and $v_i = v_n^{r_i} e_i$ for $1 \le i < n$ together with all $v \in S$ such that $v_n < 0$ and $v_i = -|v_n|^{r_i} e_i$ for $1 \le i < n$. Moreover, if $e \ne \mathbf{0}$, Then $\mathcal{H}_e$ is isomorphic to $\mathbb{R}_M$, the multiplicative group of nonzero real numbers.*

Proof. It follows immediately from Theorem 6.3 that $\mathcal{H}_e = \{\mathbf{0}\}$ if $e = \mathbf{0}$. Suppose $e \ne \mathbf{0}$. Then $e_n = 1$ by Theorem 6.4. Suppose $v_n > 0$ and $v_i = v_n^{r_i}$ for $1 \le i < n$. Take $c = w_n$ and it follows from Theorem 6.3(i) that $v\mathcal{H}e$. Suppose, conversely, that $v\mathcal{H}e$. Then $v_n \ne 0$ by Theorem 6.3. Suppose $v_n > 0$. Take $v_n = c$. Then $v_n = c = ce_n$ and by (2.3), $(ev)_i = e_i v_i^{r_i} = c^{r_i} e_i$ for $1 \le i < n$. Thus Theorem 6.3(i) is satisfied. It follows in a similar manner that if $v\mathcal{H}e$ and $v_n < 0$, then Theorem 6.3(ii) is satisfied. Define a surjection φ from $\mathcal{H}_e$ onto $\mathbb{R}_M$ by $\varphi(v) = v_n$. It is immediate that φ is a homomorphism. Suppose $\varphi(v) = 1$. Then $v_i = e_i$ for all i. That is, $v = e$ and we conclude that φ is an isomorphism from $\mathcal{H}_e$ onto $\mathbb{R}_M$. □

Our next result is the analogue of Corollary 5.6.

Corollary 6.6. *Let S be a Type III semigroup. Then S is the union of its nonzero maximal subgroups, each of which is isomorphic to $\mathbb{R}_M$, together with a subsemigroup Z which has the property that $vw = \mathbf{0}$ for each $v \in S$ and each $w \in Z$. Moreover, the identity of each nonzero maximal subgroup of S is a right identity for all of S.*

Proof. Let $Z = \{v \in S : v_n = 0\}$. Suppose $v \notin Z$. Then $v_n \ne 0$. Define $e_n = 1$. If $v_n > 0$, define $e_i = v_i / v_n^{r_i}$ and it follows from Theorem 6.3(i) that $v \in \mathcal{H}_e$. If $v_n < 0$, define $e_i = -v_i / |v_n|^{r_i}$ and it follows from Theorem 6.3(ii) that $v \in \mathcal{H}_e$. Thus, we conclude that S is the union of its nonzero maximal subgroups, each of which is isomorphic to $\mathbb{R}_M$, together with the subsemigroup Z. It is immediate that $vw = \mathbf{0}$ for each $v \in S$ and each $w \in Z$ and that each identity of a nonzero maximal subgroup of S is a right identity for all of S. □

Lemma 6.7. *Let S be a Type III semigroup. Let $v, w \in S$ and suppose that $v \ne \mathbf{0}$. Then $v \in SwS$ if and only if $w_n \ne 0$.*

Proof. Suppose $v \in SwS$. Then $v = xwy$ for some $x, y \in S$. Then $v_n = x_n w_n y_n$ and $v_i = x_i (w_n y_n)^{r_i}$ for $1 \le i < n$ whenever $w_n y_n \ge 0$. Then $w_n \ne 0$ since $v \ne \mathbf{0}$. Of course, $w_n \ne 0$ in the remaining case where $w_n y_n < 0$.

Suppose, conversely, that $w_n \ne 0$. If $w_n > 0$, Take $y_n = 1$ and if $w_n < 0$, take $y_n = -1$. Take y_i to be arbitrary for $1 \le i < n$, $x_n = v_n / |w_n|$, and $x_i = v_i / |w_n|^{r_i}$ for $1 \le i < n$. It follows that $v = xwy$. □

Lemma 6.8. *Let S be a Type III semigroup, let $v,w \in S$ and suppose $v \neq \mathbf{0}$. Then $v \in wS$ if and only if there exists a positive number c such that $v_n = cw_n$ and*

$$v_i = c^{r_i} w_i \quad \text{for } 1 \leq i < n \tag{6.3}$$

or a negative number c such that $v_n = cw_n$ and

$$v_i = -|c|^{r_i} w_i \quad \text{for } 1 \leq i < n. \tag{6.4}$$

Proof. Suppose there exists a positive number c such that $v_i = c^{r_i} w_i$ for $1 \leq i < n$. Define $y_n = c$ and $y_i = v_i/c^{r_i}$ for $1 \leq i < n$. Then $v = wy \in wS$. Now suppose there exists a negative number c such that $v_n = cw_n$ and $v_i = -|c|^{r_i} w_i$ for $1 \leq i < n$. Again, let $y_n = c$ but this time let $y_i = -v_i/|c|^{r_i}$ for $1 \leq i < n$. In this case also we have $v = wy \in wS$.

Now suppose $v \neq \mathbf{0}$ and $v \in wS$. Then $v = wy$ for some $y \in S$. Suppose $y_n = 0$. Then $v_n = w_n y_n = 0$ and $v_i = w_i (y_n)^{r_i} = 0$ for $1 \leq i < n$. But this contradicts the fact that $v \neq \mathbf{0}$. Thus, $y_n \neq 0$. Take $c = y_n$. Condition (6.3) is satisfied if $c > 0$ and condition (6.4) is satisfied if $c < 0$. □

Theorem 6.9. *Let S be a Type III nearring and let $v,w \in S$. Then $v \mathcal{J} w$ if and only if $v = w$ or one of the following three conditions is satisfied:*

(i) $v_n \neq 0 \neq w_n$,

(ii) *there exists a positive number c such that $v_n = cw_n$, $v_i = c^{r_i} w_i$ for $1 \leq i < n$, or*

(iii) *there exists a negative number c such that $v_n = cw_n$, $v_i = -|c|^{r_i} w_i$ for $1 \leq i < n$.*

Proof. If (i) holds, then $v \mathcal{L} w$ by Theorem 6.1 and thus, we have $v \mathcal{J} w$ as well. Suppose (ii) holds. Then $v \in wS$ by Lemma 6.8. Take $b = 1/c$. Then $w_n = bv_n$ and $w_i = b^{r_i} v_i$ and it follows from Lemma 6.8 that $w \in vS$. Consequently, $v \mathcal{R} w$ which implies $v \mathcal{J} w$. It follows in much the same manner that $v \mathcal{J} w$ if (iii) is satisfied.

Now suppose that $v \mathcal{J} w$ and suppose further that $v \neq w$. Then either $v \neq \mathbf{0}$ or $w \neq \mathbf{0}$ and there is no loss of generality if we assume that $v \neq \mathbf{0}$. Again, we use the fact that $S^1 w S^1 = S^1 w S$ to conclude that $v \in S^1 w S$. Thus $v = xwy$ where $y \in S$.

Case 1 ($x \in S$). Then $v \in SwS$ which means $w_n \neq 0$ by Lemma 6.7. Since $w \in S^1 v S$, we have either $w = xvy$ or $w = vy$ where $x, y \in S$. In the former case, $w_n = x_n v_n y_n$ and in the latter case, $w_n = v_n y_n$. In either case, $v_n \neq 0$ since $w_n \neq 0$ and we conclude that (i) is satisfied.

Case 2 ($x \notin S$). then $x = 1$ and $v = wy \in wS$ and it follows immediately from Lemma 6.8 that (ii) holds or (iii) holds. This completes the proof. □

7. The multiplicative semigroups of Type IV nearrings. In the case of Type IV nearrings, $a^2 + b^2 \neq 0$ so we have three cases to consider: (1) $a = 0$, (2) $b = 0$, and (3) $a \neq 0 \neq b$. Of course, whenever $a = 0$, $b \neq 0$ and the multiplication (2.4) becomes

$$(vw)_i = \begin{cases} 0 & \text{for } w_n \leq 0, \\ v_i(bw_n)^{r_i} & \text{for } w_n > 0, \end{cases} \tag{7.1}$$

for $i \neq n$,

$$(vw)_n = \begin{cases} 0 & \text{for } w_n \leq 0, \\ bv_n w_n & \text{for } w_n > 0, \end{cases} \tag{7.2}$$

where $r_i > 0$ and $b > 0$. Similarly, when $b = 0$, then $a \neq 0$ and the multiplication (2.4) becomes

$$(vw)_i = \begin{cases} v_i(aw_n)^{r_i} & \text{for } w_n < 0, \\ 0 & \text{for } w_n \geq 0, \end{cases} \tag{7.3}$$

for $i \neq n$,

$$(vw)_n = \begin{cases} av_n w_n & \text{for } w_n < 0, \\ 0 & \text{for } w_n \geq 0, \end{cases} \tag{7.4}$$

where $r_i > 0$ and $a < 0$.

In particular, $vw = \mathbf{0}$ whenever $a = 0$ and $w_n \leq 0$ and $vw = \mathbf{0}$ whenever $b = 0$ and $w_n \geq 0$.

Theorem 7.1. *Let S be a Type IV semigroup and let $v, w \in S$. If $a = 0$, then $v\mathcal{L}w$ if and only if $v = w$ or $v_n, w_n > 0$. If $b = 0$, then $v\mathcal{L}w$ if and only if $v = w$ or $v_n, w_n < 0$ and if $a \neq 0 \neq b$, then $v\mathcal{L}w$ if and only if $v = w$ or $v_n \neq 0 \neq w_n$.*

Proof. We consider first the case where $a = 0$. Suppose $v\mathcal{L}w$ and suppose further that $v \neq w$. Then, either $v \neq \mathbf{0}$ or $w \neq \mathbf{0}$ and there is no loss of generality if we assume $v \neq \mathbf{0}$. We have $v = xw$ and $w = yv$ for $x, y \in S$. Since $v \neq \mathbf{0}$, it follows from (7.1) that $w_n > 0$. Then $w \neq \mathbf{0}$ and since $w = yv$, it follows from (7.1) that $v_n > 0$. Suppose, conversely, that $v_n, w_n > 0$. Define $x_n = v_n/bw_n$ and $x_i = v_i/(bw_n)^{r_i}$ for $1 \leq i < n$. Then $v = xw \in S^1w$. In a similar manner, one produces a $y \in S$ such that $w = yv$ and we conclude that $v\mathcal{L}w$. The case where $b = 0$ is similar so no details will be given.

Now consider the case where $a \neq 0 \neq b$. Suppose $v\mathcal{L}w$ and suppose further that $v \neq w$. Then, either $v \neq \mathbf{0}$ or $w \neq \mathbf{0}$ and, again, there is no loss of generality if we assume $v \neq \mathbf{0}$. Here also we have $v = xw$ for some $x \in S$ and since $v \neq \mathbf{0}$, we must have $w_n \neq 0$. Thus $w \neq \mathbf{0}$ and since $w = yv$ for some $y \in S$, we conclude that $v_n \neq 0$. Suppose, conversely, that $v_n \neq 0 \neq w_n$. If $w_n > 0$, define $x_n = v_n/bw_n$ and $x_i = v_i/(bw_n)^{r_i}$ for $1 \leq i < n$. If $w_n < 0$, define $x_n = v_n/aw_n$ and $x_i = v_i/(aw_n)^{r_i}$ for $1 \leq i < n$. In either event, $v = xw \in S^1w$. In a similar manner, one shows that $w \in S^1 v$ which means $v\mathcal{L}w$ and the proof is complete. □

Let $N(v) = \{i : v_i \neq 0\}$ for $v \in \mathbb{R}^n$.

Theorem 7.2. *Let S be a Type IV semigroup and let $v, w \in S$. Then $v\mathcal{R}w$ if and only if $v = w$ or $N(v) = N(w)$, and $v_i w_i \geq 0$ for $1 \leq i \leq n$ and either*

$$v_n = 0 = w_n, \qquad \left(\frac{v_i}{w_i}\right)^{1/r_i} = \left(\frac{v_j}{w_j}\right)^{1/r_j} \quad \forall i, j \in N(v)\backslash\{n\} \tag{7.5}$$

or

$$v_n \neq 0 \neq w_n, \qquad \left(\frac{v_i}{w_i}\right)^{1/r_i} = \frac{v_n}{w_n} \quad \forall i \in N(v)\backslash\{n\}. \tag{7.6}$$

Proof. Consider first the case where $a \neq 0 \neq b$. Suppose $N(v) = N(w)$ and $v_i w_i \geq 0$ for $1 \leq i \leq n$ and suppose (7.5) holds. If $N(v) = \emptyset$, then $v = \mathbf{0} = w$ so we need only to consider the case where $N(v) \neq \emptyset$. Choose any $k \in N(v)$ and define $x_n = (1/b)(v_k/w_k)^{1/r_k}$. Let x_i be arbitrary for $1 \leq i < n$. It follows from (7.5) that

$$\left(\frac{v_i}{w_i}\right)^{1/r_i} = \left(\frac{v_k}{w_k}\right)^{1/r_k} = bx_n \tag{7.7}$$

for all $i \in N(v)$. It follows readily from (7.7) that $v_i = w_i(bx_n)^{r_i}$ for all $i \in N(v)$. Since $v_i = 0 = w_i$ for $i \notin N(v)$, we conclude that $v_i = w_i(bx_n)^{r_i}$ for $1 \leq i < n$ and, of course, $v_n = 0 = bv_n w_n$. This implies that $v = wx$. In a similar manner, one produces a $y \in S$ such that $w = vy$ and we conclude that $v\mathcal{R}w$. Now suppose that (7.6) holds. Then $v_n/bw_n > 0$. Define $x_n = v_n/bw_n$ and take x_i for $1 \leq i < n$ to be arbitrary. It then follows from (7.6) that

$$\frac{v_i}{w_i} = \left(\frac{v_n}{w_n}\right)^{r_i} = (bx_n)^{r_i} \tag{7.8}$$

for all $i \in N(v)$. Then $v_n = bw_n$, $v_i = w_i(bx_n)^{r_i}$ for $i \in N(v)$ in view of (7.8). Since $v_i = 0 = bw_i x_i$ for $i \notin N(v)$, it readily follows that $v = wx$. One shows, in a similar manner, that $w = vy$ for some $y \in S$ and we conclude that $v\mathcal{R}w$.

Suppose, conversely, that $v\mathcal{R}w$ and suppose further that $v \neq w$. Then either $v \neq \mathbf{0}$ or $w \neq \mathbf{0}$ and there is no loss of generality if we assume that $v \neq \mathbf{0}$. Now $v = wx$ for some $x \in S$ and since $v \neq \mathbf{0}$, we must have $x_n \neq 0$. Consider the case where $x_n < 0$. Then $v_n = aw_n x_n$. Since $ax_n > 0$, it readily follows that $v_n w_n \geq 0$ and $v_n = 0$ if and only if $w_n = 0$. Suppose $v_n = 0 = w_n$. Since $v_i = w_i(ax_n)^{r_i}$ for $1 \leq i < n$ and it readily follows that $v_i = 0$ if and only if $w_i = 0$, thus, $N(v) = N(w)$ and for all $i \in N(v) = N(w)$, we have

$$\left(\frac{v_i}{w_i}\right)^{1/r_i} = ax_n \tag{7.9}$$

and it follows that (7.5) is satisfied in this case. It follows in much the same manner that (7.5) is satisfied whenever $v_n = 0 = w_n$ and $x_n > 0$. It remains to consider the case where $v_n \neq 0 \neq w_n$. We again have $v = wx$ and $x_n \neq 0$ since $v \neq \mathbf{0}$ and we consider two subcases.

Subcase 1 ($x_n < 0$). Then $v_n = aw_nx_n$ and

$$v_i = (wx)_i = w_i(ax_n)^{r_i} = w_i\left(\frac{v_n}{w_n}\right)^{r_i} \quad \text{for } 1 \leq i < n. \tag{7.10}$$

It follows from (7.10) that (7.6) is satisfied in this case.

Subcase 2 ($x_n > 0$). We have $v_n = bw_nx_n$ and

$$v_i = (wx)_i = w_i(bx_n)^{r_i} = w_i\left(\frac{v_n}{w_n}\right)^{r_i} \quad \text{for } 1 \leq i < n \tag{7.11}$$

and it follows that (7.6) is satisfied in this case also.

The case where $a = 0$ and the case where $b = 0$ both differ somewhat from the previous case but since they are similar, we give the details in the latter case only. So we consider the case where $b = 0$. Suppose that $N(v) = N(w)$, $v_iw_i \geq 0$ for $1 \leq i < n$ and suppose further that (7.5) holds. Again, if $N(v) = \emptyset$, we have $v = \mathbf{0} = w$ so we need only to consider the case where $N(v) \neq \emptyset$ and we choose any $k \in N(v)$. Then $v_kw_k > 0$. We define $x_n = (1/a)(v_k/w_k)^{1/r_k}$ and we take x_i to be arbitrary for $1 \leq i < n$. In view of (7.5), for any $i \in N(v)\backslash\{n\}$, we have

$$\left(\frac{v_i}{w_i}\right)^{1/r_i} = \left(\frac{v_k}{w_k}\right)^{1/r_k} = ax_n. \tag{7.12}$$

It follows from (7.12) that $v_i = w_i(ax_n)^{r_i}$ for all $i \in N(v)\backslash\{n\}$ and since $v_n = 0 = aw_nx_n$ and $v_i = 0 = w_i(ax_n)^{r_i}$ for $i < n$ and $i \notin N(v)$, we conclude that $v = wx$. In the same manner, one can produce a $y \in S$ such that $w = vy$ and we conclude that $v\mathcal{R}w$ whenever (7.5) is satisfied.

Suppose (7.6) is satisfied. Then $v_nw_n > 0$ and this time we define $x_n = v_n/aw_n$ and we take x_i to be arbitrary for $1 \leq i < n$. Note that $x_n < 0$. Evidently, $v_n/w_n = ax_n$ and it follows from (7.6) that

$$\left(\frac{v_i}{w_i}\right)^{1/r_i} = \frac{v_n}{w_n} = ax_n \quad \forall i \in N(v)\backslash\{n\}. \tag{7.13}$$

It follows from (7.13) that $v_i = w_i(ax_i)^{r_i}$ for $i \in N(v)\backslash\{n\}$. Since $v_n = aw_nx_n$ and $v_i = 0 = w_i(ax_n)^{r_i}$ for $i \notin N(v)$, we conclude that $v = wx$. Similarly, $w = vy$ for some $y \in S$ and we conclude that $v\mathcal{R}w$.

Suppose, conversely, that $v\mathcal{R}w$ and suppose further that $v \neq w$. Then, not both v and w can be $\mathbf{0}$ and there is no loss of generality if we assume $v \neq \mathbf{0}$. It also follows from our assumption that $v \neq w$ that $v = wx$ and $w = vy$ for appropriate $x, y \in S$. Then $x_n < 0$ since $v \neq \mathbf{0}$. Since $v_n = aw_nx_n$ and $v_i = w_i(ax_n)^{r_i}$ for $1 \leq i < n$, we see that $v_i = 0$ if and only if $w_i = 0$ for

$1 \le i \le n$. Thus, $N(v) = N(w)$. It follows from the latter assertion that $w \ne \mathbf{0}$. Since $ax_n > 0$, we conclude that $v_i w_i \ge 0$ for $1 \le i \le n$. Suppose $v_n = 0$. Then $w_n = 0$ as well and since $v_i = w_i (ax_n)^{r_i}$ for $1 \le i < n$, we conclude that

$$\left(\frac{v_i}{w_i}\right)^{1/r_i} = ax_n \quad \forall i \in N(v)\backslash\{n\}, \tag{7.14}$$

and it follows from (7.14) that (7.5) holds in this case. Now suppose $v_n \ne 0$. Then $w_n \ne 0$ since $N(v) = N(w)$. In this case, we have $v_n = aw_n x_n$ and since $v_i = w_i (x_n)^{r_i}$ for $1 \le i < n$, we conclude that

$$\left(\frac{v_i}{w_i}\right)^{1/r_i} = ax_n = \frac{v_n}{w_n} \quad \forall i \in N(v)\backslash\{n\}. \tag{7.15}$$

Thus, (7.6) holds in this case. □

Our next result is an immediate consequence of Theorems 7.1 and 7.2.

Theorem 7.3. *Suppose S is a Type IV semigroup. Suppose $a = 0$ and suppose $v, w \in S$. Then $v\mathcal{H}w$ if and only if $v = w$ or $v_n, w_n > 0$, $N(v) = N(w)$, $v_i w_i \ge 0$ for $1 \le i < n$ and*

$$\left(\frac{v_i}{w_i}\right)^{1/r_i} = \frac{v_n}{w_n} \quad \forall i \in N(v)\backslash\{n\}. \tag{7.16}$$

Theorem 7.4. *Suppose S is a Type IV semigroup and suppose $a = 0$. Then $v \in S$ is a nonzero idempotent of S if and only if $v_n = 1/b$.*

Proof. If $v_n = 1/b$, then $(vv)_n = bv_n v_n = v_n$ and $(vv)_i = v_i (bv_n)^{r_i} = v_i$ for $1 \le i < n$ and we see that v is a nonzero idempotent. Suppose, conversely, that v is a nonzero idempotent. If $v_n \le 0$, it follows from (7.1) that $v^2 = \mathbf{0}$ which contradicts the fact that v is idempotent. Thus $v_n > 0$ and from (7.1) we conclude that $v_n = (vv)_n = bv_n^2$ which implies that $v_n = 1/b$. □

In what follows, we will denote by $\mathbb{R}_M^+$ the multiplicative group of positive real numbers.

Theorem 7.5. *Let e be a nonzero idempotent of a Type IV semigroup where $a = 0$. Then $v \in \mathcal{H}_e$ if and only if $v_n > 0$, $N(v) = N(e)$, $v_i e_i \ge 0$ for $1 \le i < n$, and*

$$\left(\frac{v_i}{e_i}\right)^{1/r_i} = bv_n \quad \forall i \in N(v)\backslash\{n\}. \tag{7.17}$$

Furthermore, $\mathcal{H}_e$ is isomorphic to $\mathbb{R}_M^+$ for each nonzero idempotent e and S is the union of its nonzero maximal subgroups, each of which is isomorphic to $\mathbb{R}_M^+$, together with a subsemigroup T with the property that $vw = \mathbf{0}$ for all $v \in S$ and $w \in T$. Finally, each nonzero idempotent of S is a right identity for S.

Proof. The fact that $\mathcal{H}_e$ consists of all $v \in S$ such that $v_n > 0$, $N(v) = N(e)$, $v_i e_i \ge 0$ for $1 \le i \le n$ and satisfies (7.17) is an immediate consequence of

Theorems 7.3 and 7.4. Define a surjection φ from $\mathcal{H}_e$ onto $\mathbb{R}_M^+$ by $\varphi(v) = bv_n$. Then $\varphi(vw) = b(vw)_n = b^2 v_n w_n = \varphi(v)\varphi(w)$ and we see that φ is an epimorphism. Suppose $\varphi(v) = 1$. Then, we must have $v_n = 1/b$ which means $v_n = e_n$ and it follows from (7.17) that $v_i = e_i$ for $1 \le i < n$ as well. Thus $v = e$ and we conclude that φ is an isomorphism from $\mathcal{H}_e$ onto $\mathbb{R}_M^+$. Note that for any nonzero idempotent e and any $v \in S$ we have $(ve)_n = bv_n e_n = v_n$ and $(ve)_i = v_i(be_n)^{r_i} = v_i$ for $1 \le i < n$ so that e is a right identity for S. Finally, let $T = \{v \in S : v_n \le 0\}$, it follows from (7.1) that $vw = \mathbf{0}$ for all $v \in S$ and $w \in T$. Now suppose $v \in S \backslash T$. Then $v_n > 0$. Define $e_n = 1/b$ and $e_i = v_i/(bv_n)^{r_i}$. Then it follows that e is a nonzero idempotent and it follows from our previous considerations that $v \in \mathcal{H}_e$. Consequently, we conclude that S is the union of its nonzero maximal subgroups, each of which is isomorphic to $\mathbb{R}_M^+$, together with the subsemigroup T and the proof is now complete. □

Our next result is an immediate consequence of Theorems 7.1 and 7.2.

Theorem 7.6. *Suppose S is a Type IV semigroup. Suppose $b = 0$ and $v, w \in S$. Then $v\mathcal{H}w$ if and only if $v = w$ or $v_n, w_n < 0$, $N(v) = N(w)$, $v_i w_i \ge 0$ for $1 \le i \le n$, and*

$$\left(\frac{v_i}{w_i}\right)^{1/r_i} = \frac{v_n}{w_n} \quad \forall i \in N(v)\backslash\{n\}. \tag{7.18}$$

The proofs of our next two results are quite similar to those of Theorems 7.4 and 7.5, respectively, and, for that reason, will not be given.

Theorem 7.7. *Suppose S is a Type IV semigroup and $b = 0$. Then $v \in S$ is a nonzero idempotent of S if and only if $v_n = 1/a$.*

Theorem 7.8. *Let e be a nonzero idempotent of a Type IV semigroup where $b = 0$. Then $v \in \mathcal{H}_e$ if and only if $v_n < 0$, $N(v) = N(e)$, $v_i e_i \ge 0$ for $1 \le i \le n$, and $v_i = e_i(av_n)r_i$ for all $i \in N(v)\backslash\{n\}$. Furthermore, $\mathcal{H}_e$ is isomorphic to $\mathbb{R}_M^+$ for each nonzero idempotent e and S is the union of its nonzero maximal subgroups, each of which is isomorphic to $\mathbb{R}_M^+$, together with a subsemigroup T with the property that $vw = \mathbf{0}$ for all $v \in S$ and $w \in T$. Finally, each nonzero idempotent of S is a right identity for S.*

The proof of our next result is straightforward and will also be omitted.

Theorem 7.9. *Suppose S is a Type IV semigroup with $a \ne 0 \ne b$. Then $v \in S$ is a nonzero idempotent if and only if either $v_n = 1/a$ or $v_n = 1/b$.*

The next result follows immediately from Theorems 7.1 and 7.2.

Theorem 7.10. *Suppose S is a Type IV semigroup with $a \ne 0 \ne b$ and let $v, w \in S$. Then $v\mathcal{H}w$ if and only if $v = w$ or $v_n \ne 0 \ne w_n$, $N(v) = N(w)$, $v_i w_i \ge 0$ for $1 \le i \le n$, and*

$$\left(\frac{v_i}{w_i}\right)^{1/r_i} = \frac{v_n}{w_n} \quad \forall i \in N(v)\backslash\{n\}. \tag{7.19}$$

Theorem 7.11. *Let e be a nonzero idempotent of a Type IV semigroup where $a \neq 0 \neq b$. If $e_n = 1/a$, then $v \in \mathcal{H}_e$ if and only if $v_n < 0$, $N(v) = N(e)$, $v_i e_i \geq 0$ for $1 \leq i \leq n$, and $v_i = e_i(av_n)^{r_i}$ for all $i \in N(v)\backslash\{n\}$. If $e_n = 1/b$, then $v \in \mathcal{H}_e$ if and only if $v_n > 0$, $N(v) = N(e)$, $v_i e_i \geq 0$ for $1 \leq i \leq n$, and $v_i = e_i(bv_n)^{r_i}$ for all $i \in N(v)\backslash\{n\}$. Furthermore, $\mathcal{H}_e$ is isomorphic to $\mathbb{R}_M^+$ for each nonzero idempotent e and S is the union of its nonzero maximal subgroups, each of which is isomorphic to $\mathbb{R}_M^+$, together with a subsemigroup T with the property that $vw = \mathbf{0}$ for all $v \in S$ and $w \in T$. Finally, each nonzero idempotent of S is a right identity for S.*

Proof. The proof of this result is quite similar to the proof of Theorem 7.5 so we will omit most of the details. A few remarks, however, are appropriate. If $e_n = 1/a$, the map φ defined by $\varphi(v) = av_n$ is an isomorphism from $\mathcal{H}_e$ onto $\mathbb{R}_M^+$ and if $e_n = 1/b$, then the map φ defined by $\varphi(v) = bv_n$ is an isomorphism from $\mathcal{H}_e$ onto $\mathbb{R}_M^+$. Finally, let $T = \{v \in S : v_n = 0\}$. Then $vw = \mathbf{0}$ for all $v \in S$ and $w \in T$ and S is the union of T, together with all the nonzero maximal subgroups of S. □

We are now in a position to prove a result mentioned in Section 3.

Theorem 7.12. *If two linear right ideal nearrings are isomorphic, then they must be of the same type.*

Proof. We observed, following the proof of Theorem 4.4, that the maximal subgroups of a Type I semigroup are all singletons. Theorem 5.5 assures that the nonzero maximal subgroups of a Type II semigroup are all isomorphic to $\mathbb{R}_M$ and Theorem 6.5 assures that the nonzero maximal subgroups of a Type III semigroup are all isomorphic to $\mathbb{R}_M$. Theorem 7.11 tells us that the nonzero maximal subgroups of a Type IV semigroup are all isomorphic to $\mathbb{R}_M^+$. Since a group of order one, $\mathbb{R}_M$, and $\mathbb{R}_M^+$ are all mutually nonisomorphic, the only possibility for a nearring of one type to be isomorphic to a nearring of another type is for a Type II nearring to be isomorphic to a Type III nearring. Let $\mathcal{N}_2$ and $\mathcal{N}_3$ be a Type II and a Type III nearrings, respectively. Let w be any element of $\mathcal{N}_3$ such that $w_n = -1$. One easily verifies that $vw = -v$ for all $v \in \mathcal{N}_3$. That is, vw is the additive inverse of v for all $v \in \mathcal{N}_3$. It is easily verified that $\mathcal{N}_2$ contains no such element w. Consequently, $\mathcal{N}_2$ and $\mathcal{N}_3$ cannot be isomorphic. □

Theorem 7.13. *In a Type IV semigroup, $\mathcal{D} = \mathcal{L} \cup \mathcal{R}$.*

Proof. The proofs of the three cases, $a = 0$, $b = 0$, and $a \neq 0 \neq b$ are all quite similar so we give the details in the case $a = 0$ only. Evidently, $\mathcal{L} \cup \mathcal{R} \subseteq \mathcal{D}$ so we must verify that $\mathcal{D} \subseteq \mathcal{L} \cup \mathcal{R}$. Suppose $v\mathcal{D}w$. Then $v\mathcal{L}u$ and $u\mathcal{R}w$ for some $u \in S$. If $v = u$, then $(v,w) \in \mathcal{R}$. If $v \neq u$, then $v_n, u_n > 0$ by Theorem 7.1 and since $N(u) = N(w)$ and $u_i w_i \geq 0$ for $1 \leq i \leq n$, it follows from Theorem 7.2 that $w_n > 0$. Thus, $v_n, w_n > 0$ and now it follows from Theorem 7.1 that $(v,w) \in \mathcal{L}$. □

Theorem 7.14. *Let S be a Type IV semigroup and suppose $v, w \in S$ with $v \neq \mathbf{0}$. If $a = 0$, then $v \in SwS$ if and only if $w_n > 0$. If $b = 0$, then $v \in SwS$ if and only if $w_n < 0$, and if $a \neq 0 \neq b$, then $v \in SwS$ if and only if $w_n \neq 0$.*

Proof. The proofs of the three cases are similar. We give the details in the latter case. Suppose $v \in SwS$. Then $v = xwy$ for appropriate $x, y \in S$. Suppose $(wy)_n = 0$. Then $v_n = ax_n(wy)_n = 0$ and $v_i = x_i(a(wy)_n)^{r_i} = 0$ for $1 \leq i \leq n$. This, of course, contradicts the fact that $v \neq \mathbf{0}$. Thus, we conclude that $(wy)_n \neq 0$. Now $(wy)_n = aw_ny_n$ if $y_n < 0$ and $(wy)_n = bw_ny_n$ if $y_n > 0$. In either event, we must have $w_n \neq 0$ since $(wy)_n \neq 0$.

Suppose, conversely, that $w_n \neq 0$. Take $y_n = 1$ and take y_i to be arbitrary for $1 \leq i < n$. If $w_n < 0$, define $x_n = v_n/abw_n$ and $x_i = v_i/(abw_n)^{r_i}$ for $1 \leq i < n$. If $w_n > 0$, define $x_n = v_n/b^2w_n$ and $x_i = v_i/(b^2w_n)^{r_i}$ for $1 \leq i < n$. In either event, $v = xwy \in SwS$. □

Theorem 7.15. *Let S be a Type IV semigroup, suppose $v, w \in S$ and suppose further that $v \neq \mathbf{0}$. Then $v \in wS$ if and only if $N(v) = N(w)$, $v_iw_i \geq 0$ for $1 \leq i \leq n$, and there exists a positive real number c such that*

$$v_n = cw_n, \qquad \left(\frac{v_i}{w_i}\right)^{1/r_i} = c \quad \forall i \in N(v)\backslash\{n\}. \tag{7.20}$$

Proof. As in previous results, there are three cases to consider: $a = 0$, $b = 0$, and $a \neq 0 \neq b$. We give the details only in the latter case. Suppose $v \in wS$. Then $v = wy$ for some $y \in S$. Now, $y_n \neq 0$ since $v \neq \mathbf{0}$.

Case 1 ($y_n < 0$). Let $c = ay_n$. Then $c > 0$ and it follows from (2.4) that $v_n = (wy)_n = aw_ny_n = cw_n$ and $v_i = (wy)_i = w_i(ay_n)^{r_i} = w_ic^{r_i}$ for $1 \leq i < n$. Thus, $v_i \neq 0$ if and only if $w_i \neq 0$ for $1 \leq i \leq n$ since $c^{r_i} \neq 0$ for $1 \leq i < n$ and we conclude that $N(v) = N(w)$. Moreover, since $c > 0$, it also follows that $v_iw_i \geq 0$ for $1 \leq i \leq n$. Finally, it readily follows that $(v_i/w_i)^{1/r_i} = c$ for all $i \in N(v)\backslash\{n\}$.

Case 2 ($y_n > 0$). Let $c = by_n > 0$. Then $c > 0$ and it follows from (2.4) that $v_n = (wy)_n = bw_ny_n = cw_n$ and $v_i = (wy)_i = w_i(by_n)^{r_i} = w_ib^{r_i}$ for $1 \leq i < n$. Thus $v_i \neq 0$ if and only if $w_i \neq 0$ for $1 \leq i \leq n$ since $c^{r_i} \neq 0$ for $1 \leq i < n$ and we conclude that $N(v) = N(w)$. Moreover, since $c > 0$, it also follows that $v_iw_i \geq 0$ for $1 \leq i \leq n$. Again, it readily follows that $(v_i/w_i)^{1/r_i} = c$ for all $i \in N(v)\backslash\{n\}$.

Suppose, conversely, that $N(v) = N(w)$, $v_iw_i \geq 0$ for $1 \leq i \leq n$, and there exists a positive real number c such that (7.20) is satisfied. Define $y_n = c/b$ and take y_i to be arbitrary for $1 \leq i < n$. Then $y_n > 0$ and $v_n = cw_n = bw_ny_n$. In addition, we have $v_i = w_ic^{r_i} = w_i(by_n)^{r_i}$ for $i \in N(v)\backslash\{n\}$. Since $v_i = 0 = w_i(bx_n)^{r_n}$ for $i \notin N(v)\backslash\{n\}$, we conclude that $v_i = w_ic^{r_i} = w_i(by_n)^{r_n}$ for $1 \leq i < n$. Thus, $v = wy \in wS$. □

Theorem 7.16. *Let S be a Type IV semigroup with $a \neq 0 \neq b$ and let $v, w \in S$. Then $v\mathcal{J}w$ if and only if $v = w$ or $v_n \neq 0 \neq w_n$ or $N(v) = N(w) \neq \emptyset$, $v_iw_i \geq 0$*

for $1 \le i < n$*, and there exists a positive real number* c *such that*

$$v_n = cw_n, \qquad \left(\frac{v_i}{w_i}\right)^{1/r_i} = c \quad \forall i \in N(v)\backslash\{n\}. \tag{7.21}$$

Proof. Suppose $v \ne w$ and $v\mathcal{J}w$. Since S has right identities, it follows that $S^1uS^1 = S^1uS$ for all $u \in S$. Thus $S^1vS = S^1wS$ and thus, $v \in S^1wS$ which means that $v = xwy$ where $x \in S^1$ and $y \in S$.

Case 1 ($x \ne 1$)**.** Then $x \in S$ and $v \in SwS$. It follows from Theorem 7.14 that $w_n \ne 0$. Now $w \in S^1vS$. If $w \in vS$, it follows from Theorem 7.15 that $v_n = cw_n \ne 0$. If $w \in SvS$, it follows from Theorem 7.14 that $v_n \ne 0 \ne w_n$. In any event, we have $v_n \ne 0 \ne w_n$ in the case where $x \ne 1$.

Case 2 ($x = 1$)**.** It follows from Theorem 7.15 that $N(v) = N(w)$, $v_iw_i \ge 0$ for $1 \le i \le n$, and there exists a positive real number c such that $v_n = cw_n$ and $(v_i/w_i)^{1/r_i} = c$ for all $i \in N(v)\backslash\{n\}$.

Now suppose $v \ne w$. If $v_n \ne 0 \ne w_n$, it follows from Theorem 7.16 that $SvS = SwS$ which implies that $v\mathcal{J}w$. Now consider the case where $N(v) = N(w)$, $v_iw_i \ge 0$ for $1 \le i \le n$, and there exists a positive real number c such that $v_n = cw_n$ and $(v_i/w_i)^{1/r_i} = c$ for all $i \in N(v)\backslash\{n\}$. It follows from Theorem 7.15 that $v \in wS$. Since $w_n = (1/c)v_n$ and $(w_i/v_i)^{1/r_i} = 1/c$ for all $i \in N(v)\backslash\{n\}$, we conclude that $w \in vS$. Thus, $v\mathcal{J}w$ and the theorem is proved. □

The proofs of our two closing results are similar to the proof of the previous theorem and, for that reason, will be omitted.

Theorem 7.17. *Let* S *be a Type IV semigroup with* $a = 0$ *and let* $v, w \in S$*. Then* $v\mathcal{J}w$ *if and only if* $v = w$ *or* $v_n, w_n > 0$ *or* $N(v) = N(w) \ne \varnothing$*,* $v_iw_i \ge 0$ *for* $1 \le i < n$*, and there exists a positive real number* c *such that*

$$v_n = cw_n, \qquad \left(\frac{v_i}{w_i}\right)^{1/r_i} = c \quad \forall i \in N(v)\backslash\{n\}. \tag{7.22}$$

Theorem 7.18. *Let* S *be a Type IV semigroup with* $b = 0$ *and let* $v, w \in S$*. Then* $v\mathcal{J}w$ *if and only if* $v = w$ *or* $v_n, w_n < 0$ *or* $N(v) = N(w) \ne \varnothing$*,* $v_iw_i \ge 0$ *for* $1 \le i < n$*, and there exists a positive real number* c *such that*

$$v_n = cw_n, \qquad \left(\frac{v_i}{w_i}\right)^{1/r_i} = c \quad \forall i \in N(v)\backslash\{n\}. \tag{7.23}$$

References

[1] J. R. Clay, *Nearrings. Geneses and Applications*, Oxford Science Publications, Clarendon Press, Oxford University Press, New York, 1992.

[2] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups. Vol. I*, Mathematical Surveys, vol. 7, American Mathematical Society, Rhode Island, 1961.

[3] K. D. Magill Jr., *Linear right ideal nearrings*, Int. J. Math. Math. Sci. **27** (2001), no. 11, 663–674.

[4] J. D. P. Meldrum, *Near-Rings and Their Links with Groups*, Research Notes in Mathematics, vol. 134, Pitman, Massachusetts, 1985.

[5] G. Pilz, *Near-Rings*, revised ed., North-Holland Mathematics Studies, vol. 23, North-Holland Publishing, Amsterdam, 1983.

K. D. Magill Jr.: Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260-2900, USA

E-mail address: kdmagill@acsu.buffalo.edu