ON THE PRIME SUBMODULES OF MULTIPLICATION MODULES

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By considering the notion of multiplication modules over a commutative ring with identity, first we introduce the notion product of two submodules of such modules. Then we use this notion to characterize the prime submodules of a multiplication module. Finally, we state and prove a version of Nakayama lemma for multiplication modules and find some related basic results.

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1. Introduction. Let R be a commutative ring with identity and let M be a unitary R -module. Then, M is called a multiplication R -module provided for each submodule N of M ; there exists an ideal I of R such that $N = IM$. Note that our definition agrees with that of [1, 2], but in [6] the term *multiplication module* is used in a different way. (In this paper, an R -module M is a multiplication if and only if every submodule of M is a multiplication module in the above sense.) Recently, prime submodules have been studied in a number of papers; for example, see [3, 4, 5]. Now in this paper, first we define the notion of product of two submodules of a multiplication module and then we obtain some related results. In particular, we give some equivalent conditions for prime submodules of multiplication submodules. Finally, we state and prove a version of Nakayama lemma for multiplication modules.

2. Preliminaries. Throughout this paper, R denotes a commutative ring with identity and all related modules are unitary R -modules.

DEFINITION 2.1. A proper submodule K of M is called *prime* if $rm \in K$, for $r \in R$ and $m \in M$, then $r \in (K : M)$ or $m \in K$, where $(K : M) = \{r \in R \mid rM \subseteq K\}$.

THEOREM 2.2 (see [5]). *Let K be a submodule of M . Then the following statements are satisfied:*

- (i) *K is prime if and only if $P = (K : M)$ is a prime ideal of R and R/P -module M/K is torsion-free,*
- (ii) *if $(K : M)$ is a maximal ideal of R , then K is a prime submodule of M .*

For any R -module M , let $\text{Spec}(M)$ denote the collection of all prime submodules of M . Note that some modules M have no prime submodules (i.e., $\text{Spec}(M)$

is empty); such modules are called *primeless*. For example, the zero-module is primeless. In [5], some nontrivial examples are shown and some conditions for primeless modules are given.

DEFINITION 2.3. An R -module M is a multiplication module if for every submodule N of M , there is an ideal I of R such that $N = IM$.

LEMMA 2.4 (see [1]). *Let M be a multiplication module and let N be a submodule of M . Then $N = (\text{ann}(M/N))M$.*

LEMMA 2.5 (see [1, Proposition 1.1]). *An R -module M is a multiplication if and only if for each m in M , there exists an ideal I of R such that $Rm = IM$.*

LEMMA 2.6 (see [1]). *An R -module M is a multiplication if and only if*

$$\cap_{\lambda \in \Lambda} (I_\lambda M) = (\cap_{\lambda \in \Lambda} [I_\lambda + \text{ann}(M)])M \quad (2.1)$$

for any collection of ideals I_λ ($\lambda \in \Lambda$) of R .

THEOREM 2.7 (see [1, Theorem 2.5]). *Let M be a nonzero multiplication R -module. Then,*

- (i) *every proper submodule of M is contained in a maximal submodule of M ;*
- (ii) *K is a maximal submodule of M if and only if there exists a maximal ideal P of R such that $K = PM \neq M$.*

THEOREM 2.8 (see [1, Corollary 2.11]). *The following statements are equivalent for a proper submodule N of M :*

- (i) *N is a prime submodule of M ;*
- (ii) *$\text{ann}(M/N)$ is a prime ideal of R ;*
- (iii) *$N = PM$ for some prime ideal P of R with $\text{ann}(M) \subseteq P$.*

THEOREM 2.9 (see [1, Theorem 3.1]). *Let M be a faithful multiplication R -module. Then the following statements are equivalent:*

- (i) *M is finitely generated;*
- (ii) *$AM \subseteq BM$ if and only if $A \subseteq B$;*
- (iii) *for each submodule N of M , there exists a unique ideal I of R such that $N = IM$;*
- (iv) *$M \neq AM$ for any proper ideal A of R ;*
- (v) *$M \neq PM$ for any maximal ideal P of R .*

DEFINITION 2.10. Let N be a proper submodule of M . Then, the radical of N denoted by $M\text{-rad}(N)$ or $r(N)$ is defined in [1] to be the intersection of all prime submodules of M containing N .

THEOREM 2.11 (see [1, Corollary 2.11]). *Let N be a proper submodule of a multiplication R -module M . Then $M\text{-rad}(N) = \sqrt{AM}$, where $A = \text{ann}(M/N)$.*

DEFINITION 2.12. Let M be an R -module. Then, the radical of M denoted by $\text{rad}(M)$ is defined to be the intersection of the maximal submodules of M if such exists, and M otherwise.

Let $\mathcal{M}$ denote the collection of all maximal ideals of R . Define $P_1(M) = \{P \in \mathcal{M} \mid M \neq PM\}$ and $P_2(M) = \{P \in \mathcal{M} \mid \text{ann}(M) \subseteq P\}$. Now, define $J_1(M) = \cap \{P \mid P \in P_1(M)\}$ and $J_2(M) = \cap \{P \mid P \in P_2(M)\}$.

THEOREM 2.13 (see [1, Theorem 2.7]). *Let M be a multiplication R -module. Then $\text{rad}(M) = J_1(M)M = J_2(M)M$.*

3. The product of multiplication submodules

DEFINITION 3.1. Let M be an R -module and let N be a submodule of M such that $N = IM$ for some ideal I of R . Then, we say that I is a *presentation ideal* of N or, for short, a *presentation* of N . We denote the set of all presentation ideals of N by $\text{Pr}(N)$.

Note that it is possible that for a submodule N , no such presentation ideal exists. For example, if V is a vector space over an arbitrary field with a proper subspace W ($\neq 0$ and V), then W does not have any presentations. By [Lemma 2.4](#), it is clear that every submodule of M has a presentation ideal if and only if M is a multiplication module. In particular, for every submodule N of a multiplication module M , $\text{ann}(M/N)$ is a presentation for N .

Let $L(R)$ and $L(M)$ denote the lattices of ideals of R and submodules of M , respectively. Define the relation $\sim$ on $L(R)$ as follows:

$$I \sim J \iff IM = JM. \quad (3.1)$$

It is easy to verify that this relation is an equivalence relation on $L(R)$. We denote the equivalence class of $I \in L(R)$ by $[I]$.

THEOREM 3.2. *Let M be a faithful multiplication R -module. Then, the following statements are equivalent:*

- (i) M is finitely generated;
- (ii) each equivalence class of the relation $\sim$ is a singleton;
- (iii) the map

$$\varphi : L(R) \longrightarrow L(M) \quad (3.2)$$

defined by $\varphi(I) = IM$ is a lattice isomorphism;

- (iv) for every proper ideal I of R , $[I] = \{I\}$;
- (v) for any maximal ideal P of R , $[P] = \{P\}$.

PROOF. (i) $\Rightarrow$ (ii) follows from [Theorem 2.8](#), [Definition 3.1](#), and [Theorem 2.9](#).

(ii) $\Rightarrow$ (iii). By [Theorem 2.8](#), we conclude that φ is bijective and order-preserving. Obviously, $(I + J)M = IM + JM$ and by [Lemma 2.5](#), $(I \cap J)M = IM \cap JM$ since M is faithful. Therefore, ϕ is a lattice isomorphism.

(iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v), and (v) $\Rightarrow$ (i) are an immediate consequence of [Theorem 2.8](#). $\square$

DEFINITION 3.3. Let $N = IM$ and $K = JM$ for some ideals I and J of R . The product of N and K is denoted by $N \cdot K$ or NK is defined by IJM .

Clearly, NK is a submodule of M and contained in $N \cap K$. Now, we show that the product of two submodules is defining an operation on submodules of M .

THEOREM 3.4. Let $N = IM$ and $K = JM$ be submodules of a multiplication R -module M . Then, the product of N and M is independent of presentations of N and K .

PROOF. Let $N = I_1M = I_2M = N'$ and $K = J_1M = J_2M = K'$ for ideals I_i and J_i of R , $i = 1, 2$. Consider $rs m \in NK = I_1J_1M$ for some $r \in I_1$, $s \in J_1$, and $m \in M$. From $J_1M = J_2M$, we have

$$sm = \sum_{i=1}^n r_i m_i, \quad r_i \in J_2, \quad m_i \in M. \quad (3.3)$$

Then,

$$rsm = \sum_{i=1}^n r_i (r m_i). \quad (3.4)$$

From $r m_i \in I_1M = I_2M$, we conclude that

$$r m_i = \sum_{j=1}^k t_{ij} m'_{ij}, \quad t_{ij} \in I_2, \quad m'_{ij} \in M. \quad (3.5)$$

Thus,

$$rsm = \sum_{i=1}^n \sum_{j=1}^k r_i t_{ij} m'_{ij}. \quad (3.6)$$

Therefore, $rsm \in I_2J_2M$, and hence $I_1J_1M \subseteq I_2J_2M$. Similarly, we have $I_2J_2M \subseteq I_1J_1M$. This completes the proof. $\square$

PROPOSITION 3.5. Let M be a multiplication module N , and let K and L be submodules of M . Then the following statements are satisfied:

- (i) $L(M)$, the lattice of submodules of M with operation product on submodules, is a semiring;
- (ii) the product is distributive with respect to the sum on $L(M)$;
- (iii) $(K + L)(K \cap L) \subseteq KL$;
- (iv) $K \cap L = KL$ provided $K + L = M$ (in this case, K and L are said to be coprime or comaximal).

PROOF. (i), (ii), (iii) are obtained from [Definition 3.3](#), [Lemma 2.5](#), the well-known related results of the ideals theory, and the fact that $\sum_{k \in K} I_k M = (\sum_{k \in K} I_k)M$.

(iv) $K + L = M$ implies that $M(K \cap L) \subseteq KL$ by (iii), and hence $K \cap L \subseteq KL$. Clearly $KL \subseteq K \cap L$. Therefore $KL = K \cap L$. $\square$

LEMMA 3.6. *Let N and K be submodules of a multiplication module M . Then,*

(i) *the ideals $\text{ann}(M/N) \cdot \text{ann}(M/K)$ and $\text{ann}(M/NK)$ are presentations of NK ;*

(ii) *if M is finitely generated, then $\text{ann}(M/N) \cdot \text{ann}(M/K) = \text{ann}(M/NK)$.*

PROOF. (i) By [Lemma 2.4](#) and [Theorem 3.4](#), $\text{ann}(M/N)$ and $\text{ann}(M/K)$ are presentations for N and K , respectively. Thus, by [Definition 3.3](#), $MN = [\text{ann}(M/N) \cdot \text{ann}(M/K)]M$. Therefore, $(\text{ann}(M/N) \cdot \text{ann}(M/K))$ is a presentation for MN .

(ii) By [Lemma 2.4](#), we have $MN = \text{ann}(M/NK)$ and hence by [Theorem 2.8](#) and (i), we conclude that

$$\text{ann}(M/N) \cdot \text{ann}(M/K) = \text{ann}(M/NK). \quad (3.7)$$

$\square$

REMARK 3.7. (i) Recall that by [Lemma 2.5](#), for any $m \in M$, we have $Rm = IM$ for some ideal I of R . In this case, we say that I is a presentation ideal of m or, for short, a presentation of m and denote it by $\text{Pr}(m)$. In fact, $\text{Pr}(m)$ is equal to $\text{Pr}(Rm)$.

(ii) For $m, m' \in M$, by mm' , we mean the product of Rm and Rm' , which is equal to IJM for every presentation ideals I and J of m and m' , respectively.

PROPOSITION 3.8. *Let M be a multiplication R -module. Let $N, K, N_i \in I$ be submodules of M , $s \in R$, and k any positive integer. Then the following statements are satisfied:*

- (i) $\text{Pr}(\sum_{i \in I} N_i) = \sum_{i \in I} \text{Pr}(N_i)$;
- (ii) $\text{Pr}(\cap_{i \in I} N_i) = (\cap_{i \in I} [\text{Pr}(N_i) + \text{ann}(M)])M$;
- (iii) $\text{Pr}(\sum_{i=1}^k m_i) \subseteq \sum_{i=1}^k \text{Pr}(m_i)$;
- (iv) $\text{Pr}(sm) = s \text{Pr}(m)$;
- (v) $\text{Pr}(NK) = \text{Pr}(N) \cdot \text{Pr}(K)$;
- (vi) $\text{Pr}(N^k) = (\text{Pr}(N))^k$;
- (vii) $\text{Pr}(m^k) = (\text{Pr}(m))^k$;
- (viii) $\text{Pr}(M\text{-rad}(N)) = M\text{-rad}(\text{Pr}(N))$.

PROOF. (i) Let I_i be presentation ideals of N_i for every $i \in I$. Then it is easy to verify that

$$\sum_{i \in I} N_i = \sum_{i \in I} (M_i) = \left(\sum_{i \in I} I_i \right) M. \quad (3.8)$$

Thus, $\text{Pr}(\sum_{i \in I} N_i) = \sum_{i \in I} \text{Pr}(N_i)$.

(ii) It is an immediate consequence of [Lemma 2.6](#).

(iii) By [Remark 3.7](#)(i), we have

$$\Pr\left(\sum_{i=1}^k m_i\right) = \Pr\left(R \sum_{i=1}^k m_i\right) \subseteq \Pr\left(R \sum_{i=1}^k R m_i\right) = \Pr\left(\sum_{i=1}^k R m_i\right) = \sum_{i=1}^k \Pr(m_i). \quad (3.9)$$

(iv), (v), (vi), and (vii) are an immediate consequence of [Theorem 3.4](#) and [Remark 3.7](#).

(viii) It follows from [Theorem 2.11](#). $\square$

DEFINITION 3.9. Let M be a multiplication R -module and let N be a submodule of M . Then,

- (i) N is called *nilpotent* if $N^k = 0$ for some positive integer k , where N^k means the product of N , k times;
- (ii) an element m of M is called nilpotent if $m^k = 0$ for some positive integer k .

The set of all nilpotent elements of M is denoted by N_M .

THEOREM 3.10. Let M be a multiplication module. A submodule N of M is nilpotent if and only if for every presentation ideal I of N , $I^k \subseteq \text{ann}(M)$ for some positive integer $k \in \mathbb{N}$.

PROOF. Let I be a presentation ideal of N . If N is nilpotent, then $N^k = 0$ for some positive integer k , that is, $N^k = I^k M = 0$. Thus, $I^k \subseteq \text{ann}(M)$. Conversely, suppose that $I^k \subseteq \text{ann}(M)$ for some presentation ideal I of N . Then,

$$N^k = I^k M \subseteq \text{ann}(M)M = 0. \quad (3.10)$$

Therefore, N is nilpotent. $\square$

COROLLARY 3.11. Let M be a faithful R -multiplication module and let N be a submodule of M . Then, N is nilpotent if and only if every presentation ideal of N is a nilpotent ideal.

THEOREM 3.12. Let M be a multiplication module. Then, N_M is a submodule of M and M/N_M has no nonzero nilpotent element.

PROOF. Let $x, y \in N_M$, say $x^m = 0$ and $y^n = 0$. Consider presentation ideals I and J of x and y , respectively. Then $x^m = I^m M = 0$ and $y^n = J^n M = 0$. Since $Rx = IM$ and $Ry = JM$, then by [Lemma 2.5](#), we have $R(x + y) \subseteq Rx + Ry = IM + JM = (I + J)M$, then $I + J$ is a presentation ideal for $x + y$. Let $l = m + n$. Then,

$$(x + y)^{m+n} = (I + J)^{m+n} M = \left(\sum_{i=0}^l \binom{l}{i} (I)^i (J)^{l-i} \right) M = (0)M = (0), \quad (3.11)$$

and hence $x + y \in N_M$. Now, let $m \in N_M$ and $r \in R$. Consider presentation ideal I of m . Thus, $m^k = I^k M = 0$ since $Rrm = (rI)M \subseteq IM$. Thus, $(rm)^k = (rI)^k M \subseteq I^k M = (0)$ and hence $rm \in N_M$. Therefore, N_M is a submodule of M .

Let $\bar{x} \in M/N_M$ be represented by x . Then, $\bar{x}^n$ is represented by x^n so that $\bar{x}^n = 0$. Thus, $x^n \in N_M$ and hence $(x^n)^k = 0$ for some $k \geq 0$. Therefore, $x \in N_M$ and so $\bar{x} = 0$. $\square$

THEOREM 3.13. *Let N be a submodule of a multiplication R -module M . Then $M\text{-rad}(N) = \{m \in M \mid m^k \subseteq N \text{ for some } k \geq 0\}$.*

PROOF. Let

$$B = \{m \in M \mid m^k \subseteq N \text{ for some } k \geq 0\}. \quad (3.12)$$

First, we show that B is a submodule of M . Let $x, y \in B$, and let I and J be presentation ideals of x and y , respectively. Then, $x^n = I^n$ and $y^m = JM \subseteq N$ for some positive integers m and n , and presentation ideals I, J of x and y , respectively. Let $k = \max\{m, n\}$. Then

$$\begin{aligned} (x + y)^k &= (IM + JM)^k = ((I + J)M)^k \\ &= (I + J)^k M = \sum_{i=0}^k \binom{k}{i} (IM)^i (JM)^{k-i}, \end{aligned} \quad (3.13)$$

that is, $x + y \in B$. Also, for $x \in B$ and $r \in R$, we have $(rx)^n \subseteq N$ since $x^n \subseteq N$. Thus, B is a submodule of M . Suppose that $m \in B$ and A is a presentation of m . Then, $m^k = A^k M \subseteq N$ for some $n \geq 1$ and hence by [Theorem 2.11](#), we have

$$M\text{-rad}(m^k) = \sqrt{A^k M} = \sqrt{AM} \subseteq M\text{-rad}(N). \quad (3.14)$$

Thus, $M\text{-rad}(Rm) = M\text{-rad}(AM) \subseteq M\text{-rad}(N)$ and this implies that $B \subseteq M\text{-rad}(N)$.

Conversely, let $m \in M\text{-rad}(N) = \sqrt{IM}$, where $I = \text{ann}(M/N)$. Then, $m = \sum_{i=1}^n r_i m_i$ for $r_i \in \sqrt{I}$ and $m_i \in M$. Thus, $r_i^{n_i} \in I$ for some $n_i \geq 1$. Thus, for a sufficiently large n , we have $m^k \subseteq IM = N$ and hence $M\text{-rad}(N) \subseteq B$. Therefore, $B = M\text{-rad}(N)$. $\square$

COROLLARY 3.14. *Let M be a multiplication R -module. Then N_M is the intersection of all prime submodules of M .*

PROOF. By [Theorem 2.11](#), we have $M\text{-rad}(0) = \sqrt{AM}$, where $A = \text{ann}(M)$, and by [Theorem 3.13](#), $M\text{-rad}(N) = N_M$. $\square$

COROLLARY 3.15. *Let M be a faithful multiplication R -module. Then $N_M = \mathcal{N}M$, where $\mathcal{N}$ is the nilradical of R .*

THEOREM 3.16. *Let P be a proper submodule of a multiplication module M . Then P is prime if and only if*

$$UV \subseteq P \Rightarrow U \subseteq P \quad \text{or} \quad V \subseteq P \quad (3.15)$$

for each submodule U and V of M .

PROOF. Let P be prime and $UV \subseteq P$, but $U \not\subseteq P$ and $V \not\subseteq P$ for some submodules U and V of M . Suppose that I and J are presentations of U and V , respectively. Then $UV = IJM \subseteq P$. Thus, there are $ry \in U - P$ and $sx \in U - P$ for some $r \in I$ and $s \in J$. Thus, $rsx \in P$ and hence $rM \subseteq P$, that is, $ry \in P$, which is a contradiction.

Conversely, suppose that condition (3.15) is true. Let $rx \in P$ for some $r \in R$ and $x \in M - P$, but $rM \not\subseteq P$; then, $rm \notin P$ for some $m \in M$. Let I and J be presentation ideals of rx and m , respectively. Then

$$R(rx) \cdot (Rm) = (Rx) \cdot (Rrm) = IM \cdot JM = IJM \subseteq P. \quad (3.16)$$

Now, by hypothesis, we must have $Rx \subseteq P$ or $Rrm \subseteq P$, which implies that $x \in P$ or $rm \in P$, which is a contradiction. Therefore, P is prime. $\square$

COROLLARY 3.17. *Let P be a proper submodule of M . Then P is prime if and only if*

$$m \cdot m' \subseteq P \Rightarrow m \in P \quad \text{or} \quad m' \in P \quad (3.17)$$

for every $m, m' \in M$.

PROOF. If P is prime, then, clearly, (3.17) is true. Conversely, suppose that (3.17) is true, and $UV \subseteq P$ for submodules U and V of M , but $U \not\subseteq P$ and $V \not\subseteq P$. Thus, there are $u \in U - P$ and $v \in V - P$. Then $uv = RuRv \subseteq UV \subseteq P$ and hence by (3.17), we must have $u \in U$ or $v \in V$, which is a contradiction. Therefore, P is prime. $\square$

DEFINITION 3.18. An element u of an R -module M is said to be a *unit* provided that u is not contained in any maximal submodule of M .

THEOREM 3.19. *Let M be a multiplication R -module. Then $u \in M$ is a unit if and only if $\langle u \rangle = M$.*

PROOF. The sufficiency is clear. For a necessary part, let u be a unit element. Then $\langle u \rangle$ is not contained in any maximal submodule of M . Thus, by Theorem 2.7, we must have $\langle u \rangle = M$. $\square$

THEOREM 3.20. *Let M be an R -module (not necessarily multiplicative) such that M has a unit u . Then $m \in \text{rad}(M)$ if and only if $u - rm$ is unit for every $r \in R$.*

PROOF. See [7, Theorem 4.8]. $\square$

THEOREM 3.21. *Every homomorphic image of a multiplication module is a multiplication module.*

PROOF. Let M be a multiplication R -module, $\phi : M \rightarrow M'$ an R -module homomorphism, and $K = \phi(M)$. Let $k \in K$, then $k = (\phi m)$ for some $m \in M$. Since M is a multiplication, then by [Lemma 2.5](#), there is an ideal I of R such that $Rm = IM$. Thus,

$$\varphi(IM) = I\varphi(M) = IK = \varphi(Rm) = R\varphi(m) = Rk. \quad (3.18)$$

Therefore, by [Lemma 2.5](#), K is a multiplication R -module. $\square$

COROLLARY 3.22. *Let M be a multiplication R -module and N a submodule of M . Then, M/N is a multiplication R -module.*

THEOREM 3.23 (a version of Nakayama lemma). *Let M be a faithful multiplication R -module such that M has a unit u . Then, for every submodule N , the following conditions are equivalent:*

- (i) N is contained in every maximal submodule of M ;
- (ii) $u - rx$ is a unit for all $r \in R$ and for all $x \in N$;
- (iii) if M is a finitely generated R -module such that $NM = M$, then $M = 0$;
- (iv) if M is finitely generated and K is a submodule of M such that $M = NM + K$, then $M = K$.

PROOF. (i) $\Rightarrow$ (ii) is an immediate consequence of [Theorem 3.19](#).

(ii) $\Rightarrow$ (iii). Since M is finitely generated, there must be a minimal generating set $X = \{m_1, \dots, m_n\}$ of M . If $M \neq 0$, then $m_1 \neq 0$ by minimality. Now, let I be a presentation of N . Then, $NM = M$ implies that $M = IM \cdot M = M$, and since M is faithful, then by [Theorem 2.13](#), we have $N \subseteq \text{rad}(M) = J_1(M)M \subseteq J(R)M$. Thus, $m_1 = j_1 m_1 + j_2 m_2 + \dots + j_n m_n$ ($j_i \in J(R)$) whence $j_1 m_1 = m_1$ so that $(1 - j_1)m_1 = 0$ if $n = 1$, and

$$(1 - j_1)m_1 = j_2 m_2 + \dots + j_n m_n, \quad n > 1. \quad (3.19)$$

Since $1 - j_1$ is a unit in R , $m_1 = (1 - j_1)^{-1}(1 - j_1)m_1 + \dots + (1 - j_1)^{-1}j_n m_n$. Thus, if $n = 1$, then $m_1 = 0$, which is a contradiction. If $n > 1$, then m_1 is a linear combination of $m_2, m_3, \dots, m_n$; consequently, $\{m_2, \dots, m_n\}$ generates M , which contradicts the choice of X .

(iii) $\Rightarrow$ (iv). Since for every submodule K/N of M/N , we have $K/N = \text{ann}(M/N/K/N)M/N = \text{ann}(M/K)M/N$; then by [Corollary 3.22](#), M/N is a multiplication R -module. Now, it is easy to verify that $\text{rad}(M/N) = M/N$ and hence, by (iii), we must have $M = K$.

(iv) $\Rightarrow$ (i). Let K be any maximal submodule of M , then $K \subseteq NM = K$. Consequently, $NM + M = M$ by maximality of K , otherwise $M = K$ by (iv) a contradiction. Therefore, $N = NM \subseteq K$. $\square$

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