IJMMS 2003:13, 817–825
PII. S0161171203207109
http://ijmms.hindawi.com

SECTIONAL REPRESENTATION OF BANACH MODULES AND THEIR MULTIPLIERS

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Received 9 July 2002

Let X be a Banach module over the commutative Banach algebra A with maximal ideal space Δ. We show that there is a norm-decreasing representation of X, as a space of bounded sections in a Banach bundle $\pi : \mathscr{E} \to \Delta$, whose fibers are quotient modules of X. There is also a representation of $M(X)$, the space of multipliers $T : A \to X$, as a space of sections in the same bundle, but this representation may not be continuous. These sectional representations subsume results of various authors over the past three decades.

2000 Mathematics Subject Classification: 46H25, 46M20.

In this paper, A will always be a commutative Banach algebra. Denote by $\Delta = \Delta_A$ the space of multiplicative functionals on A, and for $h \in \Delta$, let $K_h = \ker h \subset A$ be the corresponding maximal ideal. We give Δ its weak-$*$ topology. Let X be a Banach A-module and for $h \in \Delta$, let $K_h X$ be the closure in X of $\operatorname{span}\{ax : a \in K_h, x \in X\}$. As usual, $C_0(\Delta)$ is the space of continuous complex-valued functions on Δ which vanish at infinity and $\hat{}: A \to C_0(\Delta)$ is the Gelfand representation of A. Following the notation of Takahasi [9], if $h \in \Delta$, we choose $e_h \in A$ such that $\widehat{e_h}(h) = h(e_h) = 1$, and we let X^h be the closure in X of $K_h X + (1 - e_h)X$; it is easy to show that X^h is independent of the choice of e_h. We set $X_h = X/X^h$. Denote by X_e the essential part of X, that is, X_e is the closed span of $\{ax : a \in A, x \in X\}$. If $X = X_e$, then $X^h = K_h X$. We denote by $M(X)$ the space of continuous multipliers $T : A \to X$, that is, the space of continuous A-module homomorphisms from A to X. (So, if $T \in M(X)$, then $T(ab) = aT(b)$ for all $a, b \in A$.) If $x \in X$, denote by T_x the multiplier defined by $T_x(a) = ax$.

We refer the reader to [1, 2, 3] for fundamental notions regarding bundles of Banach spaces and Banach modules. If $\pi : \mathscr{E} \to \Delta$ is a Banach bundle, we denote by $\mathscr{C}(\mathscr{E})$ (resp., $\mathscr{C}^b(\mathscr{E})$) the spaces of all (respectively, bounded) selections (= choice functions) $\sigma : \Delta \to \mathscr{E}$, by $\Gamma(\pi)$ the space of sections (= continuous choice functions) $\sigma : \Delta \to \mathscr{E}$, and by $\Gamma^b(\pi)$ and $\Gamma_0(\pi)$ the subspaces of $\Gamma(\pi)$ which, respectively, are bounded and vanish at infinity. We especially draw upon the following result, which is a special case of [2, Corollary 3.7].

Proposition 1 (see [3, Proposition 1.3]). *Let U be a topological space and let $\{X^p : p \in U\}$ be a collection of closed subspaces of the Banach space X. Let*

$\mathscr{E} = \dot{\bigcup}\{X/X^p : p \in U\}$ be the disjoint union of the quotient spaces X/X^p. Then, $\mathscr{E}$ can be topologized uniquely in such a way that the conditions (1) $\pi : \mathscr{E} \to U$ is a bundle of Banach spaces; and (2) for each $x \in X$, the selection $\tilde{x} : U \to \mathscr{E}$, $\tilde{x}(p) = x + X^p$ is a bounded section of the bundle $\pi : \mathscr{E} \to U$ are satisfied if and only if the function $p \mapsto \|\tilde{x}(p)\|$ is upper semicontinuous on U for each $x \in X$.

Let A be a commutative Banach algebra and let X be a Banach A-module. Let $\mathscr{E} = \dot{\bigcup}\{X_h : h \in \Delta\}$ be the disjoint union of the X_h. We give an element $x + X^h \in X_h \subset \mathscr{E}$ its quotient norm $\|x + X^h\|$ and we let $\pi : \mathscr{E} \to \Delta$ be the obvious projection.

In [10], it was shown that if Δ is a compact Hausdorff space, then any (unital) module X over $A = C(\Delta)$ can be represented as a space of sections in a bundle $\pi : \mathscr{E} \to \Delta$ of Banach spaces, in this case, with fibers $E_h = X_h = X/K_hX$. This (clearly norm-decreasing) representation is given by $\tilde{}: X \to \Gamma(\pi)$, $\tilde{x}(h) = x + X^h \in X_h$, and satisfies the equation $\widetilde{ax}(h) = \hat{a}(h)\tilde{x}(h)$. In [3], it was shown that when X is an essential module over a commutative algebra A with bounded approximate identity, then there is a bundle $\pi : \mathscr{E} \to \Delta$, again, with fibers $E_h = X_h = X/K_hX$, and a (again, norm-decreasing) representation $\tilde{}: X \to \Gamma_0(\pi)$ satisfying the same equation. Using the quotient modules suggested in [9], it was shown in [7] that when A has a bounded approximate identity and X is any Banach A-module, not necessarily essential, then there is in fact a bundle $\pi : \mathscr{E} \to \Delta$ with fibers $E_h = X_h = X/X^h$ and a norm-decreasing representation $\tilde{}: M(X) \to \Gamma^b(\pi)$ given by $\tilde{T}(h) = T(e_h) + X^h$. Again, $\widetilde{aT}(h) = \hat{a}(h)\tilde{T}(h)$.

The purpose of this paper is to show that this notion of sectional representation can be extended to modules X over arbitrary commutative Banach algebras A, that is, the earlier conditions that X is essential or that A have a bounded approximate identity can be removed. Thus, the representation obtained will be norm decreasing. We also show that $M(X)$ can be represented by sections in the same bundle as X, and give an example to show that this representation need not be continuous.

We define a map $\tilde{}: X \to \mathscr{C}^b(\mathscr{E})$ by $\tilde{x}(h) = x + X^h$. We also define a map $\tilde{}: M(X) \to \mathscr{C}(\mathscr{E})$, the space of all choice functions from Δ to $\mathscr{E}$, by $\tilde{T}(h) = T(e_h) + X^h$. From a remark in [7], we note that the equations $\widetilde{ax}(h) = \hat{a}(h)\tilde{x}(h)$ and $\widetilde{aT}(h) = \hat{a}(h)\tilde{T}(h)$ still hold. For $x \in X$ and for $h \in \Delta$, we have

$$\widetilde{T_x}(h) = T_x(e_h) + X^h = e_hx + X^h = \widetilde{e_hx}(h) = \widehat{e_h}(h)\tilde{x}(h) = \tilde{x}(h), \tag{1}$$

that is, $\tilde{x} = \widetilde{T_x}$.

We now demonstrate that the selections $\{\widetilde{T_x} : x \in X,\ \widetilde{T_x}(h) = e_hx + X^h = x + X^h \text{ for } h \in \Delta\}$ generate a (unique) bundle topology on $\mathscr{E} = \dot{\bigcup}\{X_h : h \in \Delta\}$ in the most general situation (although the next proposition actually shows a little more than we need).

Proposition 2. *Let A be a commutative Banach algebra, let X be a Banach module over A, and let $T \in M(X)$. Then, the mapping $h \mapsto \|\widetilde{T}(h)\| = \|T(e_h) + X^h\|$ is upper semicontinuous on Δ.*

Proof. The proof follows that of [7, Proposition 2.5] with only one alteration. Suppose that $\varepsilon > 0$ is given and that $\|\widetilde{T}(h)\| < \varepsilon$. Choose $a_i \in K_h$, $y_i \in X$, $(i = 1,\dots,n)$, and $z \in X$ such that

$$||\widetilde{T}(h)|| \le \left\|T(e_h) + \sum a_i y_i + (1 - e_h)z\right\| < \varepsilon \tag{2}$$

and set

$$\varepsilon' = \varepsilon - \left\|T(e_h) + \sum a_i y_i + (1 - e_h)z\right\|. \tag{3}$$

The functions $h' \mapsto \|a_i + K_{h'}\|$ are all upper semicontinuous on Δ, the function $h' \mapsto \|h'\|$ is lower semicontinuous on Δ, and $\|a + K_{h'}\| = |\hat{a}(h')|/\|h'\|$ (see [4]). (In particular, the lower semicontinuity of $h' \mapsto \|h'\|$ is what allows us to obtain the results in all what follows in this paper.) The function $\widehat{e_h}$ is also continuous on Δ. We can therefore find a neighborhood V of h such that when $h' \in V$, all of the following hold:

$$\begin{gathered} \sum ||a_i + K_{h'}|| < \frac{\varepsilon'}{3(\sum ||y_i|| + 1)}; \\ \left|\frac{1}{\widehat{e_h}(h)} - \frac{1}{\widehat{e_h}(h')}\right| = \left|1 - \frac{1}{\widehat{e_h}(h')}\right| < \frac{\varepsilon'}{3(||T(e_h)|| + 1)}; \\ \|h'\| > \frac{\|h\|}{2} > 0; \qquad |1 - \widehat{e_h}(h')| < \frac{\varepsilon'\|h\|}{6(\|z\| + 1)}. \end{gathered} \tag{4}$$

Since the definition of $X^{h'}$ is independent of the choice of $e_{h'}$ for $h' \in V$, we may as well take $e_{h'} = (1/\widehat{e_h}(h'))e_h$.

Then for $h' \in V$, we have

$$\begin{aligned} ||\widetilde{T}(h')|| &= ||T(e_{h'}) + X^{h'}|| \\ &\le ||T(e_{h'}) - T(e_h) + X^{h'}|| + \left\|T(e_h) + \sum a_i y_i + (1 - e_h)z + X^{h'}\right\| \\ &\quad + \left\|\sum a_i y_i + (1 - e_h)z + X^{h'}\right\| \\ &\le ||T(e_{h'}) - T(e_h)|| + \left\|T(e_h) + \sum a_i y_i + (1 - e_h)z\right\| \\ &\quad + \left\|\sum a_i y_i + X^{h'}\right\| + ||(e_h - e_{h'})z + X^h|| \end{aligned}$$

$$\begin{aligned}
&= |1-\widehat{e_h}(h')|\,\|T(e_h)\| + \left\|T(e_h) + \sum a_i y_i + (1-e_h)z\right\| \\
&\quad + \left\|\sum a_i y_i + X^{h'}\right\| + \|(e_h - e_{h'})z + X^{h'}\| \\
&\le \frac{\varepsilon'}{3} + \left\|T(e_h) + \sum a_i y_i + (1-e_h)z\right\| + \left\|\sum a_i y_i + K_{h'}X\right\| \\
&\quad + \|(e_h - e_{h'}) + K_{h'}\|\,\|z + K_{h'}X\| \\
&\le \frac{\varepsilon'}{3} + \left\|T(e_h) + \sum a_i y_i + (1-e_h)z\right\| + \sum \|a_i + K_{h'}\|\,\|y_i\| \\
&\quad + \frac{|\widehat{e_{h'}}(h') - \widehat{e_h}(h')|}{\|h'\|}\|z\| \\
&< \varepsilon.
\end{aligned} \tag{5}$$

□

Note that the key to the above generalization is the observation that the function $h' \mapsto \|h'\|$ is locally bounded away from 0. This will also play a part in the sequel.

Corollary 3. *Let A be a commutative Banach algebra and X a Banach module over A. Then there is a unique topology on $\mathscr{E} = \dot{\bigcup}\{X_h : h \in \Delta\}$ which makes $\pi : \mathscr{E} \to \Delta$ a bundle of Banach spaces, and such that for each $x \in X$, $\tilde{x} : \Delta \to \mathscr{E}$ is an element of $\Gamma^b(\pi)$.*

Proof. Note that for $x \in X$, we have $\widetilde{T_x} = \tilde{x} : \Delta \to \mathscr{E}$, and apply Propositions 1 and 2. □

We call this mapping $\tilde{\ } : X \to \Gamma(\pi)$ the *Gelfand representation of* X and $\pi : \mathscr{E} \to \Delta$ the *canonical bundle* for X. (In [7], $\pi : \mathscr{E} \to \Delta$ is also called the multiplier bundle for X. The reason for this nomenclature shift has to do with the universal property that is to be discussed later.)

Recall now that in the bundle topology on $\mathscr{E}$, neighborhoods of a point $x + X^h$ are described by tubes. Let $\sigma \in \Gamma^b(\pi)$ be such that $\sigma(h) = x + X^h$, let V be a neighborhood of h in Δ, and let $\varepsilon > 0$. Then, $\mathscr{T} = \mathscr{T}(V, \sigma, \varepsilon) = \{z + X^{h'} : h' \in V,\ \|\sigma(h') - (z + X^{h'})\| < \varepsilon\}$ is a neighborhood of $x + X^h$, and in fact sets of this form, as V ranges over all neighborhoods of h and $\varepsilon > 0$ varies, form a fundamental system of neighborhoods of $\sigma(h) = x + X^h$. We rely on this description to prove the following corollary.

Corollary 4. *Assume that A and X are as generally given and let $T \in M(X)$. Then $\tilde{T} \in \Gamma(\pi)$.*

Proof. Let $h \in \Delta$ be fixed, and set $\tilde{T}(h) = x + X^h$ for some fixed $x \in X$. ($x = T(e_h) \in X$ will do.) Let $\sigma \in \Gamma^b(\pi)$ be such that $\sigma(h) = x + X^h$ ($\sigma = \widetilde{T(e_h)} \in \Gamma^b(\pi)$ will do.) Let V be a neighborhood in Δ of h, and let $\varepsilon > 0$. We need to find a neighborhood V' of h such that $\widetilde{T}(V') \subset \mathcal{T}(V,\sigma,\varepsilon)$. Since $\widetilde{x} \in \Gamma^b(\pi)$ is continuous, there exists a neighborhood $V' \subset V$ of h such that if $h' \in V'$, then $\|\widetilde{x}(h') - \sigma(h')\| < \varepsilon/2$. Since $T - T_x$ is a multiplier of X, the map $h' \mapsto \|\widetilde{(T - T_x)}(h')\|$ is upper semicontinuous on Δ, by Proposition 2. Hence, there is a neighborhood $V'' \subset V'$ of h such that if $h' \in V''$, then by our choice of x, we have $\|\widetilde{(T - T_x)}(h')\| < \varepsilon/2$ since $\widetilde{(T - T_x)}(h) = 0$. Then, it follows immediately that for $h' \in V''$, we have

$$||\widetilde{T}(h') - \sigma(h')|| \le ||\widetilde{(T - T_x)}(h')|| + ||\widetilde{x}(h') - \sigma(h')|| < \varepsilon \tag{6}$$

so that $\widetilde{T}(h') \in \mathcal{T}(V,\sigma,\varepsilon)$. □

In particular, if $T \in M(X)$, then $\widetilde{T}$ is "locally close" in $\Gamma(\pi)$ to sections of the form $\widetilde{T_x} = \widetilde{x}$.

It was shown in [7] that if A has a bounded approximate identity and if $T \in M(X)$, then in fact $\widetilde{T}$ is bounded. In the absence of a bounded approximate identity, the boundedness of $\widetilde{T}$ cannot be guaranteed, as in the next example.

Example 5. For each $n \in \mathbb{N}$, let $E_n = \mathbb{C}$, with norm $\|\alpha\|_n = n|\alpha|$, and let $A = \{x : \mathbb{N} \to \mathbb{C} : \lim_n \|x(n)\|_n = 0\}$. Then A is a Banach algebra under the pointwise operations and norm $\|x\| = \sup_n\{\|x(n)\|_n\}$, and by [5], we have $\Delta = \Delta_A = \{\phi_n : n \in \mathbb{N}\}$ where $\phi_n(x) = x(n)$. We have $\|\phi_n\| = \sup_{\|x\|=1}\{|\phi_n(x)|\} = \sup_{\|x\|=1}\{|x(n)|\}$. If $\|x\| = 1$, then for each n, we have $|x(n)| \le 1/n$; on the other hand, if e_n is the standard basis vector $(e_n(j) = \delta_{nj})$, then $\phi_n((1/n)e_n) = 1/n$ and $\|(1/n)e_n\| = 1$. Hence, $\|\phi_n\| = 1/n$, and so by [7, Lemma 2.3], A has no bounded approximate identity (although $\{\sum_{k=1}^n e_k : n \in \mathbb{N}\}$ does form an approximate identity). Consider A as a module over itself. The sequence y given by $y(n) = 1/\sqrt{n}$ defines a multiplier T_y on A, $T_y(x)(n) = x(n)y(n) = (1/\sqrt{n})x(n)$, and it is easy to see that T_y is norm decreasing. Note that $y \notin A$ since $\|y\| = \infty$. For each n, $\|\widetilde{T_y}(n)\| = \|T_y(e_n) + A^{\phi_n}\| = \inf\{\|(1/\sqrt{n})e_n + a\| : a \in A^{\phi_n}\}$. Now, $A^{\phi_n} = \ker\phi_n = \{a \in A : a(n) = 0\}$ and so

$$\inf\left\{\left\|\frac{1}{\sqrt{n}}e_n + a\right\| : a \in A^{\phi_n}\right\} = \inf\left\{\left\|\frac{1}{\sqrt{n}}e_n\right\| + \|a\| : a \in A^{\phi_n}\right\} = \sqrt{n}. \tag{7}$$

Thus, $\widetilde{T_y}$ is unbounded.

Moreover, if $T \in M(A)$ and if $\widetilde{T}$ is bounded, then $T = T_y$ for some $y \in A$. To see this, let $n,k \in \mathbb{N}$ and let $T \in M(A)$. Then $(e_nT(e_n))(k) = e_n(k)Te_n(k) = \delta_{nk}Te_n(k) = 0$ if $n \ne k$. Define y by $y(n) = (Te_n)(n)$; clearly, for $a \in A$, we have

$$T(a)(n) = e_n(n)(Ta)(n) = a(n)(Te_n)(n) = a(n)y(n) = (T_y a)(n), \tag{8}$$

so that $T = T_y$.

Now, suppose that $\widetilde{T} = \widetilde{T_y}$ is bounded; we will show that $y \in A$. In fact, for all n, we have

$$\begin{aligned} ||y(n)||_n &= ||y(n)e_n(n)||_n = ||T_y(e_n)|| \\ &= \inf\{||T_y(e_n)+a|| : a \in A^{\phi_n}\} = ||\widetilde{T_y}(\phi_n)|| \\ &\le ||T_y||||\phi_n|| = \frac{||T_y||}{n} \longrightarrow 0 \end{aligned} \tag{9}$$

as $n \to \infty$. Hence, $y \in A$.

On the basis of this example, we ask the following: let $T \in M(X)$. If $\widetilde{T} \in \Gamma^b(\pi)$, what conditions on A and X will guarantee that there is some $x \in X$ such that $\widetilde{T_x} = \widetilde{x}$? If $\widetilde{T} \in \Gamma_0(\pi)$, what conditions on A and X will guarantee that $\widetilde{T_x} = \widetilde{x}$ for some $x \in X_e$?

Now, let A and X be as generally given. Following [3, Section 2] , if $\Psi : X \to \Gamma(\rho)$ is a bounded map, where $\rho : \mathcal{F} \to \Delta = \Delta_A$ is a Banach bundle, we will call Ψ a *sectional representation of Gelfand type* provided that $\Psi(ax) = \hat{a}\Psi(x)$. In [3, Theorem 2.7], it is shown that if A has a bounded approximate identity and if X is an essential A-module, then the representation $\sim : X \to \Gamma_0(\pi)$, where $\pi : \mathcal{E} \to \Delta$ is the canonical bundle for X described above, is universal with respect to all sectional representations of X of Gelfand type. In that context, this means that if $\rho : \mathcal{F} \to \Delta$ is a bundle of Banach spaces and if $\Psi : X \to \Gamma_0(\rho)$ is a sectional representation of Gelfand type, then there is a unique continuous map $\widetilde{\Psi} : \mathcal{E} \to \mathcal{F}$ which is fiber-preserving (meaning $\widetilde{\Psi}(E_h) \subset F_h$) and linear on each fiber. Moreover, $\|\widetilde{\Psi}\| = \sup_h\{\|\widetilde{\Psi} \restriction E_h\|\} \le \|\Psi\|$ and $\Psi(x) = \widetilde{\Psi} \circ \widetilde{x}$. When A has a bounded approximate identity and X is essential, this universal property characterizes the canonical bundle $\pi : \mathcal{E} \to \Delta$ up to isomorphism. The same universal property can now be shown to obtain in the general case.

Proposition 6. *Let A and X be as generally given and let $\pi : \mathcal{E} \to \Delta$ be the canonical bundle for X constructed in Corollary 3. Let $\rho : \mathcal{F} \to \Delta$ be a Banach bundle with fibers $\{F_h : h \in \Delta\}$ and suppose that $\Psi : X \to \Gamma(\rho)$ is a sectional representation of Gelfand type. Then, there exists a unique fiber-preserving continuous map $\widetilde{\Psi} : \mathcal{E} \to \mathcal{F}$ such that $\|\widetilde{\Psi}\| \le \|\Psi\|$ and such that $\Psi(x) = \widetilde{\Psi} \circ \widetilde{x}$.*

Proof. For $h \in \Delta$, define $\Psi_h : X \to F_h$ by $\Psi_h(x) = [\Psi(x)](h)$. Then the kernel of Ψ_h contains X^h (Justification: $\Psi_h(ax + (1-e_h)z) = [\Psi(ax+(1-e_h)z)](h) = 0$ for $a \in K_h$ and $x, z \in X$, because Ψ is of Gelfand type and because $\widehat{e_h}(h) = 1$). This induces a map $\widetilde{\Psi_h} : X_h = X/X^h \to F_h$ such that $\widetilde{\Psi_h}(x + X^h) = \Psi_h(x) = [\Psi(x)](h)$, and we define $\widetilde{\Psi}$ on all of $\mathcal{E}$ by $\widetilde{\Psi}(x + X^h) = \widetilde{\Psi_h}(x + X^h)$. Clearly, $\|\widetilde{\Psi}\| = \sup\{\|\widetilde{\Psi_h}\| : h \in \Delta\} \le \|\Psi\|$. We can now essentially repeat the proof of [3, Theorem 2.7] to get the desired result. □

Corollary 7. *Let A and X be as generally given. Suppose that $\rho_k : \mathcal{F}_k \to \Delta$ are Banach bundles and that $\phi_k : X \to \Gamma(\rho_k)$ $(k = 1,2)$ are Gelfand representations of X. Suppose also that if $\phi : X \to \Gamma(\xi)$ $(\xi : \mathcal{G} \to \Delta)$ is any sectional representation of Gelfand type, then there exist unique continuous, fiber-preserving, and linear-on-fibers maps $\widetilde{\phi_k} : \mathcal{F}_k \to \mathcal{G}$ such that $\|\widetilde{\phi_k}\| \le \|\phi\|$ and such that $\phi(x) = \widetilde{\phi_k} \circ \phi_k(x)$ $(k = 1,2)$. Then, there exists a continuous map $\Phi : \mathcal{F}_1 \to \mathcal{F}_2$ such that $\phi_2(x) = \Phi \circ \phi_1(x)$ for all $x \in X$, and such that Φ is fiber-preserving and a linear isomorphism on each fiber.*

Proof. Here, we may repeat the proof of [3, Proposition 2.8]. □

In [7, Proposition 2.8], it is shown that in the presence of a bounded approximate identity in A, the canonical bundles $\pi : \mathcal{E} \to \Delta$ for X and $\pi' : \mathcal{E}' \to \Delta$ for X_e are homeomorphic. The proof uses the facts that (1) with or without a bounded approximate identity in A, for $h \in \Delta$, X_h and $X_h^{'} = X_e/(X_e)^h$ are topologically isomorphic via the maps $\psi_h : X_h \to X_h^{'}$, $\psi_h(x + X^h) = e_h x + K_h X_e$, and $\phi_h : X_h^{'} \to X_h$, $\phi_h(ax + K_h X_e) = ax + X^h$, with $\|\psi_h\| \le \|e_h\|$ and $\|\phi_h\| \le 1$ [7, Proposition 2.7]; and (2) when A has a bounded approximate identity, the set $S = \{e_h : h \in \Delta\}$ can be chosen to be bounded. The bound on S is then used to obtain the homeomorphism. Even without the bounded approximate identity, we can easily modify the proof of [7, Proposition 2.8] to obtain the following proposition.

Proposition 8. *Let A and X be as generally given and let $\pi : \mathcal{E} \to \Delta$ and $\pi' : \mathcal{E}' \to \Delta$ be the canonical bundles for X and X_e, respectively. Then, $\mathcal{E}$ and $\mathcal{E}'$ are homeomorphic in their bundle topologies.*

Proof. We will show that the map $\Psi : \mathcal{E} \to \mathcal{E}'$ given by $\Psi(x + X^h) = \psi_h(x + X^h) = e_h x + K_h X_e$ is continuous; the proof of the continuity of the inverse map $\Phi : \mathcal{E}' \to \mathcal{E}$, $\Phi(ax + K_h X_e) = \phi_h(ax + K_h X_e)$ will be similar.

Fix $h \in \Delta$ and let $x + X^h \in \mathcal{E}$. Let $\mathcal{T}_1 = \mathcal{T}_1(V, \widetilde{e_h x}, \varepsilon)$ be a tube around $e_h x + K_h X_e = \Psi(x + X^h) = \psi_h(x + X^h) \in \mathcal{E}'$, and let $V' \subset V$ be a neighborhood of h such that for $h' \in V'$, we have $|\widehat{e_h}(h')| > 1/2$. In V', set $e_{h'} = (1/\widehat{e_h}(h'))e_h$; then $\|e_{h'}\| < 2\|e_h\|$. Then, $\mathcal{T}_2 = \mathcal{T}_2(V', \tilde{x}, \varepsilon/(2\|e_h\|))$ is a neighborhood of $x + X^h = \tilde{x}(h)$ in $\mathcal{E}$. Taking $y + X^{h'} \in \mathcal{T}_2$, we have $h' \in V'$ and

$$||(y + X^{h'}) - \tilde{x}(h')|| = ||(y + X^{h'}) - (x + X^{h'})|| < \frac{\varepsilon}{2||e_h||}. \tag{10}$$

Then

$$\begin{aligned} ||\Psi(y + X^{h'}) - \Psi(x + X^{h'})|| &= ||\psi_{h'}(y + X^{h'}) - \psi_{h'}(x + X^{h'})|| \\ &\le ||e_{h'}||||(y + X^{h'}) - (x + X^{h'})|| < \varepsilon \end{aligned} \tag{11}$$

so that $\Psi(y + X^{h'}) \in \mathcal{T}_1$. □

We noted above that in the most general case, elements of the form $\widetilde{T}$ ($T \in M(X)$) are locally close to elements of the form $\widetilde{x}$ ($x \in X$). With an additional assumption on A, local closeness can be replaced by local equality.

Proposition 9. *Let A be a completely regular algebra and let X be an A-module. Let $h \in \Delta$ and let $T \in M(X)$. Then for each compact set $V \subset \Delta$ containing h, there exists $x \in X_e$ such that $\widetilde{T}(h') = \widetilde{x}(h')$ for $h' \in V$.*

Proof. Let V be such a compact set containing h. It then follows from [6, Theorem 2.7.12] that there exists $e \in A$ such that $\hat{e} \equiv 1$ on V. Set $x = eT(e) \in X_e$, and for $h' \in V$, let $e_{h'} = e$. Then for $h' \in V$, we have

$$\widetilde{T}(h') = T(e_{h'}) + X^{h'} = T(e) + X^{h'} = eT(e) + X^{h'} = \widetilde{eT(e)}(h') = \widetilde{x}(h'). \quad (12)$$

□

Compare this to [8, Theorem 4.1 and Corollary 4.2]. In our case, for $T \in M(X)$, it cannot be guaranteed that there might exist $x \in X$ such that $\widetilde{T} = \widetilde{x}$ on a neighborhood of infinity, because $\widetilde{T}$ can be unbounded, while $\widetilde{x}$ is bounded. However, in the event that there actually does exist $x \in X$ such that $\widetilde{T} = \widetilde{x}$ on some neighborhood of infinity, we obtain the following corollary, which is similar to [8, Corollary 4.2].

Corollary 10. *Suppose that A is completely regular and that $T \in M(X)$. If there exists a neighborhood V of infinity such that for some $x \in X$, we have $\widetilde{T}(h') = \widetilde{x}(h')$ for $h' \in V$, then there exists $y \in X$ such that $\widetilde{T} = \widetilde{y}$ on all of Δ.*

Proof. We can repeat the partition of unity proof of [8, Theorem 4.1]. □

Acknowledgment. The authors thank the referee for his careful reading of the manuscript.

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